Harmony of the c-theorem and the a-theorem

Ann Arbor 2011

<u>Overture</u>: A typical scenario in a given QFT is that the correlation fns. approach those of a conformal thy, the CFT_{UV}, at short distance, and those of another conformal thy, the CFT_{IR}, at long distance. In $D=2$, ~~the~~ conformal theories are characterized by the central charges c_{UV} and c_{IR} of their Virasora algebras. Zamolodchikov used properties of the 2-pt fn. of the stress tensor to prove that $\boxed{c_{UV} > c_{IR}.}$ The intuition behind this general and fundamental result is that the effective number of degrees of freedom of the thy. decreases in the RG flow toward the infrared.

Zam's result motivated the search for a similar property in $D=4$ QFT. For several reasons, this proved to be a difficult problem. Although it did not meet its goal, the 25 year effort has taught us quite a bit of interesting physics. Very recently a new and interesting proof was proposed by

Komargodski + Schwimmer ArXiV 1107.3987

Typical scenario in QFT: the RG paradigm

Correlation fns:

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle_\Lambda \nearrow \langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle_{CFT_{UV}} \quad |x_{ij}| \ll 1/\Lambda$$

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle_\Lambda \searrow \langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle_{CFT_{IR}} \quad |x_{ij}| \gg 1/\Lambda$$

< Corrs in a CFT_D are invariant under $SO(D,2)$

P_μ, $M_{\mu\nu}$ of Poincaré_D + D, K_μ

CFT corr are simpler:

i) no scale

ii) power laws for 2-point + 3-point fns

$$\langle \mathcal{O}_\Delta(x_1)\, \mathcal{O}_\Delta(x_2) \rangle = \frac{c}{(x_1 - x_2)^{2\Delta}}$$

$$\langle \mathcal{O}_{\Delta_1}(x_1)\, \mathcal{O}_{\Delta_2}(x_2)\, \mathcal{O}_{\Delta_3}(x_3) \rangle = \frac{c_{123}}{(x_{12})^{\nu_{12}} (x_{23})^{\nu_{23}} (x_{31})^{\nu_{31}}}$$

iii) fewer variables for $n \geq 4$

cross ratios $\dfrac{|x_{ij}|\,|x_{kl}|}{|x_{il}|\,|x_{jk}|}$

A 4-pt corr in QFT_4 depends on 6 variables

in CFT_4 " " 2 cross ratios

Critical phenomena in planar condensed matter: they are nicely described by CFT_2's

RG paradigm also useful in QFT_4:

In (Euclidean) QFT_2 $\quad z = x + iy$

$$T_{\mu\nu} \longrightarrow T_{zz}, \quad T_{\bar{z}\bar{z}}, \quad T_{z\bar{z}} = g^{\mu\nu} T_{\mu\nu} \equiv \Theta(x)$$

The operator $\Theta(x)$ describes breaking of scale inv.

$$\Theta(x) = 0 \quad \text{in } CFT_D$$

A CFT_2 is characterized by its <u>central charge</u> c

It appears in several ways:

1) power law $\langle T_{zz}(z)\, T_{zz}(0)\rangle = \dfrac{c}{2z^4}$

2) embed the CFT in curved bkgd metric $g_{\mu\nu}(x)$

then we have trace anomaly:

$$\langle \Theta(x)\rangle = \langle g^{\mu\nu} T_{\mu\nu}\rangle = \frac{-c}{12} R$$

Ricci scalar

Zamolodchikov 1986 (3 pages!) proved

$c_{UV} > c_{IR}$ for any QFT_2 that satisfies

i) Euclidean invariance

ii) $T_{\mu\nu}$ is well defined and conserved

iii) unitarity

Zam. constructed a <u>c-function</u> with properties

i) $c(\Lambda) \geq 0$ + ve

ii) $\Lambda \frac{\partial}{\partial\Lambda} c(\Lambda) \geq 0$ ← monotone decreasing toward IR

iii) $c(\Lambda) \xrightarrow[\Lambda\to\infty]{} c_{UV}$, $c(\Lambda) \xrightarrow[\Lambda\to 0]{} c_{IR}$ approaches central charges at end-pts of flow

iv) if $\Lambda \frac{\partial}{\partial\Lambda} c(\Lambda) = 0$, then $\Theta(x) = 0$ ← stationary at CFT's.

Implications: a) RG flow is irreversible

b) agrees with intuition that # of active d.f of a QFT decreases as $E \to 0$.

c) Wilsonian renormalization → loops of particles of mass m have no effect on couplings $g(\Lambda)$ for $\Lambda < m$

d) Basic problem of D=4 QFT. What is long distance realization of an asympt. free gauge thy?

If there is a c-thm for $d=4$, the rigorous result $c_{IR} < c_{UV}$ could help determine the strong coupling behavior in the IR.

Proof of Zam. thm — Polchinski Sec 15.8

Euclid. inv. in any QFT_2 $\Rightarrow$ the 3 2pt fns of T_{zz}, Θ depend only on $r^2 = z\bar{z}$:

$$F(\Lambda r) = z^4 \langle T_{zz}(x)\, T_{zz}(0)\rangle_\Lambda$$

$$G(\Lambda r) = 4 z^3 \bar{z} \langle T_{zz}(x)\, \Theta(0)\rangle_\Lambda$$

$$H(\Lambda r) = 16 z^2 \bar{z}^2 \langle \Theta(x)\, \Theta(0)\rangle_\Lambda$$

Conservation law $\partial^\mu T_{\mu\nu} = 0 \Rightarrow$ the lin combination

$$C(\Lambda r) = 2F - G - \tfrac{3}{8} H$$ satisfies

$$\frac{d}{d\ln r^2} C = -\frac{3}{4} H \sim -\langle \Theta(\tfrac{x}{2})\, \Theta(-\tfrac{x}{2})\rangle \le 0$$

By reflection +ivity in Eucl QFT_2 $\Longleftrightarrow$ unitarity in Lorentzian.

Important "substructure" $S^{eff}_{QFT_{UV}} = S_{CFT_{UV}} + \sum_i g^i \Phi_i(x)$

where $\Phi_i(x)$ is a relevant op of scale dim $\Delta_i \leq 2$

Then $\Theta(x) = \sum_i \beta^i(g^j) \Phi_i(x)$

RG flow is gradient flow in the space of couplings:

$$\Lambda \frac{\partial}{\partial \Lambda} g^i(\Lambda) = \beta^i = G^{ij} \partial_j c(g)$$

G_{ij} = Zam. metric

$\Rightarrow$ couplings $g_i(\Lambda)$ flow ~~most~~ to fixed pt values g_i^* in IR.

Rules out limit cycles and other exotic behavior in IR.

This xtra structure does not yet play a role in new developments in $D=4$. Deemphasize.

Friedan 1990 has recast the c-thm using Lehmann rep of $\langle \Theta(x)\Theta(0)\rangle = FT \int_0^\infty d\mu\, \rho(\mu) \frac{1}{p^2+\mu^2}$

$\rho(\mu)$ = "density of states" $|\langle 0|\Theta(x)|p\rangle|^2_{p^2=\mu^2}$

The flow of c is given by

$$c_{UV} - c_{IR} = \int_0^\infty d\mu\, \rho(\mu) = \int d^2x\, x^2 \langle \Theta(x)\Theta(0)\rangle$$

Cardy sum rule

Integrals converge as $\mu \to \infty, 0 \sim x \to 0, \infty$ because

$$\langle \Theta(x)\, \Theta(0)\rangle = \sum_i (\beta^i)^2 \langle \Phi_i(x)\, \Phi_i(0)\rangle$$

$\xrightarrow[x\to 0]{}$ $1/x^{2\Delta_i}$ Δ is dim. of least relevant op

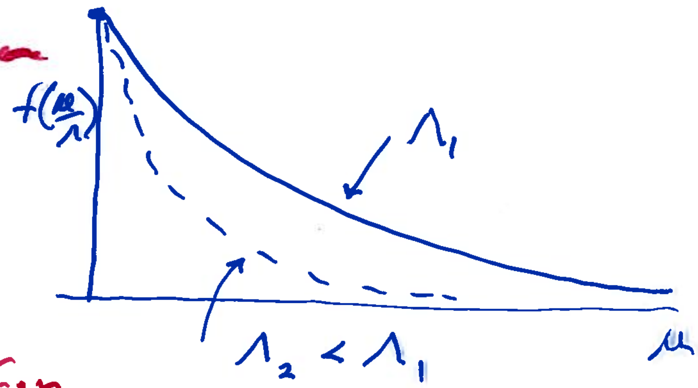
$\xrightarrow[x\to\infty]{}$ $1/x^{2\Delta_j}$ Δ " " " least irrelevant op.

Convergence was studied by Cappelli, Friedan, Latorre 1991

We will ask a similar question about K-S formula for the flow $a_{UV} - a_{IR}$ for $D=4$.

There are many ways to define an interpolating c-function

Given any $f(u)$ which satisfies $f(0)=1$, $f(u)>0$ $u>0$, $f'(u)<0$, $f \sim e^{-ku}$ $u \to \infty$



Then

$$C(\Lambda) \equiv \int_0^\infty d\mu\, \rho(\mu)\, f(\mu/\Lambda) + c_{IR}$$

$$\xrightarrow[\Lambda\to\infty]{} c_{UV}$$

$$\xrightarrow[\Lambda\to 0]{} c_{IR}$$

Hence a good c-function

$$\Lambda \frac{\partial}{\partial \Lambda} C(\Lambda) > 0$$

This ends the $D=2$ part of this talk. I have tried to raise issues relevant to new work in $D=4$.

II. D=4 Cardy 1988 (3.5 pages)

1. Zam's argument based on $\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle$ does not work bc. the decomp. formula based on Lorentz inv. requires 5 fns of r^2 rather than 3. $\nexists$ any lin comb. with mon~~oton~~ic behavior.

2. The curved space trace anom. in a CFT_4 has two contribs.

$$\langle \Theta(x)\rangle = \langle T^{\mu}_{\mu}\rangle = \frac{1}{16\pi^2}\left[c\, W^2 - a\, E_4\right]$$

(Weyl tensor)^2 $\quad W^2 = (R_{\mu\nu\rho\sigma})^2 - 2(R_{\mu\nu})^2 + \frac{1}{3}R^2$

Euler density $\quad E_4 = (R_{\mu\nu\rho\sigma})^2 - 4(R_{\mu\nu})^2 + R^2$

CFT_4's characterized by values of two central charges c, a.

In any CFT_4, c is determined by 2-pt fn (at $x \neq 0$)

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle = -\frac{c}{48\pi^4}\, \Pi_{\mu\nu\rho\sigma}\frac{1}{x^4}$$

$$\Pi_{\mu\nu\rho\sigma} = \partial_\mu \partial_\nu \partial_\rho \partial_\sigma + \cdots$$ transverse traceless projection op.

a is determined by 3 pt fn $\langle T_{..}(x)\, T_{..}(y)\, T_{..}(z)\rangle$ also at $x \neq y \neq z$ Osborn

3. Free field values of of c, a have been known for many years. 1974 Capper + Duff dim. reg. also Pauli-Villars

Values are indep of regularization scheme. Vilenkin

spin	0	1/2 (Weyl)	1
$360c$	3	9	36
$360a$	1	11/2	62

These free field can be used to calc. c, a in interacting thys provided the dynamics becomes free in appropriate limit.

4. Cardy effectively conjectures that the Euler central charge a satisfies c-thm in $D=4$. No mention of c.

$$a_{UV} - a_{IR} > 0$$

No proof but he tests the conjecture in models:

Simplest model QCD with gauge gp $SU(N_c)$ with

N_f flavors of massless quarks in fundamental

$\mathcal{L}$ has $SU(N_f)_L \times SU(N_f)|_R$ chiral sym. (non-anomalous)

a. Thy is asym. free if $N_f < \frac{11}{2} N_c$

So free field results can be used to calc a_{UV} (or c_{UV})

b. Assume that IR thy has spont. breaking of chiral sym with N_f^2-1 Goldstone bosons + massive gauge inv. states. Since Golds. bosons are weakly coupled at low energy, we can use free field values to calc a_{IR}, c_{IR}

Result: $360\, a_{UV} = 62(N_c^2-1) + 11 N_c N_f$

$360\, a_{IR} = N_f^2 - 1$

$\Longrightarrow \quad a_{UV} - a_{IR} > 0 \quad \forall\, N_f < \frac{11}{2} N_c$ ← region of asymp. freedom where UV calc. is justified

In this model $c_{UV} - c_{IR} > 0$, also, so one does not discriminate between a, c.

$\exists$ many other models with $c_{UV} - c_{IR} < 0$ (violate c-thm):

Simplest in same model with $N_f = \frac{11}{2} N_c - k$ with large N_c and k small: barely asymp. free

The β fn $\beta(\alpha)$ has a close 2-loop IR fixed pt

$$\alpha = g^2/4\pi$$

Pert thy is valid through the flow.

CFL calculate $c_{UV} - c_{IR} = -\frac{2k}{N} < 0$

III. Anomaly flows in $\mathcal{N}=1$ SUSY gauge thys.

Two features make Δa and Δc calculable:

1. Seiberg ($\gtrsim 1994$) dynamics tells us what IR realization is.

Conformal window of $\mathcal{N}=1$ $SU(N_c)$ gauge thy with $N_c^2 - 1$ gauge mults and $2N_cN_f$ chiral mults Q, $\tilde{Q}$

Cond for asymp. freedom $N_f < 3N_c$

If $\frac{3}{2}N_c < N_f < 3N_c$ thy has a strongly interacting $SCFT_4$ in the IR. (dual descrip as IR free magnetic gauge thy will not be relevant for us.)

2. The supercomf group $SU(2,2|\mathcal{N}=1)$ contains a conserved axial $U(1)_R$ current R_μ in same supermult as stress tensor $T_{\mu\nu}$. $\Rightarrow$ If one can calculate the chiral anomaly $\langle \partial^\mu R_\mu \rangle$, then one also learns what $\langle T_\mu{}^\mu \rangle$ is. Hence one learns c, a.

The values a_{UV}, c_{UV} can be found from free field content bc of asympt freedom

In two papers Anselmi et al 9708042 and 9711033 used 't Hooft anomaly matching + subtle args to calc a_{IR}, c_{IR} in MANY models:

Result $\Delta a > 0$ in all (renorm) models

but Δc can have either sign ($\Rightarrow$ no universal c-thm.)

Example in model on prev. slide

$$a_{UV} - a_{IR} = \frac{1}{48} N_f N_c \left(1 - \frac{3N_c}{N_f}\right)^2 \left(2 + \frac{3N_c}{N_f}\right) > 0$$

$$c_{UV} - c_{IR} = \frac{-1}{48} N_f N_c \left(1 - \frac{3N_c}{N_f}\right)\left(\frac{3N_c}{N_f} + \frac{9N_c^2}{N_f^2} - 4\right) \begin{matrix} >0 \\ \text{and} \\ <0 \end{matrix}$$

Conclusion: strong evidence for universal a thm but no universal c-thm

Important fact 1. The words "a-theorem" were used for the first time in 9711033.

Imp fact 2: a-maximization Intriligator + Wecht 0304128

V. The Holographic c-theorem:

Certain RG flows in D=4, such as relevant perturbations of $\mathcal{N}=$ SYM have gravity duals described by AdS-CFT correspondence.

AdS-CFT relates the trace anomaly to the curvature scale of AdS_5 by $a = c = \frac{\pi}{8G_5} L^3$

Henningson & Skenderis 9806087

The gravity duals of RG flows are domain wall metrics in D=5: $ds^2 = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + dr^2$

It is easy to derive from the Einstein eqns that

$C(r) = \frac{\pi}{8G_5}\left(\frac{1}{A'(r)}\right)^3$ is a perfect c-function which interpolates monotonically between the central charges and shows that

$$a_{UV} - a_{IR} = C(+\infty) - C(-\infty) \geq 0$$

Girardello, Petrini, Porrati, Zaffaroni 9810126

DZF, Gubser, Pilch, Warner 9904017

The Bible Sec 23.12