Complex algebraic surfaces

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1 Introduction

Projective nonsingular algebraic varieties over $\mathbb{C}$ motivate and clarify a wide range of more general or more abstract techniques in algebraic geometry, and offer a useful window into the historical development of algebraic geometry. In this paper, we discuss some basic tools and results in the study of complex algebraic surfaces. We begin by extending what we know about intersections of curves on a smooth surface X to a symmetric bilinear form $\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$. As an immediate consequence of this definition we obtain the Riemann–Roch theorem for surfaces, and then we discuss several examples and applications, including the Hodge index theorem, the genus formula, and Noether's theorem. We then shift focus to cubic surfaces; we show that every cubic surface contains a line and that smooth cubic surfaces contain exactly 27 lines and are rational. In particular, we exhibit any smooth cubic as the blowup of $\mathbb{P}^2$ at six points. We then briefly discuss some of general properties of blowups and do an example, and then proceed to place the results on smooth cubic surfaces into the broader context of del Pezzo surfaces.

Throughout, we work over $\mathbb{C}$, although many results extend to any algebraically closed field (we try to indicate where this is easy to see). We use "surface" to mean a smooth, projective, irreducible 2-dimensional variety over the complex numbers, and "curve" to mean an irreducible projective 1-dimensional variety, *not* necessarily smooth. We focus our efforts on examples and calculations, and for the most part simply cite the general theory or results not specific to our work here.

2 Riemann–Roch on surfaces and its consequences

We begin by extending our discussion of intersection on surfaces. Much of the initial development in this paper roughly follows the arguments of [Bea96, Chapter 1]. Recall that we have defined $C \cdot D$ for planar curves C, D by $C \cdot D = \sum_P I(P, C, D) = \sum_P \dim_k \mathcal{O}_{\mathbb{P}^2,P}/(f, g)$, where f, g are local equations for C, D. Now, let X be any nonsingular (irreducible) surface, and let C, D be distinct curves (by which we mean irreducible codimension-1 closed subvarieties). Since X is nonsingular, a codimension-1 subvariety is cut out locally by a single equation, so the definition of

$$C \cdot D = \sum_P I(P, C, D) = \sum_P \dim_k \mathcal{O}_{X,P}/(f, g)$$

carries over unchanged ($C \cap D$ is still finite, since they have no common component, and $\mathcal{O}_{X,P}/(f, g)$ is finite-dimensional by the exact same argument as previously).

Now, we want to extend this pairing on curves to a pairing on $\operatorname{Pic} X$. By the equivalence of Cartier and Weil divisors on nonsingular varieties, C and D correspond to effective Cartier divisors, and hence to line bundles $O(C)$, $O(D)$ (with sections!). Every line bundle on X arises as $O(C)$ for some C, which may not be effective; we thus want to give a definition of intersection number which covers this case.

We start by motivating this definition. Again, let C, D be effective Cartier divisors, and let $\mathcal{J}$ be the ideal sheaf of $C \cap D$. As previously, we have a complex of sheaves

$$0 \to \mathcal{O}(-C - D) \to \mathcal{O}(-C) \oplus \mathcal{O}(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_{C\cap D} = \mathcal{O}_X/\mathcal{J} \to 0.$$

Now, we again have that $\mathcal{O}_X/\mathcal{J}$ is the direct sum of the skyscraper sheaves $\mathcal{O}_{X,P}$ for each $P \in C \cap D$, and thus we have immediately that $C \cdot D := \sum_P \dim_k \mathcal{O}_{X,P}/(f,g) = \Gamma(X, \mathcal{O}_X/\mathcal{J}) = \chi(\mathcal{O}_X/\mathcal{J})$ (since $\mathcal{O}_X/\mathcal{J}$ is flabby and has no higher cohomology). Then we can use additivity of χ to see that

$$\begin{aligned} C \cdot D &= \chi(\mathcal{O}_X) - \chi(\mathcal{O}(-C) \oplus \mathcal{O}(-D)) + \chi(\mathcal{O}(-C-D)) \\ &= \chi(\mathcal{O}) - \chi(\mathcal{O}(-C)) + \chi(\mathcal{O}(-D)) + \chi(\mathcal{O}(-C-D)). \end{aligned}$$

This motivates the following definition (where by $\chi(L)$ for a line bundle L we really mean $\chi(\mathcal{L})$ for $\mathcal{L}$ its sheaf of sections; no confusion will result):

Definition. For $M, N \in \operatorname{Pic}(X)$, define

$$M.N = \chi(\mathcal{O}_X) - \chi(M^{-1}) - \chi(N^{-1}) + \chi(M^{-1} \otimes N^{-1}).$$

Example. Two distinct lines in $\mathbb{P}^2$ meet in exactly one point; this follows immediately since two distinct hyperplanes in a three-dimensional vector space always meet in a line, but it can also be seen here: a line in $\mathbb{P}^2$ corresponds to the line bundle $\mathcal{O}(1)$, and we thus get

$$\mathcal{O}(1)^2 := \mathcal{O}(1).\mathcal{O}(1) = \underbrace{\chi(\mathcal{O}_{\mathbb{P}^2})}_{1} - 2\underbrace{\chi(\mathcal{O}(-1))}_{0} - \underbrace{\chi(\mathcal{O}(-2))}_{0} = 1.$$

More generally, we then immediately can define D^2 for any curve or divisor D, without having to make an argument on moving one copy of D to meet the other transversely (although this is certainly the geometric intuition behind this algebraic definition). Note that exactly as in class we can recover Bézout's theorem from this pairing.

Theorem. *The pairing* $\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$ *is symmetric bilinear, and we have that for* $O(C), O(D)$ *effective divisors that* $O(C).O(D) = C \cdot D$.

Symmetry is clear, and we've already shown that we recover the intersection of curves in the case of effective divisors. Thus, we just need to show this is bilinear; we already know this in the case of effective divisors, so we just need to extend this to the general case.

First, we extend to the case where one of the line bundles is effective and represents a smooth curve. Then we can restrict the other line bundle to the smooth curve. We thus have a divisor on a smooth curve, and thus a well-defined degree.

Lemma. *Say* C *be a smooth curve and* D *any divisor. Then* $\mathcal{O}(D).\mathcal{O}(C) = \deg \mathcal{O}(D)|_C$.

In particular, this implies if one argument of the intersection pairing is represented by a smooth curve, then the pairing is linear in the other argument.

Proof. First, a general observation: we know that if $i : C \to X$ is a closed immersion and L is a line bundle on X, with sheaf of sections $\mathcal{L}$, then $f^*\mathcal{L}$ is an $\mathcal{O}_C$-module, and is the sheaf of sections of $f^{-1}L = L|_C$. Note that we then have by the projection formula that

$$i_*(i^*\mathcal{L}) = i_*(i^*\mathcal{L} \otimes \mathcal{O}_C) = \mathcal{L} \otimes i_*\mathcal{O}_C.$$

Recall also that the direct image functor for a closed embedding doesn't change the cohomology, so that $\chi(L|_C) = \chi(i^*\mathcal{L}) = \chi(\mathcal{L} \otimes i_*\mathcal{O}_C)$.

Now the proof follows immediately from additivity of the Euler characteristic on short exact sequences and the Riemann–Roch formula for curves. We have a short exact sequence

$$0 \to \mathcal{O}(-C) \to \mathcal{O}_X \to i_*\mathcal{O}_C \to 0,$$

and then by tensoring with $\mathcal{O}(-D)$ we get

$$0 \to \mathcal{O}(-C-D) \to \mathcal{O}(-D) \to \mathcal{O}(-D) \otimes i_*\mathcal{O}_C \to 0.$$

Note that by the preliminary comment we have $\mathcal{O}(-D) \otimes i_*\mathcal{O}(C) = i_*(\mathcal{O}(-D|_C))$. Now, by definition

$$\mathcal{O}(D).\mathcal{O}(C) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}(-D)) - \chi(\mathcal{O}(-C)) + \chi(\mathcal{O}(-C-D)).$$

From the relation $\chi(\mathcal{O}_X) = \chi(i_*\mathcal{O}_C) + \chi(\mathcal{O}(-C))$ obtained from the first exact sequence (and using that $\chi(i_*\mathcal{O}_C) = \chi(\mathcal{O}_C))$ this becomes

$$\chi(\mathcal{O}_C) - \chi(\mathcal{O}(-D)) - \chi(\mathcal{O}(-C - D)).$$

Now we use that $\chi(\mathcal{O}(-C-D)) - \chi(\mathcal{O}(-D)) = \chi(\mathcal{O}(-D|_C))$ to write this as $\chi(\mathcal{O}_C) - \chi(\mathcal{O}(-D|_C))$. Now, we apply the Riemann–Roch theorem for curves, which says that $\chi(E) - \chi(\mathcal{O}_C) = \deg E$ for any line bundle E, to see that this is $-\deg(-D|_C) = \deg(D|_C)$, and thus our result is shown. □

Finally, we handle the general case.

Proof. Since X is projective, there is a very ample divisor H on X (just take the intersection of X with a general hyperplane). There is then n such that $nH + D$ is very ample (and nH is very ample as well) (see, e.g., [Har77, p. 169]).

Now, we claim there are smooth curves representing nH and $D + nH$. To see this, note that $D + nH$, nH are both hyperplane sections of X in some embedding in projective space. We can then apply Bertini's theorem (since X is smooth and we're working over $\mathbb{C}$; this holds more generally over an algebraically closed field, or I believe any infinite one) to see that the intersection of a general hyperplane with X is nonsingular, and thus $D + nH$ and nH both have smooth representatives.

Now, consider the expression

$$s(A, B, C) = (A.B + C) - (A.B) - (A.C).$$

Writing this out, we see that this is

$$-\chi(\mathcal{O}_X) + \chi(-A) + \chi(-B) + \chi(-C) - \chi(-A - B) - \chi(-A - C) - \chi(-B - C) + \chi(-A - B - C),$$

and thus this is symmetric in A, B, C. Now, let $A = E$, $B = D$, and $C = nH$. Via the equality $s(E, D, nH) = s(nH, E, D)$, we obtain

$$\begin{aligned}(E.D + nH) - (E.D) - (E.nH) &= (nH.E + D) - (nH.E) - (nH.D)\\ &= \deg(O(E + D)|_{nH}) - \deg(O(E)|_{nH}) - \deg(O(E)|_{nH}) = 0,\end{aligned}$$

since nH is effective and represented by a smooth curve. Thus we have that

$$(E.D) = (E.D + nH) - (E.nH).$$

Since both nH and $D + nH$ are effective and have representatives that are smooth curves, we have that the right side is linear in E, and thus the left side is as well. Thus by symmetry we have that the pairing is bilinear. □

From here on, then, we'll simply write $C.D$ in place of $C \cdot D$ for curves C, D, and interchangeably use $\mathcal{O}(C).\mathcal{O}(D)$ and $C.D$.

Remark. Note that given a map of surfaces $f : X \to Y$, if f is surjective we know how to pull back line bundles, and we denote the resulting map $\operatorname{Pic} Y \to \operatorname{Pic} X$ by f^*. If f is surjective we can also pull back divisors (we can pull back a divisor as long as its support does not contain f), and this is compatible with the pullback of line bundles.

We can also push-forward an irreducible curve under a generically finite map $f : X \to Y$ by defining $f_*(C) = 0$ if C is a point, of $f_*(C) = dC'$ if $\operatorname{im} C = C'$ with the map $C \to C'$ of degree d (note that since f is projective the image of C must be either a curve or a point). We can then extend this by linearity to obtain a pushforward $f_* : \operatorname{Pic} X \to \operatorname{Pic} Y$.

Finally, we note that given $f : X \to Y$ generically finite of degree d, if D, D' are divisors on Y, hen we have that $d(D.D') = g^*D.g^*D'$. We only use this once going forward and so we omit the proof, but it can be found in [Bea96, 1.8].

Example. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. First, we claim $\operatorname{Pic} X = \mathbb{Z} \oplus \mathbb{Z}$. First, note that every divisor on X corresponds to a bihomogeneous polynomial of bidegree (m, n), giving a surjective group homomorphism $\operatorname{Div} X \to \mathbb{Z} \oplus \mathbb{Z}$. Since the rational functions on X have bidegree $(0, 0)$, clearly this descends to a surjective homomorphism $\operatorname{Pic} X \to \mathbb{Z} \oplus \mathbb{Z}$. Now, we claim this is injective, which will following from showing than an element of $\operatorname{Div} X$ mapping to 0 is a principal divisor. Say $D = \sum a_i D_i \mapsto 0$, with the D_i prime divisors. Since X is smooth, each D_i is in fact $\operatorname{div} F_i$ for $\deg F_i = \deg D_i$. Then $\varphi := \prod F_i^{a_i}$ is a rational function with $\operatorname{div} \varphi = D$, showing our claim.

The geometry of the situation is quite clear: $\mathbb{P}^1 \times \mathbb{P}^1$ has two families of lines, the image of $\mathbb{P}^1 \times \{\mathrm{pt}\}$ and $\{\mathrm{pt}\} \times \mathbb{P}^1$. A line of the first kind is defined by an element of bidegree $(1, 0)$, and of the latter is defined by an element of bidegree $(0, 1)$. Thus the classes of these two lines, which we denote by l_1, l_2 generate $\operatorname{Pic} X$, so any curve on X is linearly equivalent to some $ml_1 + nl_2$.

Finally, let's examine the pairing on $\operatorname{Pic} X$, which is also geometrically intuitive. Clearly lines representing l_1 and l_2 meet in a point (if one line is the image of $\mathbb{P}^1 \times \{[\lambda_1, \lambda_2]\}$ and the other is the image of $l_2 = \{[\mu_1, \mu_2]\} \times \mathbb{P}_1$, then the lines meet transversely at $([\mu_1, \mu_2], [\lambda_1, \lambda_2])$). Thus we have $l_1.l_2 = 1$. In contrast, the lines in the class l_1 are all disjoint, so $l_1^2 = 0$, and likewise for l_2. Thus, via bilinearity we have

$$(ml_1 + nl_2)(m'l_1 + n'l_2) = mn' + m'n.$$

Now, we can use this pairing on $\operatorname{Pic} X$ to get an immediate proof of Riemann–Roch for surfaces (there is also a proof via Hirzebruch–Riemann–Roch, which comes up below in our discussion of Noether's formula):

Theorem (Riemann–Roch for surfaces)**.** *Let L be a line bundle on smooth projective variety X. Then*

$$\chi(L) = h^0(L) - h^1(L) + h^2(L) = h^0(L) - h^1(L) + h^0(K - L) = \chi(\mathcal{O}_X) + \frac{1}{2} L.(L - K).$$

The first equality above is by definition and the second is since $H^2(X, L) = H^0(X, K - L)$ via Serre duality. Thus, the only thing to prove is the last equality:

Proof. Consider $(-L.(L - K))$, for K a divisor corresponding to the line bundle ω_X of meromorphic 2-forms on X. By definition this is $\chi(\mathcal{O}_X) - \chi(L) - \chi(K - L) + \chi(K)$. Note that $\chi(K) = \chi(\mathcal{O}_X)$, since by Serre duality $H^i(X, \mathcal{O}_X) = H^{2-i}(X, \omega_X)$, and likewise $\chi(K - L) = \chi(L)$. Thus we get

$$-(L.(L - K)) = (-L.(L - K)) = 2\chi(\mathcal{O}_X) - 2\chi(L)$$

and thus

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{1}{2}(L.(L - K)),$$

concluding the proof. □

This immediately gives the genus formula; recall the (arithmetic) genus $g(C)$ of a curve C is defined to be $h^1(C, \mathcal{O}_C)$, which for a projective irreducible curve C is $1 - \chi(\mathcal{O}_C)$ (since $h^0(C) = 1$).

Corollary (genus formula)**.** *If C is a curve in a smooth surface X, then*

$$2 - 2g(C) = C.(C + K).$$

Proof. We have a short exact sequence of sheaves

$$0 \to \mathcal{O}(C) \to \mathcal{O}_X \to i_* \mathcal{O}_C \to 0.$$

Additivity of the Euler characteristic, and the fact that direct images of sheaves under closed embeddings don't affect the cohomology, imply that $\chi(\mathcal{O}_C) = \chi(i_* \mathcal{O}_C) = \chi(\mathcal{O}(C)) - \chi(\mathcal{O}_X)$, and then using Riemann–Roch for $\mathcal{O}(C)$ gives

$$1 - g(C) = \chi(\mathcal{O}_C) = \chi(\mathcal{O}_X) + \frac{1}{2}(C.(C + K)) - \chi(\mathcal{O}_X) = \frac{1}{2}(C.(C + K)).$$

Thus we have $2g(C) - 2 = C.(C + K)$. □

Example. Let C be a cubic in $\mathbb{P}^2$. Then $O(C) = O(3)$, while $O(K) = O(-3)$ (since the canonical bundle of $\mathbb{P}^n$ is $O(-n-1)$). Then we have that

$$2g(C) - 2 = C.(C + K) = \mathcal{O}(3).\mathcal{O}_{\mathbb{P}^2} = \chi(\mathcal{O}_{\mathbb{P}^2}) - \chi(\mathcal{O}(-3)) - \chi(\mathcal{O}_{\mathbb{P}^2}) + \chi(\mathcal{O}(-3)) = 0,$$

and thus $g(C) = 1$. Thus, any cubic in $\mathbb{P}^2$ has arithmetic genus 1, even a singular cubic.

Example (genus-degree formula)**.** Let C be a curve in $\mathbb{P}^2$, defined by a homogeneous polynomial F of degree d. Then $\mathcal{O}(C) = \mathcal{O}(d)$ and $\mathcal{O}(K) = \mathcal{O}(-3)$, so using our knowledge of the pairing on $\operatorname{Pic}\mathbb{P}^n$ we recover the genus-degree formula

$$g(C) = 1 + \frac{1}{2}(d(d-3)) = \frac{d^2 - 3d + 2}{2} = \frac{(d-1)(d-2)}{2}.$$

Thus we see that only algebraic curves of certain arithmetic genera can be realized as planar curves.

Example. Let C be a curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (m, n). The canonical divisor on $\mathbb{P}^1 \times \mathbb{P}^1$ is $\mathcal{O}(K) = \mathcal{O}(-2, -2)$, so $C - K$ corresponds to $\mathcal{O}(m-2, n-2)$. From our above description of the pairing on $\operatorname{Pic}\mathbb{P}^1 \times \operatorname{Pic}\mathbb{P}^1$, we get $C.(C - K) = m(n-2) + n(m-2) = 2mn - 2m - 2n$. Thus we have that $g(C) = 1 + mn - m - n = (m-1)(n-1)$.

In particular, for bidegree $(1, 0)$ or $(0, 1)$ we get a genus-0 curve, which makes sense because these are exactly the classes of the two families of lines on $\mathbb{P}^1 \times \mathbb{P}^1$. Note also that the formula for $g(C)$ would seem to imply that $g(C)$ can be negative, for C defined by a polynomial of bidegree $(0, n)$ or $(m, 0)$. This is *not* actually the case though; a polynomial of bidegree $(m, 0)$ has vanishing locus the union of m lines, so is not irreducible and hence does not define a curve.

Note, in contrast to the previous example, that by taking $m = 1$, n arbitrary, we see that we have planar curves of any specified genus on the smooth quadric $\mathbb{P}^1 \times \mathbb{P}^1$.

We can give another expression for the quantity $\chi(X, \mathcal{O}_X)$ appearing in the Riemann–Roch theorem:

Theorem (Noether's formula)**.** *Let X be a smooth projective surface with (some choice of) canonical divisor K. Then*

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1(T_X)^2 + c_2(T_X)) = \frac{1}{12}(K^2 + \chi_{\mathrm{top}}(X)).$$

Note that since $c_1(T_X) = c_1(\bigwedge^2 T_X) = -c_1(\omega_X) = -K$, we have that $c_1(T_X)^2 = K^2$ immediately. In contrast, the equality $c_2(T_X) = \chi_{\mathrm{top}}(X)$ is much more nontrivial. We'll prove the equality of $\chi(X, \mathcal{O}_X)$ and $\frac{1}{12}(c_1(T_X)^2 + c_2(T_X))$; the second equality follows from the Poincaré–Hopf theorem relating the topological Euler characteristic to vector fields with isolated zeros. We first give a general proof, following the Hurwitz–Riemann–Roch theorem from class.

Proof from Hurwitz–Riemann–Roch. Let L be a line bundle on a smooth projective surface X. Then from the general formula $\operatorname{ch}(E) = 1 + c_1(E) + \frac{1}{2}(c_1(E)^2 - c_2(E)) + \cdots$ we get $\operatorname{ch}(L) = 1 + c_1(L) + \frac{1}{2}c_1(L)^2$ (this follows because $c_2(L) = 0$ and higher powers of $c_1(L)$ vanish since the cohomology is zero above degree 4).

We also need $\operatorname{td}(T_X)$. Let X be arbitrary for now, and let E be a vector bundle of rank r on X, with $x_1, \ldots, x_r$ the Chern roots. Then we have that

$$\begin{aligned}\operatorname{td}(E) = \prod \frac{x_i}{1 - e^{x_i}} &= \left(1 + \frac{x_1}{2} + \frac{x_1^2}{12} + \cdots\right) \cdots \left(1 + \frac{x_r}{2} + \frac{x_r^2}{12} + \cdots\right) \\ &= 1 + \frac{1}{2}(x_1 + \cdots + x_r) + \frac{1}{12}(x_1^2 + \cdots + x_r^2) + \frac{1}{4}\left(\sum_{i<j} x_i x_j\right) + \ldots.\end{aligned}$$

Re-expressing this is in the elementary symmetric polynomials e_i, we get

$$\begin{aligned}\operatorname{td}(E) &= 1 + \frac{1}{2}e_1 + \frac{1}{12}(e_1^2 - 2e_2) + \frac{1}{4}e_2 + \cdots \\ &= 1 + \frac{1}{2}e_1 + \frac{1}{12}(e_1^2 + e_2) + \cdots \\ &= 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1(E)^2 + c_2(E)) + \cdots.\end{aligned}$$

Now, for $\dim X = 2$, all the higher-order terms vanish, so we have

$$\operatorname{td}(T_X) = 1 + \frac{1}{12}(c_1(T_X)^2 + c_2(T_X)).$$

Now, we can apply Hirzebruch–Riemann–Roch: letting $\mathcal{O}(D)$ be the sheaf of sections of the bundle determined by a divisor D, we get

$$\begin{aligned}
\chi(X, \mathcal{O}(D)) &= \int \operatorname{ch}(\mathcal{O}(D)) \operatorname{td}(T_X) \\
&= \int \Big(1 + c_1(O(D)) + \frac{1}{2}c_1(O(D))^2\Big)\Big(1 + \frac{1}{2}c_1(T_X) + \frac{1}{12}(c_1(T_X)^2 + c_2(T_X))\Big) \\
&= \frac{1}{2}c_1(O(D))^2 + \frac{1}{2}c_1(O(D))c_1(T_X) + \frac{1}{12}(c_1(T_X)^2 + c_2(T_X)). \\
&= \frac{1}{2}c_1\big(O(D).O(D-K)\big) + \frac{1}{12}(c_1(T_X)^2 + c_2(T_X)). \\
&= \frac{1}{2}D.(D-K) + \frac{1}{12}(c_1(T_X)^2 + c_2(T_X)).
\end{aligned}$$

(where we used that c_1 is a homomorphism and that $c_1(T_X) = c_1(\bigwedge^2 T_X) = c_1(\omega_X^{-1}) = -K$). But by Riemann–Roch for surfaces, we also know

$$\chi(X, \mathcal{O}(D)) = \chi(X, \mathcal{O}_X) + \frac{1}{2}D.(D-K).$$

Comparing these two expressions, we see immediately that

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1(T_X)^2 + c_2(T_X)),$$

thus giving the desired expression. $\square$

We can also give a more direct proof in the case where X is a smooth surface in $\mathbb{P}^3$. This can be extended to the general case with much more work; one embeds $X \hookrightarrow \mathbb{P}^N$ for N large; then one can project to $\mathbb{P}^3$ in a generically one-to-one way, and then analyze and resolve the resulting singularities via blowing up, while keeping track of the numerical invariants involved. See [GH94, Chapter 4] for this proof, as well as the main idea behind the following:

Direct proof. Say that $X = V(F) \subset \mathbb{P}^3$ for $\deg F = d$. We know already from Problem 5 on the last homework that

$$\chi(X, \mathcal{O}_X) = 1 - \frac{(3-d)(2-d)(1-d)}{3!} = \frac{d^3 - 6n^2 + 11n}{6}. \tag{1}$$

Now, recall from Problem 6 on that homework that for V a 4-dimensional vector space we have a short exact sequence

$$0 \to O \to V \otimes O(1) \to T_{\mathbb{P}^3} \to 0$$

of bundles on $\mathbb{P}^3$ and a short exact sequence

$$0 \to T_X \to T_{\mathbb{P}^3}|_X \to O_X(d) \to 0$$

of bundles on X, where $\mathcal{O}_X(d) = \mathcal{O}(d)|_X$.

The former sequence implies that

$$c(T_{\mathbb{P}^3}) = c(V \otimes O(1)) = c(O(1)^4) = (1+t)^4 = 1 + 4t + 6t^2 + 4t^3,$$

where t is the class of a hyperplane. The latter says, by definition, that the normal bundle $d_{X/\mathbb{P}^3}$ is $O_X(d)$. Now, the Whitney sum formula says that $c(T_{\mathbb{P}^3}|_X) = c(O_X(d))c(T_X)$.

To be precise, we have a map $X \hookrightarrow \mathbb{P}^3$, which induces a restriction map $H^*\mathbb{P}^3 \to H^*X$; by naturality of the Chern class, we have $c(T_{\mathbb{P}^3}|_X) = c(T_{\mathbb{P}^3})|_X$, where by abuse of notation the right side denotes the image of $c(T_{\mathbb{P}^3})$ in H^*X. The restriction map on cohomology just takes the hyperplane class t to the class of the intersection of a generic hyperplane with X (by Bertini's theorem such an intersection is generically smooth and reduced). We denote this class by h.

Now, we have that $c(O(d)) = 1+dt$ and thus $c(O_X(d)) = 1+dh$; likewise, $c(T_{\mathbb{P}^r}) = 1+4h+6h^2$. Thus, expanding formally and using the vanishing of h^i for $h > 3$, we get

$$c(T_X) = \frac{C(T_{\mathbb{P}^3})|_X}{1+dh} = (1+4h+6h^2)(1-dh+d^2h^2) = 1+(4-d)h-(d^2-4d+6)h^2.$$

Thus we have $c_1(T_X) = (4-d)h$ and $c_2(T_X) = (d^2-4d+6)h^2$. Now, using that $h^2 = d$ (since h^2 is the image of t^2 under the map induced by restriction, and two generic hyperplanes in $\mathbb{P}^3$ meet in a line which intersects the degree-d surface X in d points), we obtain

$$c_1(T_X) = (4-d)^2h^2 = d(4-d)^2, \qquad c_2(T_X) = (d^2-4d+6)h^2 = d(d^2-4d+6).$$

Putting this all together, we get

$$\frac{c_1(T_X)^2 + c_2(T_X)}{12} = \frac{d(4-d)^2 + d(d^2-4d+6)}{12} = \frac{2d^3-12d^2+22d}{12} = \frac{d^4-6d^2+11d}{6}.$$

But this matches (1), proving the claim. □

Now, we give an application of the Riemann–Roch theorem on surfaces, the Hodge index theorem. First, we give a few motivating examples.

Example. On $\mathbb{P}^2$, we have that $\operatorname{Pic} X \cong \mathbb{Z}$ is generated by a line l (corresponding to $\mathcal{O}(1)$), with $l^2 = 1$.

Example. On $\mathbb{P}^1 \times \mathbb{P}^1$, we have that $\operatorname{Pic} X \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $l_1 = (1,0)$ and $l_2 = (0,1)$, with $l_1^2 = l_2^2 = 0$ and $l_1.l_2 = 1$. Thus the matrix representing the symmetric bilinear intersection form is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can change our generators to $h_1 = (1,1)$, $h_2 = (1,-1)$, in which case the matrix becomes

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Example. Let X be the surface obtained by blowing up $\mathbb{P}^2$ at a point, with exceptional divisor E. As we'll see later, $\operatorname{Pic} X = \operatorname{Pic} \mathbb{P}^2 \oplus \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$ is generated by $\widetilde{l}$ and E, where $\widetilde{l}$ is the strict transform of a line l in $\mathbb{P}^2$ and E. We can choose l to *not* pass through the point we're blowing up at, so it's clear $\widetilde{l}.E = 0$. Since $X \to \mathbb{P}^2$ is an isomorphism away from E, $\widetilde{l}^2 = l^2 = 1$. Finally, $E^2 = -1$, as we'll see later from the general results on blowups in section 4. Thus we have that the matrix representing our form is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Remark. Note that this shows immediately that $\mathbb{P}^1 \times \mathbb{P}^1$ is not the blowup of $\mathbb{P}^2$ at a point, since we can't take integer multiples of the orthogonal generators of $\operatorname{Pic} \mathbb{P}^1 \times \mathbb{P}^1$ to make the matrix orthonormal, whereas we see that this is true of the blowup of $\mathbb{P}^2$ at a point. Of course, this is easier to see geometrically, e.g., any two lines in the blowup of $\mathbb{P}^2$, neither of which are the exceptional divisor, intersect, whereas every line in $\mathbb{P}^1 \times \mathbb{P}^1$ has infinitely many lines it doesn't intersect with.

We can generalize this example immediately to X the blowup of $\mathbb{P}^2$ at n points. Then $\operatorname{Pic} X = \mathbb{Z}^{n+1}$ has generators $\widetilde{l}, E_1, \ldots, E_n$. We have that $\widetilde{l}^2 = l^2 = 1$, that $E_i^2 = -1$ for all i, that $E_i.E_j = 0$ (since they intersect trivially), and $E_i.\widetilde{l} = 0$, since we can move l away from the finitely many points of the blowup. Thus we have the matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 \\ 0 & 0 & \ldots & -1 \end{pmatrix}.$$

In all of these cases, we see that the orthogonalized matrix has only one positive entry, and the rest negative (and no zeros). This is a consequence of the following definition and theorem; we follow the exposition of [Har77, V.1] for the rest of this section.

Definition. For $D \in \operatorname{Pic} X$, we say D is numerically equivalent to zero, written $D \equiv 0$ if $D.E = 0$ for all $E \in \operatorname{Pic} X$, and we say D and E are numerically equivalent if $D - E \equiv 0$.

Theorem (Hodge index theorem). *Let $H \in \operatorname{Pic} X$ be an ample line bundle on a smooth projective surface X. Then for all $D \in \operatorname{Pic} X$ with $D.H = 0$ but $D \not\equiv 0$ we have $D^2 < 0$*

We prove this after showing several lemmas first.

Lemma. *If H is a very ample line bundle and C is a curve, then $H.C > 0$, and in particular $H.C$ is the degree of the image of C in the embedding defined by H.*

Proof. Let $\varphi : X \hookrightarrow \mathbb{P}^N$ be the morphism determined by H. Let $C' = \varphi(C)$. We know that $\varphi^*\mathcal{O}(1) = H$ by definition. Let $\psi : X \to \operatorname{im} X \subset \mathbb{P}^N$ and let $i : \operatorname{im} X \hookrightarrow \mathbb{P}^n$ (so $\varphi = i \circ \widetilde{\varphi}$); then $\psi^*(i^*\mathcal{O}(1)) = H$ by functoriality and $C = \psi^*(C') = \varphi^*(C')$ (since $C' \subset \operatorname{im} X$). Then

$$C.H = \psi^*(C').\psi^*(i^*\mathcal{O}(1)) = C'.i^*\mathcal{O}(1) = \deg C',$$

where the second equality is because ψ^* is an isomorphism, hence degree 1, and the last morphism is because the degree of a curve in projective space is exactly the number of points of intersection with a hyperplane section. □

Lemma. *Let H be an ample divisor on X. There exists an N such that for all D, if $H.D > N$ then $H^2(X, \mathcal{O}(D)) = 0$.*

Proof. By Serre duality $H^2(X, \mathcal{O}(D)) = H^0(X, \mathcal{O}(K - D))$. Since H is ample there is n such that nH is very ample, and thus if $K - D$ is effective, then by the previous lemma $nH.(K - D) > 0$ and thus $H.(K - D) > 0$. Choose $N = H.K$. Then if $H.D > N = H.K$, so $H.K - H.D < 0$, we have that $K - D$ is not effective, and thus $H^0(X, \mathcal{O}(K - D)) = 0 = H^2(X, \mathcal{O}(D))$ (a noneffective divisor has no nonzero global sections). □

Lemma. *If H is an ample divisor on X, and D is a divisor with $D.H > 0$ and $D^2 > 0$, then nD is linearly equivalent to an effective divisor for all n sufficiently large.*

Proof. Apply the Riemann–Roch formula to the line bundle $O(nD)$, to obtain

$$h^0(\mathcal{O}(nD)) - h^1(\mathcal{O}(nD)) + h^2(\mathcal{O}(nD)) = \chi(\mathcal{O}_X) + \frac{1}{2}((nD)^2 + nD.K).$$

Apply the previous lemma to obtain N such that if $E.H \geq N$ then $h^2(\mathcal{O}(E)) = 0$. Then we have that $nD.H > N$ for n sufficiently large, since $D.H > 0$, and thus for large n we have $h^2(\mathcal{O}(nD)) = 0$. The right side is then a quadratic polynomial in n, since $D^2 > 0$, and thus increases, eventually monotonically as $n \to \infty$. On the left side, we have

$$h^0(\mathcal{O}(nD)) - h^1(\mathcal{O}(nD)) \leq h^0(\mathcal{O}(nD)),$$

and thus we must have that $h^0(\mathcal{O}(nD))$ also monotonically increases for large enough n, and thus in particular $\mathcal{O}(nD)$ must be effective for all n large. □

Proof of the Hodge index theorem. First, say $D^2 > 0$. As we observed in the proof of the bilinearity of the pairing, there is n such that $nH + D$ is very ample. Then $D.(nH + D) = D^2 > 0$, and so we can apply the last lemma to obtain that mD is effective for m large enough, but then $mD.H > 0$ and thus $D.H > 0$, a contradiction.

Say instead that $D^2 = 0$. Since $D \not\equiv 0$, there is some E with $E.D \neq 0$. Note that if we take $E' = (H.H)E - (H.E)H$ then we have $D.E' = (H.H)D.E \neq 0$ ($H.H \neq 0$ by our first lemma, since H is very ample and hence also effective). Moreover, $E'.H = 0$. Thus without loss of generality we can take E such that $E.D \neq 0$ but $E.H = 0$. Now, set $D' = nD + E$ for some $n \in \mathbb{Z}$. Then we have $D'.H = 0$, and $(D')^2 = n^2D^2 + 2nD.E + E.E = 2nD.E + E.E$. We can thus choose n to make this positive, and thus apply the analysis of the first paragraph to obtain a contradiction. □

Remark. The import of the Hodge index theorem is as follows. For X any surface, let $\operatorname{Pic}^0 X$ be the subgroup of elements of $\operatorname{Pic} X$ numerically equivalent to 0 (note that these are just the elements in the kernel of the symmetric bilinear form). Let $\operatorname{Num} X = \operatorname{Pic} X / \operatorname{Pic}^0 X$. For X smooth projective, $\operatorname{Num} X$ is a finitely generated abelian group. This follows from the theorem of Néron–Severi that for X a smooth projective surface the Néron–Severi group $\mathrm{NS}(X)$ is finitely generated (below, we sketch a proof over the complex numbers); a straightforward calculation that $\operatorname{Num} X$ is a quotient of $\mathrm{NS}(X)$ then implies $\operatorname{Num} X$ is also finitely generated.

Now, it is immediate that the intersection pairing descends to a *nondegenerate* symmetric bilinear form on $\operatorname{Num} X$ (since we've just taken the quotient by the kernel of the form). Let X be smooth projective. Consider the $\mathbb{Q}$-vector space $\operatorname{Num} X \otimes_{\mathbb{Z}} \mathbb{Q}$. Let H be a very ample divisor on X (such a divisor exists since X is projective). Then $H^2 > 0$, so in particular H represents a nonzero class in $\operatorname{Num} X$. Via the Gram–Schmidt algorithm, we have that $\operatorname{Num} X \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}H \oplus H^\perp$.

The Hodge index theorem implies that the symmetric bilinear form is negative-definite on $H^\perp$; to see this, note that we can take an orthonormal basis for $H^\perp$. Any element of this basis corresponds to a divisor $D \in \operatorname{Pic} X$ not numerically equivalent to 0, and we have that $D.H = 0$ (since $D \in H^\perp$). Then $D^2 < 0$, so the matrix for the quadratic form on $\operatorname{Num} X \otimes_{\mathbb{Z}} \mathbb{Q}$ is diagonal with one positive entry and the rest nonnegative.

Remark. In all of our examples discussed above ($\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, the blowup of $\mathbb{P}^2$ at finitely many points), the form on $\operatorname{Pic} X$ is already nondegenerate, so $\operatorname{Num} X = \operatorname{Pic} X$.

3 Smooth cubic surfaces

We begin by considering a particular smooth cubic, where certain properties are easy to show directly that in fact hold for all smooth cubic surfaces. We follow [Mus17] fairly closely in this section, expanding on some of the arguments there; in particular the proof that the rational morphism extends to a regular map exhibiting a smooth cubic as the blowup of $\mathbb{P}^2$ at 6 points is our own.

Example (Fermat cubic)**.** Say $\operatorname{char} k \neq 3$, and consider $X = Z(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subset \mathbb{P}^3$. It's immediately seen that X is smooth, since the partials all vanish simultaneously only for all $x_i = 0$ (using here that $\operatorname{char} k \neq 3$). We claim that there are exactly 27 lines on the Fermat cubic, and that 2 of them are disjoint.

A line in $\mathbb{P}^3$ is given by the intersection of two hyperplanes. If

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$$

is the matrix with rows the coefficients of the linear forms of the hyperplanes, we know that there is a nonvanishing 2×2 minor (since the hyperplanes are distinct), and thus we can perform row operations to make that minor the identity matrix. Say this is the first minor; thus the line is of the form $L = V(x_0 - ax_2 - bx_3, x_1 - cx_2 - bx_3)$. We thus analyze this case, and obtain the others by symmetry of the defining equations of X. The condition that this line lies on X is that

$$(ax_2 + bx_3)^3 + (cx_2 + bx_3)^3 + x_2^3 + x_3^3 = 0.$$

Expanding, we get

$$(1 + a^3 + c^3)x_2^3 + (1 + b^3 + d^3)x_3^3 + 3(a^2b + c^2d)x_2^2x_3 + 3(ab^2 + cd^2)x_2x_3^2 = 0,$$

and thus we obtain four equations:

- $a^3 + c^3 = -1$.
- $b^3 + d^3 = -1$.
- $a^2b + c^2d = 0$.
- $ab^2 + cd^2 = 0$.

First, say that none of a, b, c, d are zero. Then combining the second pair of equations we get $a/b = c/d$. Then plugging $c = ad/b$ into $a^2b + c^2d$ we get

$$a^2b + \frac{d^3a^2}{b^2} = 0$$

and thus

$$a^2(b^3 + d^3) = 0$$

However, the second equation implies $b^3 + d^3 = -1$ and $a \neq 0$, so this is a contradiction, and thus one of a, b, c, d must be 0.

Now say that $a = 0$. Then we have that $c^3 = -1$, so $a^2b + c^2d = 0$ implies $d = 0$ as well and thus $b^3 = -1$. If instead $b = 0$, then we get $d^3 = -1$, $a = 0$, and $c^3 = -1$. The same holds swapping a for c and b for d.

Thus, each solution is of the form $(\xi_1, 0, 0, \xi_2)$ or $(0, \xi_1, \xi_2, 0)$ for ξ_1, ξ_2 third roots of -1, so we obtain 18 such solutions. Now, these solutions were obtained by choosing one of the six 2×2 minors of the above matrix to make the identity, and thus if we repeat for each minor we obtain $6 \cdot 18$ solutions. However, we've certainly overcounted: if we consider a line $x_0 = \xi_1 x_2$, $x_1 = \xi_2 x_3$, then we can write this as

- $x_0 = \xi_1 x_2$, $x_1 = \xi_2 x_3$.
- $x_2 = \xi_1^{-1} x_0$, $x_1 = \xi_2 x_3$.
- $x_0 = \xi_1 x_2$, $x_3 = \xi_2^{-1} x_1$.
- $x_2 = \xi_1^{-1} x_2$, $x_3 = \xi_2^{-1} x_1$.

Thus, we've overcounted by a factor of 4, and thus we obtain $6 \cdot 18/4 = 27$ total lines on the Fermat cubic.

Note, moreover, that there are in fact two *disjoint* lines on the Fermat cubic (there are in fact many more, but for now just need two): choose ξ, ξ' distinct cube roots of -1, and take L to be defined by $x_0 = \xi x_2$, $x_1 = \xi x_3$ and L' to be defined by $x_0 = \xi' x_2$, $x_1 = \xi' x_3$. The matrix representing these four equations is

$$\begin{pmatrix} 1 & 0 & \xi & 0 \\ 0 & 1 & 0 & \xi \\ 1 & 0 & \xi' & 0 \\ 0 & 1 & 0 & \xi' \end{pmatrix}$$

This matrix is clearly full rank, and thus the vanishing locus in $\mathbb{P}^3$ is empty (since the intersection of four general hyperplanes in a 4-dimensional vector space is the origin). Thus, the two lines don't intersect.

Proposition. *If X is a cubic surface in $\mathbb{P}^3$ (possibly singular) then X contains a line.*

Proof. Let $G = \mathbb{G}(1, 3) = G(2, 4)$ be the Grassmannian of lines in $\mathbb{P}^3$ (equivalently 2-planes in a 4-dimensional vector space). We know that G is a 4-dimensional projective space. Let $\mathbb{P} \cong \mathbb{P}^{19}$ be the projective space parametrizing cubics in $\mathbb{P}^3$. Let $\Sigma \subset \mathbb{P} \times G$ be the incidence correspondence $\{(X, L) : L \subset X\}$. We have projections $p : \Sigma \to \mathbb{P}$ and $q : \Sigma \to G$. We want to show that the fibers of p are all nonempty.

First, we show that Σ is a closed subset of $\mathbb{P} \times G$ (and is hence projective). It suffices to check this locally over the open charts of G, since these cover Σ. So, consider the affine open V of G consisting of the lines in $\mathbb{P}^3$ corresponding to 2-planes of the form

$$\begin{pmatrix} 1 & 0 & a_1 & a_2 \\ 0 & 1 & b_1 & b_2 \end{pmatrix}$$

(i.e., lines occurring as the intersection of the planes $x_0 = a_1x_2 + a_2x_3$ and $x_1 = b_1x_2 + b_2x_3$). On V we have coordinates (arising from the Plücker coordinates) a_1, a_2, b_1, b_2.

A cubic surface, given by the zero locus of some

$$f_\alpha = \sum_{i+j+k+l=3} \alpha_{ijkl} x_0^i x_1^j x_2^k x_3^l,$$

corresponds to the 20-tuple of coefficients $\alpha = \alpha_{ijkl}$. A point on L can be written as $s[1,0,a_1,a_2] + t[0,1,b_1,b_2] = [s,t,sa_1+tb_1,sa_2+tb_2]$, with at least one of s,t nonzero. On L, we can rewrite f_α as a homogeneous cubic polynomial in s,t, say

$$\sum_{i=0}^{3} F_i(a_1,a_2,b_1,b_2,\alpha)s^i t^{3-i},$$

where the F_i are polynomial in the coordinates a_1,a_2,b_1,b_2 on G and the coordinate α on $\mathbb{P}$. Moreover, note that the F_i are all linear in α (this follows just by writing out $f_\alpha([s,t,sa_1+tb_1,sa_2+tb_2])$ and collecting terms). A point (α, L) is in $\Sigma \cap V$ if and only f_α vanishes on L, and thus if and only if F_0,F_1,F_2,F_3 are all zero. Thus, we see that Σ is cut out of $\mathbb{P}\times V$ by four polynomials in the coordinates on $\mathbb{P}\times V$, and is thus closed. This proof carries over to the other charts as well, and thus Σ is closed in $\mathbb{P}\times G$.

Now, we claim that $q : \Sigma \to G$ has fiber $\mathbb{P}^{15}$ over every point. We can also check this locally, so consider the affine chart on the Grassmannian considered before. We can change coordinates so that the line we're considering corresponds to the origin, i.e., the intersection of the hyperplanes $V(x_0)$ and $V(x_1)$. Then the condition that f_α vanishes at this point is just that the coefficients of $x_0^3, x_1^3, x_0x_1^2, x_1x_0^2$ are all zero; this gives four independent linear conditions on $\mathbb{P}^{19}$, and thus we see that the fiber over this point is just the codimension-4 linear subspace $\mathbb{P}^{15}$.

Thus, we have that q is a map of projective varieties, and the fibers of q are all irreducible, as is G, and thus Σ is irreducible of dimension 19. Now, we return to the morphism p. Note that the theorem on the dimension of fibers (see [Sha13, p. 77]) implies that the dimension of a fiber is always $\geq \dim \operatorname{im} X - X$. Note that we have shown the existence of a cubic with only finitely many lines, and thus the fiber over this cubic is 0-dimensional. Thus we have that $\dim\Sigma = \dim p(\Sigma) = 19 = \dim Y$. Since Σ is projective, $p(\Sigma)$ is closed in Y, and hence $p(\Sigma) = Y$ and thus every cubic contains a line. □

Proposition. *If X is a smooth cubic surface in $\mathbb{P}^3$ then X contains exactly 27 lines, and two of these lines are disjoint.*

Sketch of proof. Let $U \subset \mathbb{P}$ be the open subset of smooth cubics. The map $p^{-1}(U) \to U$ is proper, since it's the restriction of a proper map to an open subset. It suffices to show that this map is étale; then by general results, since U is irreducible every fiber has the same cardinality. Since we've exhibited one smooth cubic surface with 27 lines, every smooth cubic surface will have 27 lines. The map $p^{-1}(U) \to U$ takes $(X,L) \to X$; to see that this is étale it suffices to check that the Jacobian is nonvanishing at every (X,L) with $L \subset X$. We can check this locally over the chart V considered previously. Here, The Jacobian is given by the partial derivatives of $F_i(a_1,a_2,b_1,b_2,\alpha)$ with respect to the a_i and b_j (since these polynomials cut out $p^{-1}(U)\cap q^{-1}(V)$ over U in $\mathbb{P}\times G$).

Choose coordinates so we're working at the point $a_1 = a_2 = b_1 = b_2 = 0$. Writing out

$$f_\alpha(s,t,sa_1+tb_1,sa_2+tb_2) = \sum_{i=0}^{3} F_i(a_1,a_2,b_1,b_2,\alpha)s^i t^{3-i}$$

and differentiating with respect to each variable, we see that the Jacobian has columns given by the coefficients of $s\partial f/\partial x_2$, $s\partial f/\partial x_3$, $t\partial f/\partial x_2$, $t\partial f/\partial x_2$ at $(s,t,0,0)$. Then one can show that a linear dependence among these columns gives a shared linear factor of $\partial f/\partial x_2(s,t,0,0)$ and $\partial f/\partial x_3(s,t,0,0)$, and thus a common zero; the condition that the line $V(x_2,x_3)$ lies on $V(f)$ implies that $\partial f/\partial x_0$ and $\partial f/\partial x_1$ also vanish at this point, contradicting nonsingularity of X. Thus the map is étale and every smooth cubic contains 27 lines.

Finally, we show that there are two disjoint lines on any smooth cubic. Consider

$$p^{-1}(U) \times_U p^{-1}(U) = \{(X, L_1, L_2) : L_1, L_2 \subset X\}.$$

This carries an étale map to U (since the projection to one factor is étale, and then the map from that factor to U is étale as well). Let W be the open subset $\{(X, L_1, L_2) : L_1, L_2 \subset X, L_1 \cap L_2 = \emptyset\}$. We want to show that W is the union of connected components of $p^{-1}(U) \times_U p^{-1}(U)$, since then by irreducibility of U the fibers all have the same cardinality; in particular, we've shown that the Fermat cubic has two disjoint lines and thus all smooth cubics do.

To show this, we instead work with $R := \{(X, L_1, L_2) : L_1, L_2 \subset X, L_1 \cap L_2 \neq \emptyset\} \subset \mathbb{P} \times G \times G$. The condition $L_1 \cap L_2$ is a closed condition, so this is a projective set. R carries a proper projection to $Z \subset G \times G$ the locus of intersecting lines, which carries a further proper projection to G. We now repeatedly apply the theorem on the fibers again: the projection $Z \to G$ has fiber over each line $L \in G$ the set of all lines meeting L; this set itself carries a projection to $\mathbb{P}^1$, with fiber $\mathbb{P}^2$ (the projection just takes a line through L to the point of intersection, and the fiber consists of all lines meeting L at that point, which is parametrized by $\mathbb{P}^2$). Thus the fibers of the projection $Z \to G$ are irreducible of dimension 3, so Z is irreducible of dimension 7.

Now, consider the projection $R \to Z$. We claim that the fibers are all irreducible of dimension 12. To see this, for a point $(L_1, L_2) \in Z$, choose coordinates so that $L_1 = V(x_0, x_1)$ and $L_2 = V(x_1, x_2)$. Then the condition that $L_1, L_2 \subset X = V(f)$ is that $f \in (x_0, x_1) \cap (x_1, x_2) = (x_0 x_2, x_1)$. Thus, this condition is just that the cubic monomials *not* in this ideal have coefficient zero in f. There are 13 cubic monomials in this ideal: we obtain 4 by multiplying $x_0 x_2$ by some x_i and 10 by multiplying x_1 by some $x_i x_j$. The only monomial that we've double counted is $x_0 x_2 x_1$, which yields 13 monomials, or a linear projective subspace of dimension 12. Thus each fiber is isomorphic to $\mathbb{P}^{12}$, which shows that R is irreducible of dimension 19.

Now, we have a closed embedding $R \hookrightarrow \Sigma \times_{\mathbb{P}} \Sigma = (X, L_1, L_2) : L_1, L_2 \subset X$. If we restrict R to the preimage of U in R, say R', and restrict the right side to $p^{-1}(U) \times_U p^{-1}(U)$, we have that the right side is dimension 19 (since there are only 27 lines on each smooth fiber, we have a finite projection $p^{-1}(U) \times_U p^{-1}(U) \to U$, and $\dim U = 19$). Then since R' has dimension 19 as well and is irreducible, we have that R' must embed as a component of $p^{-1}(U) \times_U p^{-1}(U)$. Then $W = (R')^c$ is the union of the other components, concluding the proof. □

Corollary. *Smooth cubic surfaces in $\mathbb{P}^3$ are rational.*

Proof. Let X be a smooth cubic surface in $\mathbb{P}^3$; then we know that X contains two disjoint lines L_1, L_2. We claim for each $p \in \mathbb{P}^3 - (L_1 \cup L_2)$ that there is a unique line in $\mathbb{P}^3$ passing through L_1, L_2, and p. This can be seen on the level of linear algebra: if $\mathbb{P}^3 = \mathbb{P}(V)$ for V 4-dimensional, we can choose a direct sum decomposition of V as $\widetilde{L}_1 \oplus \widetilde{L}_2$, where $\widetilde{L}_1, \widetilde{L}_2$ are the 2-planes corresponding to L_1, L_2. Then any $p \in \mathbb{P}^3 - (L_1 \cup L_2)$ corresponds to a unique nonzero vector $v \in V$, up to scaling; the projections of this vector to $\widetilde{L}_1, \widetilde{L}_2$ are both nonzero (since otherwise v lies on one of the 2-planes). These two projections then span a 2-plane in V which intersects $\widetilde{L}_1, \widetilde{L}_2$ and contains v. Thus this gives the desired line in $\mathbb{P}^3$ through L_1, L_2, and p. Uniqueness is easy to see; if v lies on another 2-plane meeting $\widetilde{L}_1, \widetilde{L}_2$ then we violate the uniqueness of the direct sum decomposition of v.

The expression in coordinates is immediate. Choose coordinates on $\mathbb{P}^3$ so that $L_1 = V(x_2, x_3)$ and $L_2 = V(x_0, x_1)$ (clearly $L_1 \cap L_2 = \emptyset$). For each $p = [y_0, y_1, y_2, y_3] \in \mathbb{P}^3 - (L_1 \cup L_2)$ the line through p and L_1, L_2 meets L_1 at $[y_0, y_1, 0, 0]$ and L_2 at $[0, 0, y_2, y_3]$ (note that this is well-defined because at least one each of y_0, y_1 and y_2, y_3 are nonzero by assumption that p does not lie on $L_1 \cup L_2$).

Thus, we have a well-defined surjective morphism

$$\mathbb{P}^3 - (L_1 \cup L_2) \to \mathbb{P}^1 \times \mathbb{P}^1, \qquad [y_0, y_1, y_2, y_3] \mapsto ([y_0, y_1], [y_2, y_3]).$$

We can thus restrict this to a map $X - (L_1 \cup L_2) \to \mathbb{P}^1 \times \mathbb{P}^1$. This map is dominant and is 1-to-1 over an open set, since a general line meets X in three distinct points, i.e., a line meeting L_1 and L_2 and a third point p either does not meet X anywhere else or is contained on X, and there are only 27 such lines contained on X (even fewer of which meet L_1 and L_2). Thus, we have a well-defined rational inverse (this follows either by direct computation or by applying Zariski's main theorem to the open subset over which the projection is finite), and thus we see that X is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, which is birational to $\mathbb{P}^2$ (for example, because they have identical function fields). Thus X is rational. □

Note that this proof generalizes to show that given a cubic hypersurface in $\mathbb{P}^{2m+1}$ containing two disjoint m-planes is rational.

Remark. In fact, more can be said about the above proof. As we showed, on the Fermat cubic given two disjoint lines there are exactly five distinct lines on the cubic meeting both of them. In fact, the use of incidence correspondences to show that any smooth cubic has two disjoint lines can be extended to show that these lines meet exactly as they do in the Fermat cubic, and therefore, given disjoint lines L_1, L_2 on any smooth cubic, there are exactly five lines meeting both of them.

Moreover, we can extend the rational map $X - (L_1 \cup L_2)$ to all of X, giving a surjective map $X \to \mathbb{P}^1 \times \mathbb{P}^1$. To do so, retain the choice of coordinates from above. Write $X = V(F)$, where F is a cubic polynomial; since $L_1 \cup L_2 \subset X$ we have that $F \in (x_0x_2, x_0x_3, x_1x_2, x_1x_3)$. We can then write

$$F = x_0F_0(x_0, \dots, x_3) + x_1F_1(x_0, \dots, x_3).$$

We will use this expression to expand our rational map to $L_2 = V(x_0, x_1)$. Now, we claim that F_0 and F_1 don't vanish simultaneously on L_2. This follows by nonsingularity of X; the Jacobian of F is

$$\left(F_0 + x_0\frac{\partial F_0}{\partial x_0} + x_1\frac{\partial F_1}{\partial x_0} \qquad x_0\frac{\partial F_0}{\partial x_1} + \frac{\partial F_1}{\partial x_1} + F_1 \qquad x_0\frac{\partial F_0}{\partial x_2} + x_1\frac{\partial F_1}{\partial x_2} \qquad x_0\frac{\partial F_0}{\partial x_3} + x_1\frac{\partial F_1}{\partial x_3} \right).$$

If F_0 and F_1 vanish at $p = [0 : 0 : y_2 : y_3]$ we see immediately that the Jacobian vanishes at this point as well, and thus F_0, F_1 cannot simultaneously vanish.

Thus, where F_0 is nonzero, on X we have that $x_0/x_1 = -F_1/F_0$, and where F_1 is nonzero we have $x_1/x_0 = -F_0/F_1$. Thus, we see that we have a morphism

$$[x_0 : x_1 : x_2 : x_3] \mapsto ([-F_1, F_0], [x_2, x_3])$$

wherever F_1 and F_0 don't simultaneously vanish, and that this agrees with the rational map given by projection. By what we showed above, these two open sets cover L_2, and thus we've extended our rational map to L_2; we can do likewise for L_1 by rewriting $F = x_2F_2 + x_3F_3$ and repeating the calculation.

Note also that since $X \to \mathbb{P}^1 \times \mathbb{P}^1$ is now a projective morphism (since X is projective) we have that the image is closed; we know that the image is dense in $\mathbb{P}^1 \times \mathbb{P}^1$, and thus the map is a surjection.

Now, consider the fibers of this map. Over a point (p, q) of $\mathbb{P}^1 \times \mathbb{P}^1$, the fiber is either the unique third point of intersection of the line through p and q with X, or, if this line lies on X, the entire line. As we've just observed, there are exactly five lines meeting L_1 and L_2, and thus there are five such $\mathbb{P}^1$ fibers. Thus we see that the surjective map $X \to \mathbb{P}^1 \times \mathbb{P}^1$ is an isomorphism away from these five points and has a $\mathbb{P}^1$ fiber over each such point, and thus we see that any smooth cubic can be realized as the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at five points, and thus by an example in the next section, as the blowup of $\mathbb{P}^2$ at six points.

In the final section, we'll see how this result fits into the broader topic of del Pezzo surfaces.

4 Blowups and resolution of singularities

In this section we cover some basic properties of blowups on smooth surfaces, some of which we've used previously and some of which we'll use in the next section. We present the results without proof; all can be found in [Bea96, Chapter 2].

Proposition. *Let X be a surface, and let $\epsilon : X' \to X$ be the blowup of X at a point p, with exceptional divisor E. Then:*

- *If $C \subset X$ is a curve, with multiplicity m at p (with m possibly 0 if $p \notin C$) then $\epsilon^*C = \widetilde{C} + mE$, where $\widetilde{C}$ denotes the strict transform of C in X'.*
- $\operatorname{Pic} X' \cong \operatorname{Pic} X \oplus \mathbb{Z}$, *via the map* $\operatorname{Pic} X \oplus \mathbb{Z} \to \operatorname{Pic} X'$ *sending* $(L, n) \mapsto \epsilon^*L + nE$.
- *For divisors D, D' on X, we have $\epsilon^*D.\epsilon^*D' = D.D'$, $\epsilon^*D.E = 0$, and $E^2 = -1$.*
- *If K is a canonical divisor on X, then $\epsilon^*K + E$ is a canonical divisor on X'.*

Recall that we have a bijection between linear systems of dimension n without fixed components on a surface X and rational maps to $\mathbb{P}^n$ with image not contained in any hyperplane. The points of indeterminacy of the rational map are exactly the base points of the linear system. We can use this viewpoint to resolve rational maps from surfaces to projective varieties:

Proposition. *Let $f : X \dashrightarrow Y$ be a rational map from a surface X to a projective variety Y. There is then a surface X', a morphism $X' \to Y$, and a birational morphism $X' \to X$ which is the composition of finitely many blowups such that*

$$\begin{tikzcd} X' \arrow[d] \arrow[dr] & \\ X \arrow[r, dashed] & Y \end{tikzcd}$$

commutes.

We omit the details of the proof; one key point, however, is that the proof gives a bound on the number of blowups required: To the rational map we associate a linear system, contained inside some $|D|$, with no fixed component; if this linear system has no base points then the map is already a morphism, so assume it has a base point. Then we blowup at this base point. We can then pull back the linear system from X to its blowup; we can subtract off a multiple of the exceptional divisor to obtain a linear system with no fixed component, and thus a rational map to Y which agrees with the composition of the map to X with the rational map $X \dashrightarrow Y$. We can continue doing this until there are no base points and we thus have a morphism. The key point is that it takes at most $D.D$ blowups.

Example. We claim that the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point is isomorphic to the blowup of $\mathbb{P}^2$ at two points (in particular, this is used above to show that a smooth cubic is the blowup of $\mathbb{P}^2$ at 6 points). View $\mathbb{P}^1 \times \mathbb{P}^1$ as the smooth quadric $Q = V(x_0x_4 - x_2x_3) \subset \mathbb{P}^3$ via the Segre embedding. Let $p = [0 : 0 : 0 : 1] \in Q$, and define $\varphi : \mathbb{P}^3 - \{p\} \to \mathbb{P}^2$, $[x_0, x_1, x_2, x_3] \mapsto [x_0, x_1, x_2]$ the projection from p to the hyperplane $x_4 = 0$.

Then p restricts to a rational map $Q \to \mathbb{P}^2$, defined everywhere except p. This map is dominant but not surjective; to see this, let $[y_0, y_1, y_2] \in \mathbb{P}^3$. The points in the fiber over $[y_0, y_1, y_2]$ are of the form $[y_0, y_1, y_2, t]$. For this to be a point of Q, we need $ty_0 = y_1y_2$. If $y_0 \neq 0$ we can just take $t = y_1y_2/y_0$, and thus we see that the fiber over $[y_0, y_1, y_2]$ is a single point. Likewise, if $y_0 = y_1 = 0$, then we can just take $[0 : 0 : y_2 : y_3]$, so the fiber is a whole $\mathbb{P}^1$, and likewise if $y_0 = y_2 = 0$. However, if $y_0 = 0$ but $y_1, y_2 \neq 0$ then there is no solution, so the image of Q under projection misses a copy of the punctured affine line in $\mathbb{P}^2$.

Now, we blowup Q at p to obtain $\widetilde{Q}$. By the general theory sketched above, we thus get a rational map $\widetilde{Q} \dashrightarrow \mathbb{P}^2$. In fact, this is a morphism, and in fact a surjective morphism. To see this, we show in charts that the rational map away from the exceptional fiber extends to a morphism.

We work in the chart U on Q defined by $x_4 = 1$, on which we have $x_0 = x_1x_2$. Thus, on this chart we can take coordinates $y = x_1, z = x_2$. Writing Y, Z for the coordinates on $\mathbb{P}^1$, the blowup at p is then the subset of $U \times \mathbb{P}^1$ cut out by $yZ = Yz$. Now, take the chart U' on U where $Z \neq 0$. Define $\widetilde{\varphi} : U' \to \mathbb{P}^2$ by

$$(y, z, Y/Z) \mapsto [y, Y/Z, 1].$$

Note that where $(y, z) \neq (0, 0)$ we have $[y, Y/Z, 1] = [yz, zY/Z, z] = [x, y, z]$, so this agrees with φ away from the exceptional fiber and is regular on the exceptional fiber. Likewise, on the chart U'' where $Y \neq 0$ we define

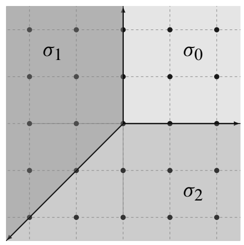
$$(y, z, Z/Y) \mapsto [z, 1, Z/Y];$$

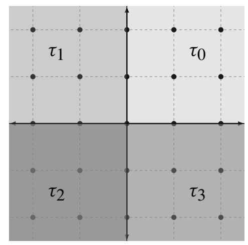
clearly these agree on the overlap $U' \cap U''$, and thus we obtain a morphism $U \to \mathbb{P}^2$. But away from the exceptional fiber $\widetilde{Q}$ is isomorphic to Q, and thus we actually have a morphism $\widetilde{Q} \to \mathbb{P}^2$. Moreover, by the above expression for the values of the morphism on the exceptional fiber we see that this is surjective.

Finally, consider the fibers of the map $\widetilde{Q} \to \mathbb{P}^2$. First, at points in the image of the map from Q itself, we have that one of two things happen: either the line through p and the point does not lie on Q, in which case it meets Q in exactly one other point, or the line lies on Q. Since we know there are exactly two lines on Q through p, we have two points over which the fiber is $\mathbb{P}^1$, and everywhere away from these points the map is an isomorphism. In fact, we saw above that these points are exactly the points $[0 : 0 : 1]$ and $[0 : 1 : 0]$, over which we have the lines $[0 : 0 : y_2 : y_3]$ and $[0 : y_1 : 0 : y_3]$. Now, considering points in the image of $\widetilde{Q}$ but not Q (the punctured affine line $[0 : y_1 : y_2]$), we see from the above expression that we have exactly one preimage, lying in the exceptional fiber.

Thus, we see that $\widetilde{Q} \to \mathbb{P}^2$ is an isomorphism away from two points, over which the fiber is a $\mathbb{P}^1$, and thus we see that $\widetilde{Q}$, the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point, is the blowup of $\mathbb{P}^2$ at two points (specifically, the points $[0 : 0 : 1]$ and $[0 : 1 : 0]$).

Remark. The above proof is trivial from the toric point of view. The fan for $\mathbb{P}^2$ is shown on the left below, and the fan for $\mathbb{P}^1 \times \mathbb{P}^1$ on the right:





Recall that refining the fan by subdividing a top-dimensional cone corresponds to taking the blowup at a point. Now, just note that we can obtain a common refinement of these two fans by:

- refining the fan for $\mathbb{P}^2$ by dividing σ_1 along the negative x-axis and σ_2 along the negative y-axis,
- refining the fan for $\mathbb{P}^1 \times \mathbb{P}^1$ by dividing τ_2 along the ray through $(-1, -1)$.

It's then immediate that the blowup of $\mathbb{P}^2$ at two points and the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point coincide.

5 del Pezzo surfaces

In this section we give a brief sketch of the topic of del Pezzo surfaces, which includes the above examples of both the blowup of $\mathbb{P}^2$ at two points and the smooth cubics (which are the blowups of $\mathbb{P}^2$ at six points). This section parallels the development given in Chapter 4 of [Bea96].

Let $1 \leq r \leq 6$, and let $p_1, \dots, p_r$ be r points in $\mathbb{P}^2$ in general position (i.e., with no six of them lying on a conic and no three of them on a line). Let L be the linear system of cubics through all the p_i; this is a subspace of the complete linear system $\mathcal{O}_{\mathbb{P}^2}(3)$ of cubics in $\mathbb{P}^2$.. Recall that this space is parametrized by $\mathbb{P}^9$; passing through each point is an independent linear condition on $\mathbb{P}^9$, and thus $\dim |L| = 9 - r$. Write $d = 9 - r$.

The linear system L defines a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^d$, with locus of indeterminancy at the r base points of the linear system. Let $P_r \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at these r points, with exceptional divisors $E_1, \dots, E_r$. Note that for each p_i, there is a cubic through p_i that is smooth at p_i (this follows by the generality hypothesis), and thus there is an element of the linear system with multiplicity 1 at each base point. Thus the linear system corresponding to the map $P_r \dashrightarrow \mathbb{P}^d$ is determined by the linear system $L' = \epsilon^* L - E_1 - \dots - E_r$ and thus has degree $L^2 - r = 9 - r = d$.

We claim that L' defines a closed embedding $P_r \hookrightarrow \mathbb{P}^d$. The image, which we'll denote S_d, is thus a smooth surface of degree d in $\mathbb{P}^d$. In particular, taking $r = 6$, so $d = 3$, we obtain another construction of smooth cubic surfaces as the blowup of $\mathbb{P}^2$ at 6 general points. Note, moreover, that since P_r is birational to $\mathbb{P}^2$ we see that each S_d is rational.

We omit the proof that L' defines a closed embedding; the key ingredient is the following lemma:

Lemma. *Let H be a linear system without fixed component on a surface X, and let X' be the blowup of X at the base points of H, and H' the corresponding linear system on X'. H defines a closed embedding $X' \hookrightarrow \mathbb{P}^r$ if and only if H separates points and separates tangent vectors, by which we mean the following:*

- *(separating points) for any two distinct points of X' there is an element of the linear system L' passing through one and not the other.*
- *(separating tangent vectors) for any point of X', the elements of L' through that point do not all have the same tangent direction.*

These conditions can be reinterpreted as conditions on the linear system L on X. To the first condition, we add that if $x \in E_i$, then there is an element of L passing through p_i with tangent direction corresponding to x. To the second condition, we add that if $x \in E_i$, and L_x is the subsystem of L passing through p_i with tangent direction corresponding to x, then for any conic Q through p_i with tangent x then there is an element of L_x with order of contact 2 with Q at p_i.

The proof that these conditions hold on P_r can be reduced immediately to showing it for $r = 6$ (since the other linear systems have more elements, so showing it for $r = 6$ shows it for all of them). Then one shows that the conditions hold, working with the linear system on $\mathbb{P}^2$, by using the fact that five general points determine a unique conic to construct a family of conics and lines, and the appropriate cubic products will separate points and tangent vectors.

Remark. The S_d are examples of *del Pezzo* surfaces. A del Pezzo surface is one whose *anticanonical divisor* $-K$ gives an embedding in projective space, i.e., those for which $-K$ is ample. Note that in our case, by the properties of blowups discussed in the previous section, we have the canonical divisor of $\mathbb{P}^2$ is $O(-3)$, and the canonical divisor of the blowup at r general points is $\epsilon^*(-O(3)) + E_1 + \cdots + E_r$. Thus the anticanonical divisor on P_r is $\epsilon^*(O(3)) - E_1 - \cdots - E_r$, which corresponds to the linear system obtained from the linear system on $\mathbb{P}^2$ of cubics through the r points $p_1, \ldots, p_r$. Thus we see that the anticanonical divisor indeed embeds P_r in projective space, so these are in fact del Pezzo surfaces. In fact, it can be shown that other than the S_d, $d = 3, \ldots, 8$, $\mathbb{P}^1 \times \mathbb{P}^1$ is the only surface embedded in projective space by its anticanonical divisor; this result depends on Castelnuovo's criterion for rationality. Thus, we have a complete list of the del Pezzo surfaces (at least over $\mathbb{C}$, or more generally any algebraically closed field).

Example (S_4)**.** As a final example, we consider the case $r = 5$, so $d = 4$ and we're considering a degree-4 surface in $\mathbb{P}^4$. We claim that this is always the intersection of two quadrics. To see this, it suffices to find two linearly independent quadrics containing S_4. If we can do so, note that the quadrics must be irreducible (if not, they decompose as lines, and S_4 cannot be contained in a line) Then we have that the intersection of these quadrics is an irreducible degree-4 codimension-2 surface containing S_4, and thus since S_4 is degree 4 and codimension 2 we have that S_4 is the intersection of these quadrics.

So, we need to show that there are two linearly independent degree-2 forms of $\mathbb{P}^4$ vanishing on S_4. We know that $h^0(\mathcal{O}_{\mathbb{P}^4}(2)) = 15$, so it suffices to show that $h^0(\mathcal{O}_{S_4}(2)) \leq 13$, since then two degree-2 forms vanish under the restriction map $h^0(\mathcal{O}_{\mathbb{P}^4}(2)) \to h^0(\mathcal{O}_{S_4}(2))$.

Let H be a hyperplane of $\mathbb{P}^4$ intersecting S_4 smoothly and irreducibly, thus in a curve C. Note that C corresponds to a smooth cubic polynomial in $\mathbb{P}^2$ via projection from $S_4 \to \mathbb{P}^2$ (since S_4 is the blowup of $\mathbb{P}^2$ at 5 points). Thus $g(C) = 1$. Consider the short exact sequence

$$0 \to \mathcal{O}_{S_4}(1) \to \mathcal{O}_{S_4}(2) \to \mathcal{O}_C(2) \to 0$$

obtained by twisting $0 \to \mathcal{O}_{S_4}(-1) \to \mathcal{O}_{S_4} \to \mathcal{O}_C \to 0$ by $\mathcal{O}_{S_4}(2)$. The long exact sequence in cohomology then gives that $h^0(\mathcal{O}_{S_4}(2)) \leq h^0(\mathcal{O}_{S_4}(1)) + h^0(\mathcal{O}_C(2))$. Clearly $h^0(\mathcal{O}_{S_4}(1)) \leq h^0(\mathcal{O}_{\mathbb{P}^4}(1)) = 5$.

Finally, we use Riemann–Roch for curves to compute $h^0(\mathcal{O}_C(2))$. We know that $g(C) = 1$ and thus $\chi(\mathcal{O}_C) = 0$. Riemann–Roch then gives

$$h^0(\mathcal{O}_C(2)) - h^1(\mathcal{O}_C(2)) = \deg \mathcal{O}_C(2) = 2;$$

note now that $\mathcal{O}_C(2) \cong \mathcal{O}_{C'}$, with C' a homogeneous quintic, so by the degree-genus formula $g(\mathcal{O}_C(2)) = g(C') = (5-1)(5-2)/2 = 6$. Thus we get $h^0(\mathcal{O}_C(2)) = 8$, and thus $h^0(\mathcal{O}_{S_4}(2)) \leq 13$, proving the result.

References

[Bea96] Arnaud Beauville. *Complex algebraic surfaces*, volume 34 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2nd edition, 1996. Translated from the 1978 French original.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. Wiley, 1994.

[Har77] Robin Hartshorne. *Algebraic geometry*. Number 52 in Graduate Texts in Mathematics. Springer, 1977.

[Mus17] Mircea Mustaţă. Cubic hypersurfaces I: Rationality of certain cubic hypersurfaces. `http://www-personal.umich.edu/~mmustata/lecture2_rationality.pdf`, 2017.

[Sha13] Igor Shafarevich. *Basic algebraic geometry I*. Springer, 3rd edition, 2013.