Singularities

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Any errors or inaccuracies are likely introduced by myself, Devlin Mallory.

1 January 28: Eric Canton, Introduction to (F)rigid geometry

References include:

(1) (characteristic 0) Jonsson–Mustaţă, "Valuations and asymptotic invariants of sequences of ideals".

(2) (characteristic 0) Boucksom–de Fernex–Favre–Urbinati, "Valuation spaces and multiplier ideals on singular varieties".

(3) (characteristic p) Canton, "Berkovich log discrepancies in positive characteristic".

For today, X will be an excellent integral (separated) scheme over a field k, and point will *not* mean closed points. We write $L = \operatorname{Frac} X$.

Definition. (1) A (real) valuation v on L is centered on X if there's some $x \in X$ such that

$$\mathcal{O}_{X,x} \subset A_v := \{f \in L : v(f) \geq 0\}.$$

(2) If v is centered on X, there's a unique $x \in X$ such that $\mathcal{O}_{X,x} \subset A_v$ is local (i.e., $m_v \cap \mathcal{O}_{X,x} = m_x$); this $x \in X$ is called the center of v on X, written $c_X(v)$.

(3) $\operatorname{Val}(X) = \{\text{valuations } v \text{ on } L \text{ centered on } X\}$.

Remark. If X is a projective (or proper) over k, then every valuation on L is centered on X. If X is not projective, then this is not true: for example, if $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$, then the valuation $\operatorname{ord}_{t^{-1}}$ is not centered on X, so $\operatorname{ord}_{t^{-1}} \notin \operatorname{Val}_{\mathbb{A}^1}$.

Remark. If $R \subset A_v$, then $m_v \cap R \in \operatorname{Spec} R$; call this point p. Then $R_p \subset A_v$ is a local inclusion and $p = c_{\operatorname{Spec} R}(v)$.

Example (examples of $v \in \operatorname{Val}(X)$)**.** (1) Say X is normal (or just regular in codimension 1); then if $E \subset X$ is a prime divisor then $\mathcal{O}_{X,E}$ is a DVR and the associated valuation $\operatorname{ord}_E : L^\times \to \mathbb{Z}$ is in $\operatorname{Val}(X)$. For $c \in (0, \infty)$, $c \cdot \operatorname{ord}_E : L^\times \to c \cdot \mathbb{Z}$ is another point of $\operatorname{Val}(X)$.

(2) If $\pi : Y \to X$ is a birational morphism and Y is normal, then for any $E \subset Y$ a prime divisor on Y and every $c > 0$ we have $c \cdot \operatorname{ord}_E \in \operatorname{Val}(X)$ (note that this is centered on X, and η is the generic point of $\overline{\pi(E)} \subset X$, then $c_X(c \cdot \operatorname{ord}_E) = \eta$ for all $c > 0$).

(3) The trivial valuation $\operatorname{triv}_X : L^\times \to \{0\}$ is a point of $\operatorname{Val} X$, centered at the generic point of X (this will be important as a sort of "limit point" of the valuation space).

(4) Suppose $R = K[[x_1, \dots, x_d]]$; pick $\underline{r} \in (\mathbb{R}_{\geq 0})^d$. Then for $f = \sum c_\alpha x^\alpha$, with $c_\alpha \in K$, we define

$$\mathrm{val}_{\underline{r}}(f) := \min\left\{ \sum_{i=1}^{d} r_i \alpha_i : c_\alpha \neq 0 \right\}.$$

Jonsson–Mustaţă, Proposition 3.1, showed that $\mathrm{val}_{\underline{r}}$ is well-defined, and that this process gives an injective map $\mathrm{val} : (\mathbb{R}_{\geq 0})^d \hookrightarrow \mathrm{Val}_{\mathrm{Spec}\, R}$, $\underline{r} \mapsto \mathrm{val}_{\underline{r}}$.

Remark. In characteristic 0 at least, we can realize the valuation space as a limit of these cones over resolutions of our variety.

(5) Suppose $\pi : Y \to X$ is birational and Y is regular, and $H = \sum_{i=1}^r H_i$ is an SNC divisor on Y (with each H_i a regular irreducible hypersurface). For $m \leq n$, consider the irreducible decomposition $\bigcap_{i=1}^m H_i = Z_1 \cup \cdots \cup Z_t$. Let ε be the generic point of Z_1. Then $\mathcal{O}_{Y,\eta}$ is a regular local ring of dimension d, and $\widehat{\mathcal{O}}_{Y,\eta}$ is isomorphic to $\kappa(\eta)[[z_1, \dots, z_d]]$; we can choose z_i to be a local equation for H_i.

Now, as in the preceding example, we can define $\mathrm{val}_H : (\mathbb{R}_{\geq 0})^d \to \mathrm{Val}_{\widehat{\mathcal{O}}_{Y,\eta}}$; moreover, we can restrict the valuation to $\mathcal{O}_{Y,\eta} \subset \widehat{\mathcal{O}}_{Y,\eta}$. We can restrict further along

$$\mathcal{O}_{X,\pi(\eta)} \hookrightarrow \mathcal{O}_{Y,\eta} \hookrightarrow \widehat{\mathcal{O}}_{Y,\eta}.$$

So, we get an actual point of $\mathrm{Val}\, X$ in this manner. We can thus consider $\mathrm{im}(\mathrm{val}\, H) \subset \mathrm{Val}\, X$, and we denote this by $\mathrm{QM}_\eta(Y, H)$. We call these *quasimonomial* valuations.

Remark. Quasimonomial valuations are always Abhyankar valuations; the converse is true in characteristic 0, or at least as long as k is perfect. We'll discuss this more next time.

(6) Let $R = k[x_1, \dots, x_d]$, and let $A = k[[t]]$. Then $\mathrm{Frac}\, A = k((t))$ has infinite transcendence degree over k (Exercise: is it true that the transcendence degree is the cardinality of the power set of the natural numbers?). Choose $d-1$ transcendental series $f_2, \dots, f_d \in k[[t]]$ that are algebraically independent and without constant terms. Define $\varphi : R \to k[[t]]$ where $\varphi(x_1) = t$, $\varphi(x_j) = f_j$ for $j \geq 2$. This is injective by construction; define $V = \mathrm{ord}_t \circ \varphi$. This is a $\mathbb{Z}$-valuation of $\mathrm{Spec}\, R$, centered at $(x_1, \dots, x_d)$. We'll show next time that this v doesn't come from ord_E for any $\pi : Y \to \mathbb{A}^d$, or even as a quasimonomial valuation. (In fact, this will fail to be an Abhyankar valuation.)

Now, we put a topology on $\mathrm{Val}(X)$. There are three equivalent ways to define this topology, which we'll do in decreasing order of abstraction:

(1) We have a set-theoretic inclusion $\mathrm{Val}(X) \hookrightarrow \prod_{f \in L^\times} (-\infty, \infty)$, $v \mapsto \prod v(f)$, where the right side carries the product topology. We can then give $\mathrm{Val}\, X$ the subspace topology with respect to this inclusion.

(2) Each $f \in L^\times$ gives a function $\widehat{f} : \mathrm{Val}(X) \to \mathbb{R}$, where $\widehat{f}(v) = v(f)$ ($\widehat{f}$ is "like the Gelfand transform of f"). Now, give $\mathrm{Val}(X)$ the coarsest topology such that $\widehat{f}$ is continuous for all $f \in L^\times$.

(3) Fix $v \in \mathrm{Val}(X)$. A basis for the topology near v is given as follows: for any s, choose $f_1, \dots, f_s \in L^\times$ and $\varepsilon > 0$, and define

$$\mathcal{U}(f_1, \dots, f_s; \varepsilon) := \bigcap_{i=1}^{s} \Big\{ w \in \mathrm{Val}(X) : |\underbrace{\widehat{f}(v) - \widehat{f}(w)}_{w(f_i) - v(f_i)}| < \varepsilon \Big\}.$$

Remark (facts)**.** (1) $\mathrm{Val}(X)$ is Hausdorff in this topology.

(2) $c_X : \mathrm{Val}(X) \to X$ is anticontinuous (the preimage of a closed set is open and vice versa).

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Recall from last time that we claimed the center map c_X is anticontinuous. A "proof": Let $X = \operatorname{Spec} R$ and $U = D(f)$. Then $v \in \operatorname{Val} X$ if and only if $R \subset A_v$, so $v(f) \geq 0$. Moreover, $c_X(v) \in U$ if and only if $f \notin m_v$ if and only if $v(f) = 0$. Since $\widehat{f} : \operatorname{Val}_X \to \mathbb{R}$ is continuous, so $\widehat{f}^{-1}(0)$ is closed, but this is $c_X^{-1}(U)$.

("Proof" is in quotes because we reduced to the affine case; we need to check that Val_X is actually covered by finitely many closed sets of this form.)

Quasimonomial valuations and retractions

Recall that given a birational map $\pi : Y \to X$ from a regular scheme Y (not necessarily proper!) and $H = \sum_{i=1}^N H_i$ an snc divisor on Y. We write $[N] = \{1, \dots, N\}$, for $J \subset [N]$ we write $H_J := \bigcap_{j \in J} H_j$ (if $J = \emptyset$ then $H_J = Y$). If H_J is nonempty than we can decompose H_J into disjoint irreducible components $Z_1 \sqcup \cdots \sqcup Z_l$, and at each generic point η_i of Z_i we've defined $\mathrm{QM}_{\eta_i}(Y, H) \cong (\mathbb{R}_{\geq 0})^d$. The η that appear as J ranges over all subsets of $[N]$ are called the strata of H. For $y \in Y$, we write $J_y := \{j \in [N] : y \in H_j\}$; then for any y we have $y \in H_{J_y}$ for a unique J. We let $\eta(y)$ be the associated strata. Thus we have a function from points of Y to strata.

Note that if η is a stratum then $\eta = \eta(\eta)$, so we don't risk ambiguity in this notation.

Definition. (1) $\mathrm{QM}_y(Y, H) := \mathrm{QM}_{\eta(y)}(Y, H)$.

(2) $\mathrm{QM}(Y, H) := \bigcup_y \mathrm{QM}_y(Y, H) \subset \operatorname{Val} X$.

We now define the retraction map. Assume that our map $\pi : Y \to X$ is actually proper. Then every $v \in \operatorname{Val}_X$ has center on Y (this is the valuative criterion for properness).

Remark. Note that in the proof the valuative criterion for properness, we consider not just valuations on the function field of a variety, but on proper closed subschemes as well. Thus properness is much stronger than valuations on X having center on Y; what it really means is that if we compactify Val_X by including the valuations on proper closed subschemes (yielding the Berkovich space $X^{\beth}$) that these have centers on Y as well.

Now, let $v \in \operatorname{Val}_X$, $y = c_Y(v)$. Let D be a (Cartier) divisor on Y, such that near y we have $D = \operatorname{div} f$ for $f \in \operatorname{Frac} X$. We define $v(D) = v(f)$. Given an snc divisor $H = \sum H_i$ on Y, we write $\alpha_i = v(H_i)$; note that $\alpha_i > 0$ if and only if $y \in H_i$. We thus have $\mathrm{QM}_y(Y, H) = \mathrm{QM}_{\eta(y)}(Y, H)$, with $\eta(y)$ the generic point of some component of $\bigcap_{j : \alpha_j > 0} H_j$.

Definition. We define $r_{(Y,H)}(v) = \mathrm{val}_{(\alpha_1, \dots, \alpha_t)}$, where we renumber such that $\alpha_1, \dots, \alpha_t > 0$ and the rest are zero. (Alternatively, we can say $r_{(Y,H)}(v) = \sum v(H_i) \operatorname{ord}_{H_i}$.)

We've thus defined a map $r_{(Y,H)} : \operatorname{Val}_X \to \mathrm{QM}(Y, H)$ for every snc pair $\pi : (Y, H) \to X$ (with π proper and birational), such that:

(1) If $v \in \mathrm{QM}(Y, H)$ then $r_{(Y,H)}(v) = v$.

(2) $r_{(Y,H)}$ is continuous.

These are called the monomialization retractions.

Inverse systems

Suppose $\mu : Y' \to Y$ is another proper birational map sitting over X, and we have snc divisors H' and H on Y' and Y respectively. Suppose furthermore that $\operatorname{Supp}(\mu^* H) \subset \operatorname{Supp}(H')$.

Lemma. *The diagram*

$$\begin{array}{ccc} \operatorname{Val}_X & \xrightarrow{\ r_{(Y,H)}\ } & \mathrm{QM}(Y, H) \\ {\scriptstyle r_{(Y',H')}}\downarrow & \nearrow {\scriptstyle r_{(Y,H)}|_{\mathrm{QM}(Y',H')}} & \\ \mathrm{QM}(Y', H') & & \end{array}$$

commutes.

Moreover, each cone $(\mathbb{R}_{\geq 0})^s \subset \mathrm{QM}(Y', H')$ *maps to a single cone* $(\mathbb{R}_{\geq 0})^r \subset \mathrm{QM}(Y, H)$ *via a matrix of nonnegative integers* b_{ij}.

Proof. Because of our assumption on the supports of H' and H, we can write $\mu^*(H_i) = \sum_{j=1}^N b_{ij} H_j'$, with $H = \sum_{i=1}^M H_i$ and $H' = \sum_{j=1}^N H_j'$. Moreover, $b_{ij} \geq 0$. Now,if $v \in \mathrm{Val}_X$, then $v(H_i) = \sum b_{ij} v(H_j')$, so that we have $r_{(Y,H)}(r_{(Y',H')}(v)) = r_{(Y,H)}(v)$. □

The point of all of this is the following:

Theorem (JM 4.9)**.** *Assume* $\mathbb{Q} \subset k$. *Then the natural map* $r : \mathrm{Val}_X \to \varprojlim_{(Y,H)} \mathrm{QM}(Y, H)$ *is a homeomorphism.*

The main idea in proving this is the following: we want to show that $v(D) = r_{(Y,H)}(v)(D)$ whenever $(Y, H) \to (X, D)$ gives a log resolution; thus, evaluating any divisor D under v can be done via the retract to a quasimonomial valuation on a log resolution. This is the only place we need characteristic 0: we need the existence of log resolutions of (X, D).

Log discrepancies

Say X is a normal variety over k and Δ is a $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi : Y \to X$ be a proper birational map, with Y regular and H an snc divisor on Y. For all divisors E on Y we have the log discrepancy $A_X(E, \Delta) = 1 + \mathrm{ord}_E(K_Y - \pi^*(K_X + \Delta))$. For all $v \in \mathrm{QM}(Y, H)$, we can define $A_X(v; \Delta)$ to be $\sum v(H_i) A_X(H_i; \Delta)$.

See Proposition 5.1 of Jonsson and Mustata for the verification that this is well-defined (independent of our choice of model (Y, H). The takeaway is that $A_X(-; \Delta)$ is well-defined and affine-linear (in particular, continuous) on each cone in $\mathrm{QM}(Y, H)$. Thus, we can define (in characteristic 0) $A_X(v, \Delta) = \sup_{(Y,H)} A_X(r_{(Y,H)}(v), \Delta)$, and $A_X(-; \Delta) : \mathrm{Val}_X \to \mathbb{R} \cup \{\infty\}$ is lower-semicontinuous. The lower-semicontinuity is still true in characteristic p, although the construction and proof are different. From these log discrepancies one can define multiplier ideals, and use the topology on the valuation space to show that these are actually coherent ideals (which is new in characteristic p, see Canton).

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Our setting today will be the following: we consider normal $\mathbb{Q}$-Gorenstein varieties X, with $Y = \sum q_i Y_i$, $q_i \in \mathbb{R}_+$ the formal sum of closed subsets of X. We refer to (X, Y) as a pair. We recall briefly the definitions of the log discrepancies and minimal log discrepancies.

Definition. Let ord_E be a divisorial valuation on $k(X)$ with center $c_X(E)$ on X. The log discrepancy of E with respect to the pair (X, Y) is

$$a_E(X, Y) := 1 + \mathrm{ord}_E(K_{X'/X}) - \sum q_i \,\mathrm{ord}_E(Y_i),$$

where $X' \to X$ is a birational morphism from a normal variety such that the center $c_{X'}(E)$ of ord_E on X' is a divisor.

Recall that this is independent of the model $X' \to X$ we take.

Definition. Let $W \subset X$ be a closed subset. The minimal log discrepancy of the pair (X, Y) along W is defined to be

$$\mathrm{mld}(W; X, Y) := \inf\{a_E(X, Y) : c_X(E) \subset W\}.$$

(If $\dim X = 1$ one must take the convention that if $\mathrm{mld}(W; X, Y) < 0$ then it's $-\infty$; this is automatic in higher dimension. We won't treat the 1-dimensional case at all in the following, so this will not arise.)

The minimal log discrepancy is quite a subtle invariant of the singularities of the pair (X, Y), and its behavior is quite important for the minimal model program. The following two conjectures concern the behavior under restriction to hyperplanes and varying W:

Conjecture (inversion of adjunction)**.** *Let (X, Y) be a pair and $D \subset X$ an effective Cartier divisor not contained in $\bigcup Y_i$. Let $W \subset D \subset X$. Then*

$$\mathrm{mld}(W; D, Y|_D) = \mathrm{mld}(W; X, Y + D).$$

We write $Y|_D = \sum q_i (Y_i \cap D)$.

Conjecture (semicontinuity)**.** *Let (X, Y) be a pair. Then*

$$x \mapsto \mathrm{mld}(x; X, Y)$$

is lower-semicontinuous on the closed points of X .

Recall that lower-semicontinuity is equivalent to the set of points where $\mathrm{mld}(x; X, Y) > \alpha$ being open for any α.

The goal of these talks is to develop the theory of arc schemes in order to establish the following:

Theorem. *Both conjectures are true when X is smooth, or more generally a local complete intersection.*

These were proved in [**?**, **?**] using the formalism of motivic integration, but we will follow the exposition in [**?**], which avoids the explicit use of motivic integration.

Arc schemes

Let k be a field and let X be an essentially finite-type k-scheme. For each $\ell \in \mathbb{N}$ consider the functor

$$T \mapsto \mathrm{Hom}_k(T \times_k \mathrm{Spec}(k[t]/t^{\ell+1}), X)$$

from k-schemes to sets. A k-morphism $T \times_k \mathrm{Spec}\, k[t]/t^{\ell+1} \to X$ is called an ℓ-jet on X.

Theorem. *This functor is representable, i.e., there exists a scheme X_ℓ such that*

$$\mathrm{Hom}(T, X_\infty) \cong \mathrm{Hom}(T \times_k \mathrm{Spec}(k[t]/t^{\ell+1}), X)$$

are naturally isomorphic functors.

Remark. Note that for $\ell = 1$ this is the same as the functor representing the tangent space of X, so that the representing scheme is just the total tangent space of X.

We call the representing k-scheme X_ℓ, the ℓ-th jet scheme of X. Note that the proof demonstrates that if X is finite-type over k, so is each X_ℓ. By definition the closed points of X_ℓ corresponds to an ℓ-jet $\mathrm{Spec}\, k[t]/t^{\ell+1} \to X$, which we think of as an "infinitesimal part of a curve" on X.

Proof. We present the proof only in the case where $X = \mathrm{Spec}\, R$, with R a finitely generated algebra over k (we'll shortly see that behaves well under localization, so the general case isn't so far off). Translating everything into the language of rings, we want to construct a k-algebra R_ℓ such that for k-algebra C, giving a k-algebra morphism $R_\ell \to C$ is the same thing as giving a k-algebra morphism $R \to C[t]/t^{\ell+1}$.

Say $R = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Giving a morphism

$$R \to C[t]/t^{\ell+1}$$

is the same as giving a map

$$k[x_1, \ldots, x_n] \to C[t]/t^{\ell+1}$$

that is zero on $(f_1, \dots, f_m)$. Giving a map $k[x_1, \dots, x_n] \to C[t]/t^{\ell+1}$ is the same as giving n elements

$$x_i(t) := \sum_{i=0}^{\ell} x_i^{(j)} t^j \in C[t]/t^{\ell+1}$$

(i.e., each $x_i^{(j)}$ is an element of C). We have that this factors through R itself exactly when each

$$f_s\Big(x_1(t) \dots, x_n(t)\Big) = 0$$

in $C[t]/t^{\ell+1}$. But this is a condition only on the coefficients of each t^i, so we can expand to

$$f_s^{(0)}(x_1(t), \dots, x_n(t)) + f_s^{(1)}(x_1(t), \dots, x_n(t))t + \dots + f_s^{(\ell)}(x_1(t), \dots, x_n(t))t^{\ell}.$$

The condition then is that each $f_s^{(i)}(x_1(t), \dots, x_n(t)) = 0$.

So, giving a map $R = k[x_1, \dots, x_n]/(f_1, \dots, f_m) \to C[t]/t^{\ell+1}$ is the same as giving elements

$$x_i^{(j)}, \quad i = 1, \dots, m, j = 1, \dots, \ell$$

of C satisfying the equations

$$f_s^{(t)}(x_1(t), \dots, x_n(t)) = 0, \quad s = 1, \dots, m, t = 1, \dots, \ell.$$

This is the same as giving a morphism from

$$k\big[x_i^{(j)}\big]/\big(f_s^{(t)}(x_1(t), \dots, x_n(t))\big) \to C,$$

so that we can just take

$$R_\ell := k\big[x_1^{(j)}, \dots, x_n^{(j)} : j = 0, \dots, \ell\big]/\big(f_s^{(t)}(x_1(t), \dots, x_n(t))\big).$$

□

The takeaway here is that constructing these rings is straightforward but the number of variables and equations involved rapidly grow. Note also that if $Z \subset X$ is a closed inclusion then we have a natural closed inclusion $Z_\ell \subset X_\ell$.

Example. (1) Consider $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$. Then $X_\ell = \operatorname{Spec} k[x_0, \dots, x_\ell, y_0, \dots, y_\ell] \cong \mathbb{A}^{2(\ell+1)}$.

(2) Consider $X = \operatorname{Spec} k[x, y]/(x^2 + y^3)$. Say we want to find X_2. From the above approach, we will construct X_2 as a closed subset of

$$\mathbb{A}^2_2 = \operatorname{Spec} k[x_0, x_1, x_2, y_0, y_1, y_2].$$

A point $(x_0, x_1, x_2, y_0, y_1, y_2) \in \mathbb{A}^2_2$, though of as the 2-jet

$$\big(x_0 + x_1 t + x_2 t^2, y_0 + y_1 t + y_2 t^2\big)$$

will lie in X_2 exactly when

$$(x_0 + x_1 t + x_2 t^2)^2 + (y_0 + y_1 t + y_2 t^2)^3.$$

We multiply out and gather powers of t to obtain

$$(x_0^2 + y_0^3) + (2x_0 x_1 + 3y_0^2 y_1)t + (x_1^2 + 2x_0 x_2 + 3y_0 y_1^2 + 3y_0^2 y_2)t^2.$$

Thus, we have that

$$X_2 = \operatorname{Spec} \frac{k[x_0, x_1, x_2, y_0, y_1, y_2]}{(x_0^2 + y_0^3, 2x_0 x_1 + 3y_0^2 y_1, x_1^2 + 2x_0 x_2 + 3y_0 y_1^2 + 3y_0^2 y_2)}.$$

Remark. One can check that X_2 is neither reduced nor irreducible, thus suggesting that jet schemes of singular varieties become bad quite quickly.

In fact, as a first glimpse at what singularities arise on X_ℓ, there is the following:

Theorem (Ein–Mustata–Yasuda)**.** *Let X be a normal lci variety. Then X is:*

(1) terminal if and only if X_ℓ is normal for all ℓ.

(2) canonical if and only if X_ℓ is irreducible for all ℓ.

(3) log canonical if and only if X_ℓ is equidimensional for all ℓ.

We won't prove this, although we'll cover many of the necessary ingredients.

Remark. Note that there is a natural map $X_\ell \to X$: a T-point of X_ℓ determines an arc

$$T \times \operatorname{Spec} k[t]/t^{\ell+1} \to X,$$

and we simply precompose with the map $T = T \times \operatorname{Spec} k \to T \times \operatorname{Spec} k[t]/t^{\ell+1}$ corresponding to killing t. In more down-to-earth terms, this just sends a jet to the closed point its based at.

Remark. Note that given any morphism $f : X \to Y$ we obtain a morphism $f_\ell : X_\ell \to Y_\ell$, defined on the level of functor of points by sending a morphism

$$T \times \operatorname{Spec} k[t]/t^{\ell+1} \to X$$

to the composition

$$T \times \operatorname{Spec} k[t]/t^{\ell+1} \to Y.$$

Proposition. *If $f : X \to Y$ is an étale morphism of k-schemes, we have a pullback square*

$$\begin{array}{ccc} X_\infty = Y_\infty \times_Y X & \longrightarrow & Y_\infty \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Exercise. Prove this (hint: it's enough to use that f is formally étale).

Corollary. *Formation of the jet schemes commutes with open immersions.*

Corollary. *If X is smooth of dimension n then X_ℓ is an $n(\ell+1)$-bundle over X (and in particular is also smooth).*

Proof. It suffices to work on some affine open $U \subset X$; since U is smooth of dimension n there is an étale morphism $U \to \mathbb{A}^n$, and thus by the proposition it suffices to work in the case of $\mathbb{A}^n$, where by our construction of the jet schemes this is immediate. □

The truncation maps $k[t]/t^{\ell+1} \to k[t]/t^{\ell'+1}$ for $\ell' < \ell$ induce morphisms $\psi^X_{\ell,\ell'} : X_\ell \to X_{\ell'}$, which are easily checked to be affine, so we obtain an inverse system $\{\cdots \to X_\ell \to X_{\ell-1} \to \cdots\}$ of affine morphisms. We can thus form the inverse limit, which we denote by X_∞ and call the arc scheme of X. In contrast to the X_ℓ, this is almost never finitely generated. We have truncation maps $\psi^X_\ell := \psi^X_{\infty,\ell} : X_\infty \to X_\ell$ for all ℓ. Again, the functor $X \to X_\infty$ is functorial, so a morphism of schemes $f : X \to Y$ induces a morphism $f_\infty : X_\infty \to Y_\infty$.

Remark. X_∞ is also straightforward to construct from X; one works on affine pieces $\operatorname{Spec} R$, where we choose some presentation $R = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$. Giving a map $R \to C[[t]]$ corresponds to giving n power series

$$x_i(t) := \sum_{i=0}^{\infty} x_i^{(j)} t^j \in C[[t]].$$

such that each $f_s(x_1(t), \dots, x_n(t))$ is zero. Thus, again we just expand this out and rewrite each $f_s(x_1(t), \dots, x_n(t))$ as

$$f_s^{(0)}(x_1(t), \dots, x_n(t)) + f_s^{(1)}(x_1(t), \dots, x_n(t))t + \dots.$$

Then we just take $R_\infty = k[x_1^{(j)}, \dots, x_n^{(j)} : j = 0, 1, 2, \dots]/(f_1^{(i)}, \dots, f_s^{(i)} : i = 0, 1, \dots)$.

Remark. One can check that if $k \hookrightarrow L$ is a field extension then

$$\operatorname{Hom}(\operatorname{Spec}(L[[t]]), X) = \operatorname{Hom}(\operatorname{Spec}(L), J_\infty(X)).$$

In fact, by a theorem of Bhargav it is true (but highly nonelementary) that if X is quasicompact and quasiseparated over k and S is a k-algebra then

$$\operatorname{Hom}\big(\operatorname{Spec} S \times_k \operatorname{Spec}(k[[t]]/t^{\ell+1}), X\big) = \operatorname{Hom}(\operatorname{Spec} S, J_\ell(X)),$$

but we do not use this in the following.

From now on, when we reference X_∞, X_ℓ, or X, we refer to their k-points only.

The following is one of the key reasons for the relevance of arc spaces to the study of birational geometry:

Theorem. *Let $f : X \to Y$ be a proper birational morphism, and say f is an isomorphism away from a closed subset $Z \subset Y$. Then*

$$f_\infty : X_\infty - (f^{-1}(Z))_\infty \to Y_\infty - Z_\infty$$

is a bijection of sets.

That is, away from the "thin" sets Z_∞ and $f^{-1}(Z)_\infty$, the map is a bijection. Note that Z_∞ consists exactly of the arcs $\operatorname{Spec} k[[t]] \to X$ that factor through the inclusion $Z \hookrightarrow X$.

Remark. Note that Z_∞ is *not* the arcs with closed point in Z, but rather the arcs lying entirely in Z! For example, if $Y = \mathbb{A}^2$ and $Z = V(y)$ is the x-axis, we have that $Y_\infty = \operatorname{Spec} k[x_0, x_1, \dots, y_0, y_1, \dots]$ and $Z_\infty = V(y_0, y_1, \dots)$. That is, an arc passes through the x-axis if it lies in the codimension-1 subset $V(y_0)$, but is in Z_∞ if it lies in the infinite-codimension subset $V(y_0, y_1, \dots)$. The difference between sets with finite and infinite codimensions is at the heart of the theory of arc spaces.

Proof. Let $\gamma \in Y_\infty - Z_\infty$. We can view γ as a map $\operatorname{Spec} k[[t]] \to Y$; moreover, since the generic point $\eta := \operatorname{Spec} k((t)) \hookrightarrow \operatorname{Spec} k[[t]] \to Y$ does not lie in Z, f is an isomorphism over η, so we can lift it to a "punctured arc" $\operatorname{Spec} k((t)) \to X$. We thus obtain the diagram

$$\begin{array}{ccc} \operatorname{Spec} k((t)) & \longrightarrow & X \\ \big\downarrow & \overset{\tilde{\gamma}}{\nearrow} & \big\downarrow f \\ \operatorname{Spec} k[[t]] & \xrightarrow{\ \gamma\ } & Y \end{array}$$

Since $k[[t]]$ is a DVR, the valuative criteria for properness gives immediately that there exists a unique $\tilde{\gamma}$ making the diagram commute. ∎

4 February 18

Cylinders in the space of arcs

We now specialize to the case where $\operatorname{char} k = 0$. We will discuss the notion of codimension of cylinders in the space of arcs, but sweep most of the technical details under the rug. First, we recall a result on the structure of the arc spaces:

Theorem (Kolchin)**.** *If X is irreducible then so is X_∞.*

We won't prove this, though it's not hard; the main idea is to use resolution of singularities, the previous theorem on behavior under proper birational morphisms, and induction on dimension.

Definition. Let X be a variety. A cylinder in X_∞ is a subset of the form

$$(\psi_\ell^X)^{-1}(S)$$

for S a constructible subset of X_ℓ (recall that a constructible subset is the union of locally closed sets).

Intuitively, a cylinder should be thought of as a set in X_∞ in which membership can be verified by looking at whether finitely many coefficients are zero or nonzero; for example, if $Z \subset X$ is a proper closed subset, Z_∞ will never be a cylinder.

One can check that the intersection and union of cylinders are still cylinders, which we'll use frequently in the following.

Note that cylinders could be open or closed in the topology of X_∞; for example, if $X = \operatorname{Spec} k[x, y]$, then we've seen that $X_\infty = \operatorname{Spec} k[x_0, x_1, \dots, y_0, y_1, \dots]$, and:

(1) $V(y_0)$, the set of arcs passing through the x-axis, is a closed cylinder.

(2) $D(y_0)$, the set of arcs *not* passing through the x-axis, is an open cylinder.

(3) $V(y_0, y_1, \dots,)$, the set of arcs contained in the x-axis, is not a cylinder.

There is a natural type of cylinder we'll consider often, which are the contact loci:

Definition. Let $Z \subset X$ be a closed subscheme (not necessarily reduced!), defined by an ideal sheaf I_Z. We define

$$\operatorname{Cont}^{\geq e}(Z) = \{\gamma \in X_\infty : \operatorname{ord}_t \gamma^*(I_Z) \geq e\}.$$

This is a (closed) cylinder, because clearly

$$\operatorname{Cont}^{\geq e}(Z) = \psi_{e+1}^{-1}(Z_{e+1}).$$

Then

$$\operatorname{Cont}^{e}(Z) = \{\gamma \in X_\infty : \operatorname{ord}_t \gamma^*(I_Z) = e\} = \operatorname{Cont}^{\geq e}(Z) - \operatorname{Cont}^{\geq e+1}(Z)$$

is a (locally closed) cylinder.

We now define the notion of the *codimension* of a cylinder in X_∞. We first consider the smooth case:

Definition. If X is smooth and $C \subset X_\infty$ a cylinder, we write $C = \psi_\ell^{-1}(S)$ for $S \subset X_\ell$, and take $\operatorname{codim}(C) := \operatorname{codim}(S)$.

Let X be a reduced scheme of pure dimension n, and say that C is a cylinder, so that we can write $\psi_p^{-1}(S)$ for $S \subset X_p$ for some p. Say first that $C \subset \operatorname{Cont}^e(\operatorname{Jac}_X)$. We define

$$\operatorname{codim}(C) := (m+1)n - \dim(\psi_m(C))$$

for any $m \geq \max(p, e)$ (i.e., for any $m \gg 0$). Note that $(m+1)n$ is the "expected" dimension of X_m, i.e., the dimension X_m would be if X were smooth (if X is singular, X_m is probably higher-dimensional, but one can check that $(m+1)n$ is still the dimension of the set of "liftable" arcs, i.e., those in $\psi_m(X_\infty)$.)

If C is instead an arbitrary cylinder we set

$$\operatorname{codim}(C) := \min\{\operatorname{codim}\big(C \cap \operatorname{Cont}^e(\operatorname{Jac}_X)\big)\}.$$

If X is smooth one can check that this notion of codimension is the same as the codimension of a cylinder C in the Krull topology on X_∞; this is not true however if X is not smooth.

Sets contained in $(Z)_\infty$ for some proper subscheme Z are called *thin*; they should be thought of as being "measure zero" in some sense (for example, one can show they have infinite codimension). The codimension of a cylinder can be computed by excising a thin subset and breaking it into disjoint pieces:

Proposition. *Say C is a cylinder in X_∞ and we have disjoint cylinders $C_i \subset C$ such that $C - \bigcup C_i$ is thin. Then*

$$\lim_{i\to\infty} \operatorname{codim} C_i = \infty, \quad \operatorname{codim} C = \min_i(\operatorname{codim} C_i).$$

Remark (aside)**.** There is another (closely related) kind of cylinder that comes up frequently: the so-called *maximal divisorial sets*. To any arc $\gamma : \operatorname{Spec} k[[t]] \to X$, say factoring through Z, we can associate a valuation v_γ of $k(Z)$, given by sending $f \in \mathcal{O}_Z$ to $\operatorname{ord}_t \gamma^*(f)$. Thus, we can associate a semivaluation to any arc of X. This extends to sets of arcs in the obvious way: if $C \subset X_\infty$ we write

$$v_C(-) = \min_{f \in C}\{v_f(-)\}.$$

This will give a valuation (as opposed to a semivaluation) exactly when C is not thin, and will be nontrivial when C does not dominate X.

Moreover, any divisorial valuation $q \cdot \operatorname{ord}_E$ can be realized in this form:one can show that if $f : Y \to X$ is a proper birational morphism from a normal variety Y, and E is an exceptional divisor appearing on Y, then

$$C := \overline{f_\infty(\operatorname{Cont}^q(E))}$$

is a cylinder such that

$$v_C = q \cdot \operatorname{ord} E;$$

moreover, C is maximal (with respect to inclusion) and independent of the choice of f. Therefore the arc scheme keeps track of the information of all the divisorial valuations on X.

Finally, if X is smooth or at least lci then we have even more information encoded in the set C associated to a divisorial valuation $c \cdot \operatorname{ord}_E$: the codimension of C is just $q(A_X(\operatorname{ord}_E) + 1)$, where $A_X(-)$ is the log discrepancy of ord_E. The arc space X_∞ is then in some sense another geometric incarnation of the valuation space of X.

The birational transformation formula

We now state the key theorem in applications of arc spaces to birational geometry. Recall that the definition of discrepancy is based around the order of valuations on the relative canonical divisor $K_{X'} - f_* K_X = K_{X'/X}$. The following theorem, due to Kontsevich in the smooth case and Denef and Loeser in the singular case, will allow us to connect information about $K_{X'/X}$ to the arc spaces; in particular, it connects the behavior of the induced morphisms of *jet* schemes to the order of vanishing along a subscheme closely connected to $K_{X'/X}$.

Theorem (birational transformation theorem, smooth case)**.** *Let $f : X' \to X$ be a proper birational morphism with X and X' nonsingular. For e, e' write*

$$C_e = \operatorname{Cont}^e(K_{X/Y}).$$

Then for $m \geq \max(2e)$ the map on m-jets

$$\psi_m^{X'}(C_e) \to f_m(\psi_m^{X'}(C_e))$$

is an $\mathbb{A}^e$-fibration.

Note that this describes what happens away from the thin set $(K_{Y/X})_\infty$.

There is also the more technical variant in the singular case:

Definition. Let $f : X' \to X$ be a proper birational morphism with X reduced and pure dimension n, and X' smooth. Pulling back n-forms gives a map

$$f^* \Omega_X^n \to \Omega_{X'} = \omega_{X'},$$

and thus defines an ideal sheaf Jac_f.

When X is nonsingular as well Jac_f is locally principal and supported on the exceptional locus (e.g., if f is the blowup of $\mathbb{A}^2$ at a point then $\operatorname{Jac}_f = \mathcal{O}(-E)$); in fact, in this setting $V(\operatorname{Jac}_f) = K_{X'/X}$.

Theorem (birational transformation theorem). *Let $f : X' \to X$ be a proper birational morphism with X' nonsingular and X reduced of pure dimension n. For e, e' write*

$$C_{e,e'} = \mathrm{Cont}^e(\mathrm{Jac}_f) \cap f_\infty^{-1}(\mathrm{Cont}^{e'}(\mathrm{Jac}\, X)).$$

Then for $m \geq \max(2e, e+e')$ the map on liftable *m-jets*

$$\psi_m^{X'}(C_{e,e'}) \to f_m(\psi_m^{X'}(C_{e,e'}))$$

is an $\mathbb{A}^e$-fibration.

Note that the sets $C_{e,e'}$ cover the complement of $V(\mathrm{Jac}_f)_\infty \cup f_\infty^{-1}(X_{\mathrm{sing}})_\infty$, so this statement tells us about the behavior on "almost all" (liftable) arcs.

This statement should be contrasted with what happens on the level of the arc scheme: the map is a *bijection* there, but on the finite-level jet schemes it is a fibration of varying dimension! Thus, one can think of the map on the arc schemes as being a bijection that changes the codimension of various cylinders. In fact, we have the following:

Corollary. *If C is a cylinder in X' of codimension c, and $e = \min\{\mathrm{Cont}^e(X') \cap C \neq \emptyset\}$ (so e is the order of a contact of a "generic" arc of C), then $\overline{f(C)}$ is a cylinder in X of codimension $c+e$.*

Example. Let $f : X' = \mathrm{Bl}_p\, \mathbb{A}^2 \to X$, with exceptional divisor e. $\mathrm{Jac}_f = \mathcal{O}(-E)$, so $\mathrm{Cont}^e(\mathrm{Jac}_f) = \mathrm{Cont}^e(E)$ for all E. Consider the cylinder $\mathrm{Cont}^1(E)$, i.e., arcs with closed point in E. By definition the order of contact of a generic arc of C with Jac_f is $e = 1$. The codimension of $\mathrm{Cont}^1(E)$ is $c = 1$ (locally this is the same as our example of $V(y_0) \subset (\mathbb{A}^2)_\infty$). Thus we expect $f_\infty(C)$ to have codimension $1+1=2$, and this is exactly what happens: $f_\infty(C)$ is easily seen to consist of all arcs through the point p, which has codimension 2 (it's cut out of $(\mathbb{A}^2)_\infty$ by two equations, say x_0, y_0).

Minimal log discrepancies and cylinders in the space of arcs

We begin in the simplified setting of a smooth ambient variety; the exact same philosophy will be in play in the proof of the above theorem, but just with more bookkeeping. Let X be a smooth variety and let $Y \subset X$ a closed subscheme. Let $f : X' \to X$ be a log resolution of Y, i.e., f is a proper birational map, an isomorphism over $X - Y$, X' is smooth, and $\mathrm{Ex}\, f \cup f^{-1}(Y)$ is a strict normal crossings divisor. Say the numerical data of the resolution is

$$f^{-1}(Y) = \mathcal{O}_{X'}\Big(-\sum r_i E_i\Big), \quad K_{X'/X} = \sum k_i E_i.$$

For $\nu \in \mathbb{N}^s$ we'll write $\mathrm{Cont}^\nu(E) = \bigcap_i \mathrm{Cont}^{\nu_i}(E_i)$; note that every $\gamma \in X'_\infty - (\mathrm{Exc}\, f)_\infty$ lies in some $\mathrm{Cont}^\nu(E)$.

Theorem.

- *For any p, we have an inclusion*

$$\bigsqcup_\nu f_\infty(\mathrm{Cont}^\nu(E)) \subset \mathrm{Cont}^p(Y),$$

  where the union is over all ν such that $\sum \nu_i r_i = p$. Moreover, the inclusion is actually an equality up to a thin set.

- *If $\mathrm{Cont}^\nu(E)$ is nonempty then $f_\infty(\mathrm{Cont}^\nu(E))$ is irreducible of codimension $\sum \nu_i(k_i+1)$.*

- *Thus* $\mathrm{codim}\,\mathrm{Cont}^p(Y) = \min(\sum \nu_i(k_i+1))$*, with the sum taken over all ν such that $\sum \nu_i r_i = p$ and $\bigcap_{\nu_i \geq 1} E_i \neq \emptyset$.*

That is to say, the *intrinsic* data of the cylinder $\mathrm{Cont}^p(Y)$ (and its codimension) can be calculated by the numeric data of a choice of a log resolution.

Proof. First, note that the union is disjoint because if $\gamma \in \mathrm{Cont}^\nu(E)$ for some ν then $\gamma \notin (K_{X'/X})_\infty = \mathrm{Exc}(f)_\infty$. Thus we know by the valuative criterion for properness that f_∞ is a bijection here.

We have that $\gamma \in \mathrm{Cont}^\nu(E)$ if and only if $\mathrm{ord}_t\, \gamma^*(E_i) = \nu_i$; we then have that $f_\infty(\gamma)(Y) = \gamma^*(\sum r_i Y_i)$ has order $\sum \nu_i r_i$, so that if this is p then $f_\infty(\gamma) \in \mathrm{Cont}^p(Y)$. It's also clear that if $\gamma \in X'_\infty - \mathrm{Exc}(f)_\infty$ and $f_\infty(\gamma) \in \mathrm{Cont}^p(Y)$ then γ must lie in some $\mathrm{Cont}^\nu(E)$, since these partition the complement of $\mathrm{Exc}(f)_\infty$. Since we know already that f_∞ is a bijection (in particular, a surjection) onto the complement of a thin set, it's clear that this inclusion is actually an equality up to a thin set.

Thus, the only statement we need to show is the statement on the codimension. We first compute the codimension of $\mathrm{Cont}^\nu(E)$ in X': this can be computed locally, so we can assume X' is affine. Because X' is smooth, there's an étale morphism $X' \to \mathbb{A}^n$, and because formation of the jet schemes commutes with basechange along étale maps we may assume $X' = \mathbb{A}^n$. Let $x_1, \dots, x_n$ be coordinates, with $E_i = V(x_i)$ locally (by assumption the E_i are snc so we can do so). We know that $X'_\infty = \mathrm{Spec}\, k[x_i^{(j)}]$. Thinking of an arc $\gamma = (x_1(t), \dots, x_n(t))$, with $x_i(t) = \sum x_i^{(j)} t^j$, we have that $\gamma \in \mathrm{Cont}^{\nu_i}(E_i)$ if and only if $x_i^{(0)} = \cdots = x_i^{(\nu_i - 1)} = 0$ and $x_i^{(\nu_i)} \neq 0$. Thus $\mathrm{Cont}^{\nu_i}(E_i)$ is an open subset of a closed subset of codimension ν_i. It's then clear that $\bigcap \mathrm{Cont}^\nu(E_i)$ has codimension $\sum \nu_i$ (since the defining equations for each $\mathrm{Cont}^{\nu_i}(E_i)$ involve entirely disjoint variables).

Now, we use the birational transformation theorem to obtain the codimension of $f_\infty(\mathrm{Cont}^\nu(E))$. Note that $\mathrm{Cont}^\nu(E) \subset \mathrm{Cont}^{\sum \nu_i k_i}(K_{X'/X})$. Thus, by the birational transformation theorem, the codimension increases by $\sum \nu_i k_i$ when we push $\mathrm{Cont}^\nu(E)$ down to X via f_∞. Thus, we have that

$$\mathrm{codim}\, f_\infty(\mathrm{Cont}^\nu(E)) = \sum \nu_i + \sum \nu_i k_i = \sum \nu_i (k_i + 1),$$

giving the theorem. □

This immediately generalizes to the case where Y is the formal sum of closed subschemes, giving the following:

Corollary. *Let (X, Y) be a pair, with X smooth and $Y = \sum c_i Y_i$ a closed subscheme. Then*

$$\mathrm{mld}(W; X, Y) = \inf_{\ell_i} \Big(\mathrm{codim}\Big(\bigcap_i \mathrm{Cont}^{\ell_i}(Y_i) \cap \mathrm{Cont}^{\geq 1}(W) \Big) - \sum c_i \ell_i \Big),$$

and if this is $\neq -\infty$ then this is a minimum.

Corollary. *Let (X, Y) be a pair, with X lci and $Y = \sum c_i Y_i$ a closed subscheme. Then*

$$\mathrm{mld}(W; X, Y) = \inf_{\ell_i} \Big(\mathrm{codim}\Big(\bigcap_i \mathrm{Cont}^{\ell_i}(Y_i) \cap \mathrm{Cont}^e\, \mathrm{Jac}\, X \cap \mathrm{Cont}^{\geq 1}(W) \Big) - \sum c_i \ell_i - e \Big),$$

and if this is $\neq -\infty$ then this is a minimum.

Corollary.

$$\mathrm{lct}(X, Y) = \min_\ell \Big(\frac{\mathrm{codim}(Y_\ell, X_\ell)}{\ell + 1} \Big).$$

Proof of inversion of adjunction. Let (X, cY) be a pair, X smooth, Y a closed subscheme, and $D \subset X$ an effective Cartier divisor not contained in Y. Let $W \subset D \subset X$. Recall we want to show that

$$\mathrm{mld}(W; D, Y|_D) = \mathrm{mld}(W; X, Y + D).$$

The inequality

$$\mathrm{mld}(W; D, Y|_D) \geq \mathrm{mld}(W; X, Y + D)$$

is elementary and does not require the use of arc schemes (one takes a log resolution of $W \cup D \cup Y$, restricts to the strict transform of D and uses the standard adjunction formula).

So, assume that

$$\tau := \mathrm{mld}(W; D, Y|_D) > \mathrm{mld}(W; X, Y + D).$$

We'll show that this forces $\tau < \mathrm{mld}(W; D, Y|_D)$, an obvious contradiction. By our theorem above, applied to compute $\mathrm{mld}(W; X, Y + D)$, there is thus some ℓ, p such that

$$\mathrm{codim}\big(\mathrm{Cont}^p(D) \cap \mathrm{Cont}^\ell(Y) \cap \mathrm{Cont}^{\geq 1}(W)\big) - p - c\ell < \tau.$$

Let $C \subset X_\infty$ be some irreducible component of $\mathrm{Cont}^p(D) \cap \mathrm{Cont}^\ell(Y) \cap \mathrm{Cont}^{\geq 1}(W)$ of minimal codimension. Let e be the order of contact of a generic arc in $C \cap D_\infty$ with $\mathrm{Jac}\, D$. By a short argument using resolution of singularities, $C \cap D_\infty \not\subset (D_{\mathrm{sing}})_\infty$, so $e < \infty$.

We now appeal to the following lemma on the behavior of codimension under restriction:

Lemma. *Let X be a smooth variety and D an effective Cartier divisor. If $C \subset X_\infty$ is an irreducible cylinder contained in $\mathrm{Cont}^p(D)$ and e is the order of contact of the generic point of C along $\mathrm{Jac}\, D$, then*

$$\mathrm{codim}(C \cap D_\infty) \leq \mathrm{codim}(C, X_\infty) + e - p.$$

Applying this in our case, we have that

$$\mathrm{codim}(C \cap D_\infty) \leq \tau + p + c\ell + e - p = \tau + c\ell + e.$$

Now, note that we can interpret $C \cap D_\infty$ as a multicontact loci on D: it's the union of components of

$$\mathrm{Cont}^\ell(Y|_D) \cap \mathrm{Cont}^e(\mathrm{Jac}\, D).$$

Applying our theorem on minimal log discrepancies on D, we see immediately that

$$\mathrm{codim}(\mathrm{Cont}^\ell(Y|_D) \cap \mathrm{Cont}^e(\mathrm{Jac}\, D)) \leq \mathrm{codim}(C \cap D_\infty) \leq \tau + c\ell + e,$$

so

$$\mathrm{mld}(W; D, Y|_D) \leq \mathrm{codim}(\mathrm{Cont}^\ell(Y|_D) \cap \mathrm{Cont}^e(\mathrm{Jac}\, D)) - c\ell - e < \tau,$$

giving us a contradiction and thus the desired equality. □

The technical details

Let (X, Y) be a pair (i.e., X is normal and $\mathbb{Q}$-Gorenstein of index r, $Y = \sum_{i=1}^s q_i Y_i$ is a $\mathbb{R}$-linear sum of closed subschemes). We need one definition:

Definition (Nash ideal)**.** Let X be a normal $\mathbb{Q}$-Gorenstein variety of dimension d, say of index r. Then the image of the natural map

$$\bigwedge^d \Omega_X^{\otimes r} \to \Big(\bigwedge^d \Omega_X^{\otimes r}\Big)^{**} = \omega_X^{\otimes r}$$

is a coherent subsheaf of the line bundle $\omega_X^{\otimes r}$. This image then defines an ideal sheaf of $\mathcal{O}_X$; this ideal sheaf is called the r-th Nash ideal of X and will be denoted Z_r.

The goal is to show the following description of the minimal log discrepancy in terms of the space of arcs:

Theorem. *If (X, Y) is a pair and $W \subset X$ a proper closed subscheme then*

$$\mathrm{mld}(W; X, Y) = \inf_{\underline{w} \in \mathbb{N}^s, \ell \in \mathbb{N}} \Big\{ \mathrm{codim}\Big(\bigcap \mathrm{Cont}^{w_i}(Y_i) \cap \mathrm{Cont}^\ell(Z_r) \cap \mathrm{Cont}^{\geq 1}(W)\Big) - \ell/r - \sum q_i w_i \Big\};$$

if $\mathrm{mld}(W; X, Y) \neq -\infty$, then this is a minimum.

This will follow from the following lemma relating contact loci to the numerical data of a resolution:

Lemma. *Let (X, Y) be a pair and $f : X' \to X$ a log resolution of singularities of $X_{\text{sing}} \cup Z_r \cup W \cup Y$. Say we have*

$$f^{-1}(Y_i) = \sum r_{ij} E_j, \quad K_{X'/X} = \sum k_j E_j, \quad f^{-1}(Z_r) = \sum z_j E_j.$$

Then for all $\underline{w} \in \mathbb{N}^s$ and $\ell \in \mathbb{N}$ we have

$$\operatorname{codim}\Big(\bigcap \operatorname{Cont}^{w_i}(Y_i) \cap \operatorname{Cont}^{\ell}(Z_r) \cap \operatorname{Cont}^{\geq 1}\Big) = \ell/r + \min_{\nu} \sum_j (k_j + 1)\nu_j,$$

with the minimum taken over ν with $\sum r_{ij}\nu_j = w_i$ for all i and $\sum z_j \nu_j = \ell$ such that $\bigcap_{\nu_j \geq 1} E_j \neq \emptyset$ and $f(E_j) \subset W$ for some $\nu_j \geq 1$.

Remark. There's a lot going on in this lemma, but it's not as complicated as it seems! The conditions in the minimum ensure the following:

(1) $\sum r_{ij}\nu_j = w_i$ means that we're looking at arcs with the right order of contact along each Y_i.

(2) $\sum z_j \nu_j = \ell$ means we're looking at arcs with the right order of contact along Z_r.

(3) $\bigcap_{\nu_j \geq 1} E_j$ ensures that the codimension isn't actually ∞ (i.e., that the arcs exist!).

(4) $f(E_j) \subset W$ for some $\nu_j \geq 1$ ensures that the above conditions are witnessed by an arc actually mapping into W.

Remark. The complexity of the statement of the theorems we'll show corresponds to the following additional features:

(1) The multicontact loci $\bigcap \operatorname{Cont}^{w_i}(Y_i)$ correspond to tracking order of arcs along multiple subschemes instead of just $\operatorname{Cont}^p(Y)$.

(2) The contact loci along Z_r correspond to tracking the influence of the singularities of X itself.

(3) The codimension is the influenced by the order of contact along not just $K_{X'/X}$ but also along the preimage of Z_r.

Inversion of adjunction

Semicontinuity

5 February 25

References are:

(1) "Singularities in mixed characteristic via perfectoid big Cohen–Macaulay algebras", L. Ma and K. Schwede.

(2) "Big Cohen–Macaulay algebras and seeds", G.D. Dietz.

Big Cohen–Macaulay algebras

Definition. Let (R, m) be a local ring, $x_1, \ldots, x_d$ a full system of parameters on R. B is a big Cohen–Macaulay R-algebra with respect to $x_1, \ldots, x_n$ if:

(1) $x_1, \ldots, x_d$ form a regular sequence on B.

(2) $mB \neq B$.

B is a (balanced) big Cohen–Macaulay algebra if B is a big Cohen–Macaulay algebra with respect to every system of parameters.

Theorem (Bartijn and Stroker)**.** *If B is a big Cohen–Macaulay algebra with respect to $x_1, \ldots, x_d$, then $\widehat{B}^m$ is a balanced big Cohen–Macaulay algebra (where $\widehat{B}^m$ is the mB-adic completion).*

From now on, we take all big Cohen–Macaulay algebras to be balanced.

Singularities

Definition (BCM/BCM$_B$-rationality)**.** Let (R, m) be an excellent local domain of dimension d and let B be a big Cohen–Macaulay algebra. (R, m) is BCM$_B$-rational if:

(1) R is Cohen–Macaulay.

(2) $H^d_m(R) \to H^d_m(B)$ is injective.

(R, m) is BCM-rational if it's BCM$_B$-rational for all big Cohen–Macaulay R-algebras B.

Proposition. *If (R, m) is an excellent local domain of dimension d and $x \in R$ is a nonzerodivisor, and B is a BCM algebra, then if R/xR is* BCM$_{B/xB}$*-rational then R is* BCM$_B$*-rational.*

Corollary. *If R/xR is* BCM*-rational so is R.*

Proof. Consider the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & R/x & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \xrightarrow{x} & B & \longrightarrow & B/x & \longrightarrow & 0 \end{array}$$

Applying local cohomology and using the vanishings guaranteed by Cohen–Macaulay condition on $R, B, R/xR, B/xB$, we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{d-1}_m(R/xR) & \xrightarrow{x} & H^d_m(R) & \longrightarrow & H^d_m(R) & \longrightarrow & 0 \\ & & \downarrow & & \hookdownarrow & & \hookdownarrow & & \\ 0 & \longrightarrow & H^{d-1}_m(B/xB) & \xrightarrow{x} & H^d_m(B) & \longrightarrow & H^d_m(B) & \longrightarrow & 0 \end{array}$$

A diagram chase then ensures that the left-side map is an injection as well. □

Proposition. *If (R, m) is an excellent local domain of characteristic p, then R is BC-rational if and only if R is F-rational.*

Proof. One can check that R is BCM-rational if and only if $\widehat{R}$ is (in arbitrary characteristics). Assume $\widehat{R}$ is not F-rational: then there's some system of parameters $x_1, \ldots, x_d \in \widehat{R}$ such that $u \in (x_1, \ldots, x_d)^* - (x_1, \ldots, x_d)$. But this implies that $[u/\overline{x_1 \cdots x_d}] \in H^d_m(R)$ goes to $H^d_m(R^+)$, and R^+ is a big Cohen–Macaulay algebra, so that R cannot be BCM-rational.

Conversely assume $\widehat{R}$ is F-rational. Then $H^d_m(\widehat{R})$ is a simple object in the category of R-modules with Frobenius action. Thus the kernel of $H^d_m(\widehat{R}) \to H^d_m(B)$ is either 0 or $H^d_m(\widehat{R})$; however, since B is big Cohen–Macaulay we have that $[1/x_1 \cdots x_d]$ is not 0 in $H^d_m(B)$, so the kernel must be 0. □

Proposition. *Let (R, m) be an excellent local domain. If R is BCM-rational then it's pseudorational, and in particular if R is essentially of finite type over a field of characteristics 0 then R has rational singularities.*

Proof. The proof uses the Sancho–de-Salas exact sequence. □

Conjecture. *The converse is true as well.*

More about (perfectoid) big Cohen–Macaulay modules

Theorem (André, Shimomoto)**.** *Let (R, m) be a complete local domain of mixed characteristic and $\underline{x} = p, x_2, \ldots, x_d$ a system of parameters, and say B is a $(pg)^{1/p^\infty}$-almost integral perfectoid R-algebra that is $(pg)^{1/p^\infty}$-almost Cohen-Macaulay. Then:*

(1) There exists an integral perfectoid big Cohen–Macaulay algebra C and a morphism $B^\# := (B^\flat)^\sharp \to C$.

(2) A further morphism C to a $\widehat{R}^+$ algebra C' which is still BCM.

Definition. B is a $(pg)^{1/p^\infty}$-almost Cohen–Macaulay algebra if $(pg)^{1/p^N}(x_1,\ldots,x_i)B : x_{i+1} \subset (x_1,\ldots,x_i)B$ for any N and $(pg)^{1/p^\infty}B \not\subset mB$.

Theorem. *Let $R \to R'$ be a local map of complete local domains, where R has mixed characteristic p. Then*

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ R^+ & \longrightarrow & (R')^+ \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

for some B, B' big Cohen–Macaulay algebras.

Lemma (Ma–Schwede)**.** *Let (R,m) be an excellent local domain of mixed characteristic, and B_1, B_2 big Cohen–Macaulay integral perfectoid $\widehat{R}^+$-algebras. Then $B_1 \otimes_{\widehat{R}^+} B_2$ maps $\widehat{R}^+$-linearly to a big Cohen–Macaulay integral perfectoid $\widehat{R}^+$-algebra C.*

Definition. An integral perfectoid R-algebra S is called a perfectoid seed if $S \to B$, with B an integral perfectoid big Cohen–Macaulay R-algebra.

Lemma. *If $\{S_\lambda\}$ is a direct system of integral perfectoid R-seeds then*

$$\widehat{\lim_\lambda S_\lambda}$$

is also an integral perfectoid seed.

This is proved by repeatedly applying the preceding lemma.

The positive-characteristic case

We recall some motivation from the paper of Dietz.

Definition. If R has characteristic p, then an S-algebra $R$is a seed if we have a map $S \to B$ for B a big Cohen–Macaulay R-algebra.

Lemma. *Let (R,m) be local noetherian and $S = \lim_\lambda S_\lambda$ a directed limit of a directed set. Then S is a seed if and only if each S_λ is.*

Proposition. *If R is a characteristic p local noetherian ring, S is a seed, and T is integral over S, then T is a seed.*

We don't yet have an analogue of this in the mixed characteristic case! This can be used to show that every noetherian local ring (R,m) of characteristic p maps to a (balanced) integrally closed quasilocal m-adically big Cohen–Macaulay algebra.

Lemma. *Let R be noetherian local of characteristic p, and say B_1, B_2 are big Cohen–Macaulay R-algebras. Then there exists a big Cohen–Macaulay R-algebra C and a morphism $B_1 \otimes_R B_2 \to C$.*

Theorem. *Let $R \to S$ be a local map between local noetherian rings of characteristic p. Let B be a big Cohen–Macaulay R-algebra. Then there exists a big Cohen–Macaulay S-algebra such that we have*

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$