the Holographic S-Matrix

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1107.1499 - Fitzpatrick, JK, Penedones, Raju, Van Rees
1111.xxxx - Fitzpatrick, JK (Analyticity, Unitarity)

(see also 1107.1504 by Miguel Paulos)

Motivations

- Give a Boundary = Holographic **theory** of flat spacetime and the S-Matrix (the only observable), defining it non-perturbatively
- think of Hawking evaporation as a scattering process, and compute it holographically
- (also: Recast CFT to make physics transparent and greatly simplify AdS/CFT computations)

What's left to understand about Black Holes?

- Large (R_s > R_AdS) BHs in AdS ~ a Hot CFT, but...
- Small (R_s << R_AdS) BHs evaporate, leading to

$$\langle n_{\rm out}\rangle \approx \left(\frac{s}{M_{pl}^2}\right)^{\frac{D-2}{2}} \qquad \langle E_{\rm out}\rangle \approx M_{pl}\left(\frac{M_{pl}^2}{s}\right)^{\frac{D-3}{2}}$$

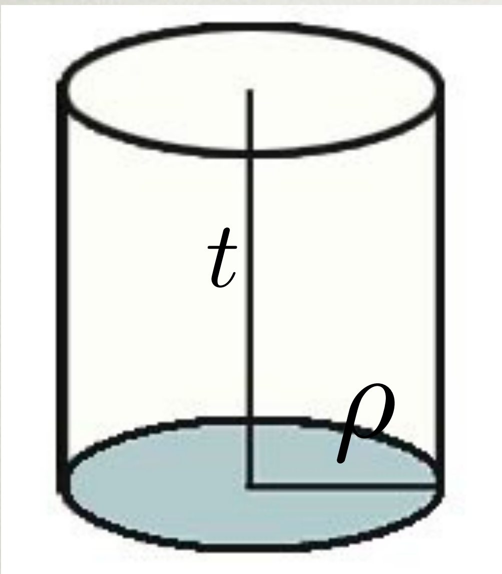
Only gravity has scattering amplitudes like this; reproducing it with AdS/CFT is a sharp question that should have a generic solution!

Planck scale should emerge as a dimension in the CFT.

Outline

- Mellin Space as `Momentum Space' for CFTs, or how to think of CFT Correlators as Scattering Amplitudes
- Mellin Amplitude as Holographic S-Matrix
- Analyticity (locality!?) from Meromorphy, some loop level examples
- Unitarity as a consequence of the OPE
- S-Matrix program as the Bootstrap program, and a peak at black holes

AdS/CFT Preliminary



With AdS in Global Coordinates

$$ds^2 = \frac{1}{\cos^2 \rho}(dt^2 + d\rho^2 + \sin^2 \rho d\Omega^2)$$

the **Dilatation Operator generates time translations.**

Bulk Effective Field Theory in **Energy** corresponds to
``Effective Conformal Theory" in **Dimension.**
One can use a cutoff on Energy/Dimension.

Let's Try to think of CFT Correlators as Scattering Amplitudes.

CFT Analog of "Free Particles"?

Scattering amplitudes involve states composed of particles that are asymptotically free.

The CFT analog is the large N expansion, because given operators $\mathcal{O}_1$ and $\mathcal{O}_2$, there must exist

$$\text{“}\mathcal{O}_1\mathcal{O}_2\text{”}$$

with dimension $\approx \Delta_1 + \Delta_2$

How should we compute Correlators?

Previous computations in AdS used position space.
Analogous to computing Feynman diagrams as...

$$\int d^d x D_F(x_1 - x) D_F(x_2 - x) D_F(x_3 - x) D_F(x_4 - x)$$

Even the 4-pt function is a box integral!!

In AdS, computations have been even worse,
with very few results beyond 4-pt.

(We will see how to compute at n-pt, easily.)

The Advantages of Momentum Space

In flat space we go to momentum space, which has several familiar advantages.

Eq. of Motion become **algebraic**

$$\nabla^2 = -p^2$$

because the Laplacian acts very simply on the momentum space representation.

We find a similar simplification in **Mellin space**, because the **Conformal Casimir** acts nicely.

Factorization and Momentum Space

Also, flat space scattering amplitudes **Factorize**

$$M(p_i) \to M_L(p_{i_L}, P_L)\frac{1}{P_L^2}M_R(-p_L, p_{i_R})$$

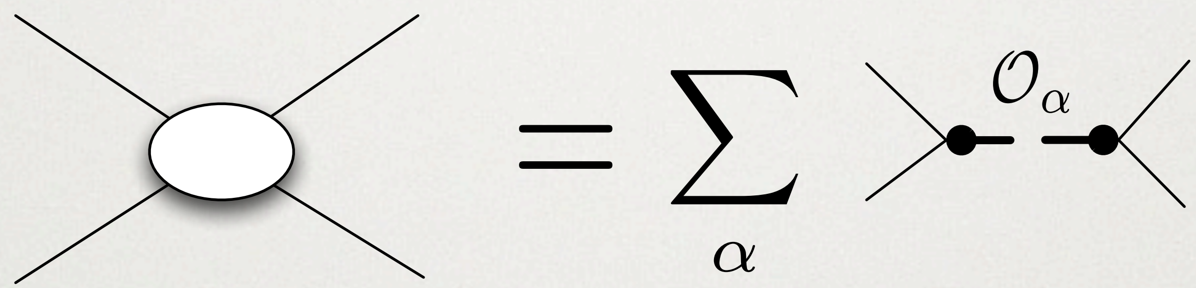
Involves **analyticity** and **unitarity**,
since factorization poles follow from the exchange of single-particle states.

Also, there are **purely algebraic Feynman Rules**.

So position space obscures a lot of physics!

The CFT Analog of Factorization

Factorization also occurs in CFTs, but this is obscure in position space.



By the operator-state correspondence, the OPE decomposition is just a sum over intermediate states:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \left(\sum_\alpha |\alpha\rangle\langle\alpha| \right) \mathcal{O}_3 \mathcal{O}_4 \rangle$$

Mellin space will display this as a sum over factorization channels.

So what is the Mellin Amplitude?

A CFT Correlator written in Mellin Space (Mack):

$$A_n(x_i) = \int [d\delta] M_n(\delta_{ij}) \prod_{i<j}^{n} (x_i - x_j)^{-2\delta_{ij}} \Gamma(\delta_{ij})$$

$$\sum_{j \neq i} \delta_{ij} = \Delta_i$$

Roughly speaking, the δ_{ij} variables are a space of relative scaling dimensions between operators.

The Mellin Amplitude for scalar operators is **Conformally Invariant**.

Mellin Space ~ Space of Mandelstam Invariants

δ_{ij} are symmetric, and with $\delta_{ii} = 0$

You can **always** think of $\delta_{ij} =$ "$p_i \cdot p_j$" with

$$\sum_{i=1}^{n} p_i = 0 \quad \text{and} \quad p_i^2 = \Delta_i$$

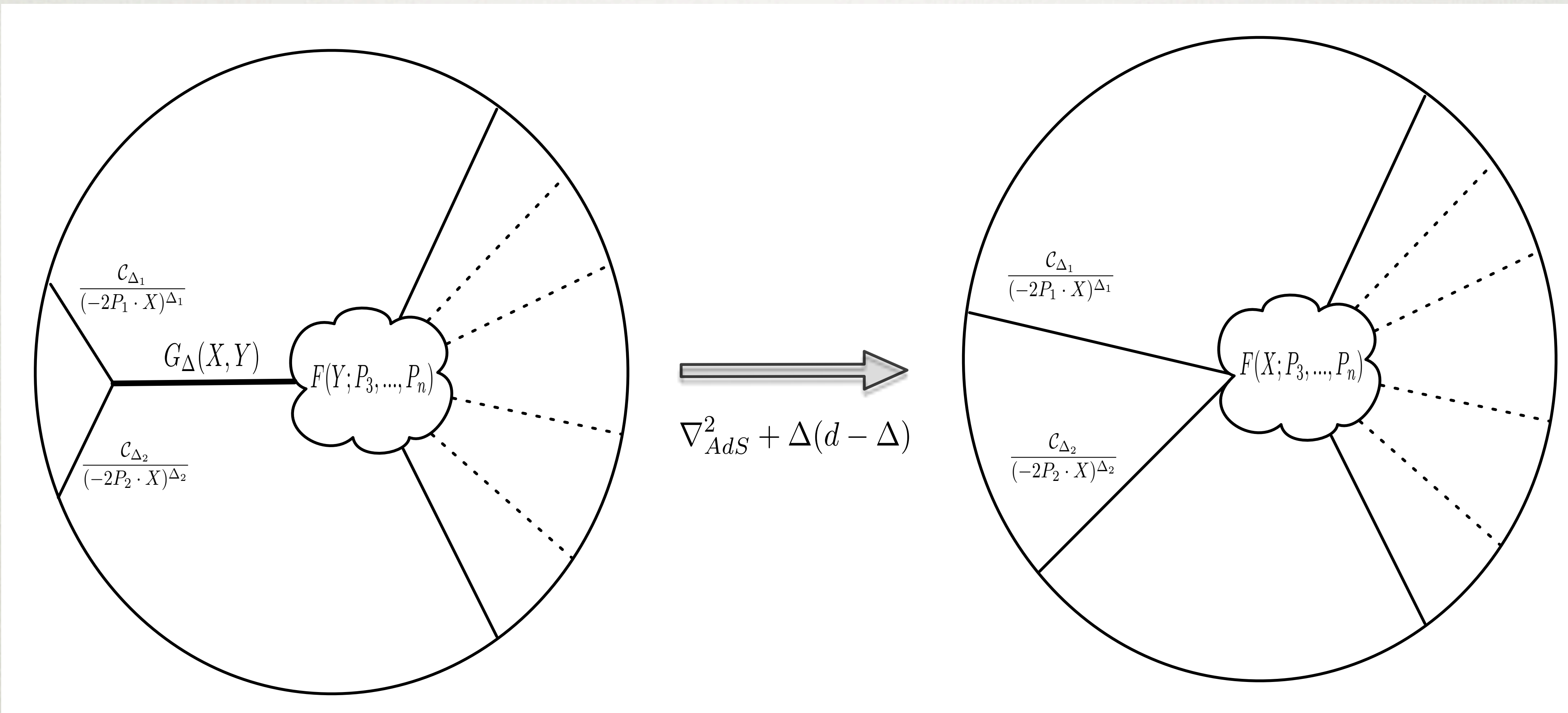
(fake) momentum conservation and on-shell conditions

We will often see combinations in propagators such as

$$\sum_{i,j=1}^{K} \delta_{ij} = (p_1 + \ldots + p_K)^2$$

How does the Mellin Amplitude mimic Scattering Amplitudes?

In Mellin Space: The Functional Equation



Find a finite difference equation for Mellin amp:

$$(\delta_{12}-a_1)(\delta_{12}-a_2)M(\delta_{12})=(\delta_{12}-a_3)(\delta_{12}-a_4)M(\delta_{12}-1)-M_0$$

(similar technique key physical point in "How to succeed at z-integrals without really trying")

OPE Factorization

The Operator Product Expansion lets us factorize:

$$A_n(x_i) \sim \sum_p \int d^d y \left\langle \prod_{i=1}^{k} \mathcal{O}_i\,(x_i)\, \mathcal{O}_p(y) \right\rangle \left\langle \widetilde{\mathcal{O}}_p(y) \prod_{i=1+k}^{n} \mathcal{O}_i\,(x_i) \right\rangle$$

We want to use variables where there is a **pole** here, with a **residue** that is the product of lower correlators.

Each $\mathcal{O}_p$ in the sum has a definite dimension, so each term scales as a **definite power law**.

Mellin space = the space of these powers.

OPE Factorization Formula for AdS/CFT

An explicit AdS/CFT factorization formula:

$$M = \sum_{m=0}^{\infty} \frac{Res(m)}{\delta_{LR} - \Delta - 2m}$$

$$Res(m) \propto \left[L_m(\delta_{ij}) R_m(\delta_{ij})\right]_{\delta_{LR}=\Delta+2m}$$

where

$$\delta_{LR} = \sum_{i,j \leq k} \delta_{ij} = \text{“}(p_1 + ... + p_k)^2\text{”}$$

Mellin Amplitudes Are Meromorphic

In general, expect Mellin amplitudes must always be meromorphic functions to get an OPE.

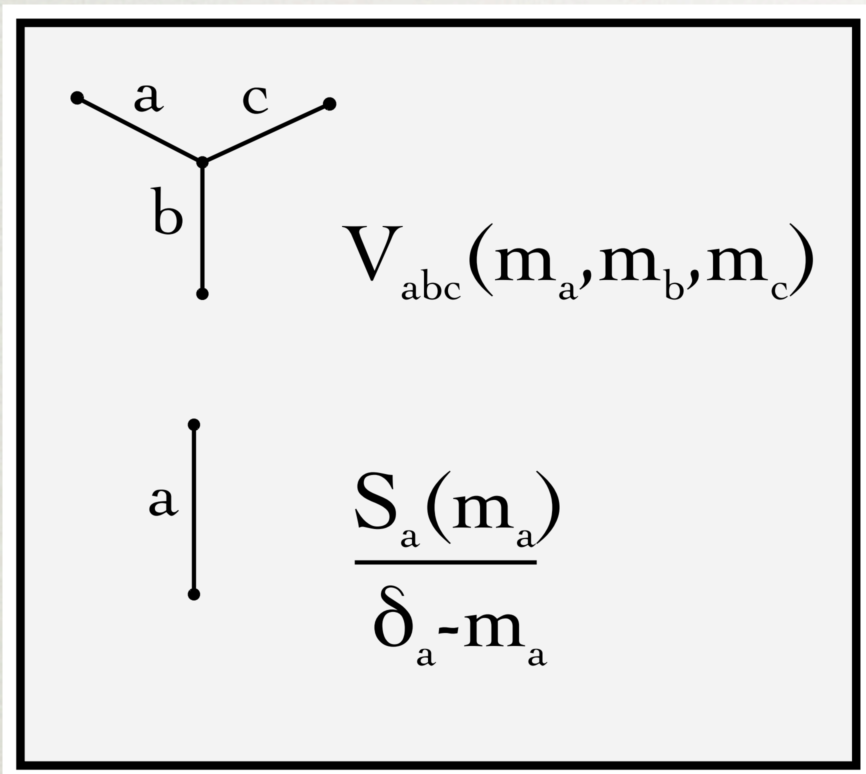
In fact, expect only simple poles, and that all poles will lie on the real axis for a unitarity CFT.

Provides a hint of analyticity for later...

Diagrammatic Rules?

We have a factorization formula, and we can factorize on **any propagator**, and reason to believe that Mellin amplitudes are basically just rational functions, so it would be surprising if there wasn't a constructive method for generating Mellin Amps.

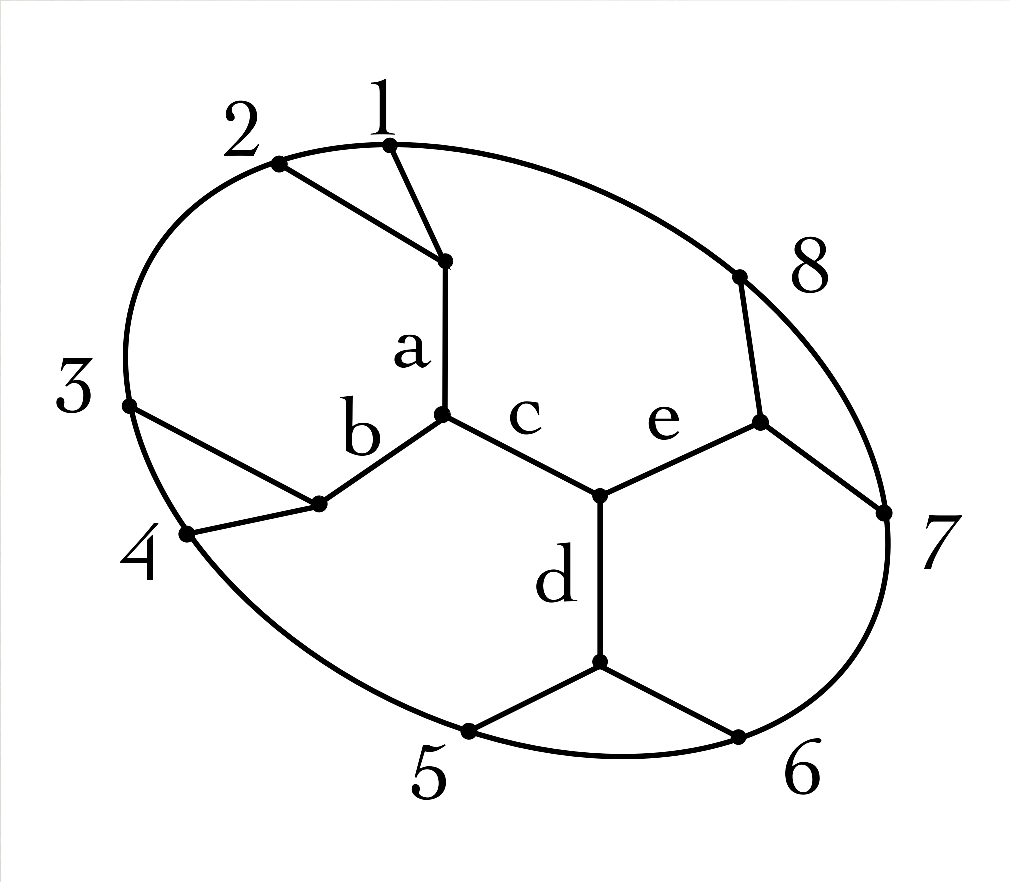
Diagrammatic Rules



Conserve fictitious ``momentum'' at all vertices.

Propagators and vertices determined and proven via the finite difference equation (very nice forms found by Paulos, 1107.1504).

So We Can Compute!



AdS/CFT Witten Diagrams such as this can be computed straightforwardly.

Previously, very few computations beyond 4-pt!!

Relation to Flat Space S-Matrix?

the Flat Space Limit

- Recall Bulk Energy = CFT Dimension
- Flat Space Limit requires

$$E_{\text{bulk}} R_{\text{AdS}} \to \infty$$

- This means that we must study CFT states of very **large dimension**, while

$$N^2 \propto (M_{d+1} R_{\text{AdS}})^{d-1} \to \infty$$

The Flat Space Limit

But we know that δ_{ij} ~ dimension.

Natural to guess (and Penedones did) that

$$\lim_{R\to\infty} M(\delta_{ij} = R^2 s_{ij}) \sim T(s_{ij})$$

And it works! Checked explicitly for theories of scalars at tree level for any number of particles, and some 1-loop examples. More precisely...

THE FLAT SPACE LIMIT

The exact relation for massless external states:

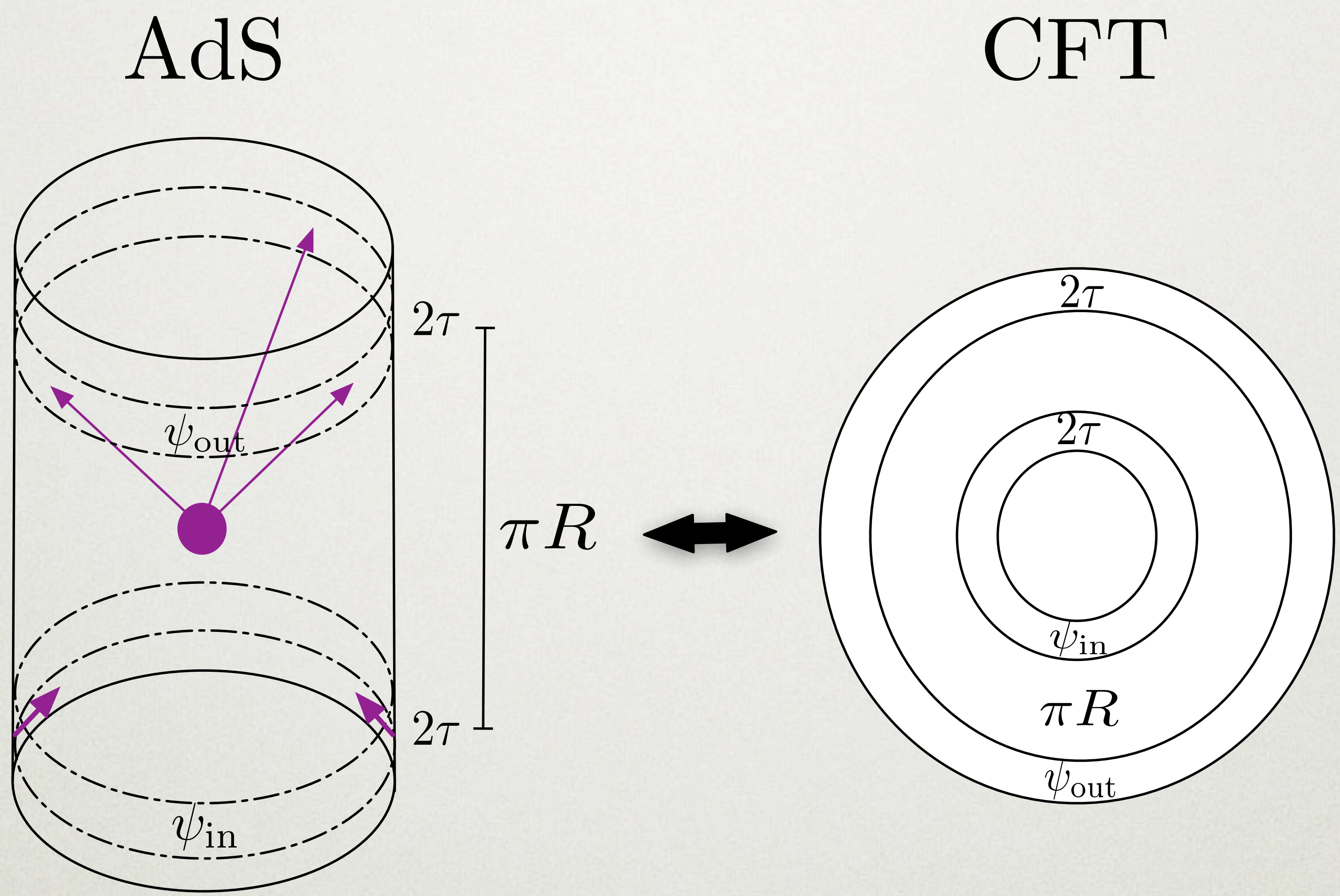
$$T(s_{ij}) = \Gamma\left(\Delta_\Sigma - h\right) \lim_{R\to\infty} \int_{-i\infty}^{i\infty} d\alpha\, e^{\alpha} \alpha^{h-\Delta_\Sigma} M\left(\delta_{ij} = \frac{R^2 s_{ij}}{2\alpha}, \Delta_a = R m_a\right)$$

A one-dimensional contour integral applied to the (meromorphic) Mellin Amplitude.

Note that as one might expect, single trace <--> single particle.

Now let's derive it...

Deriving the Flat Space Limit



Create in and out states by CFT operator smearing:

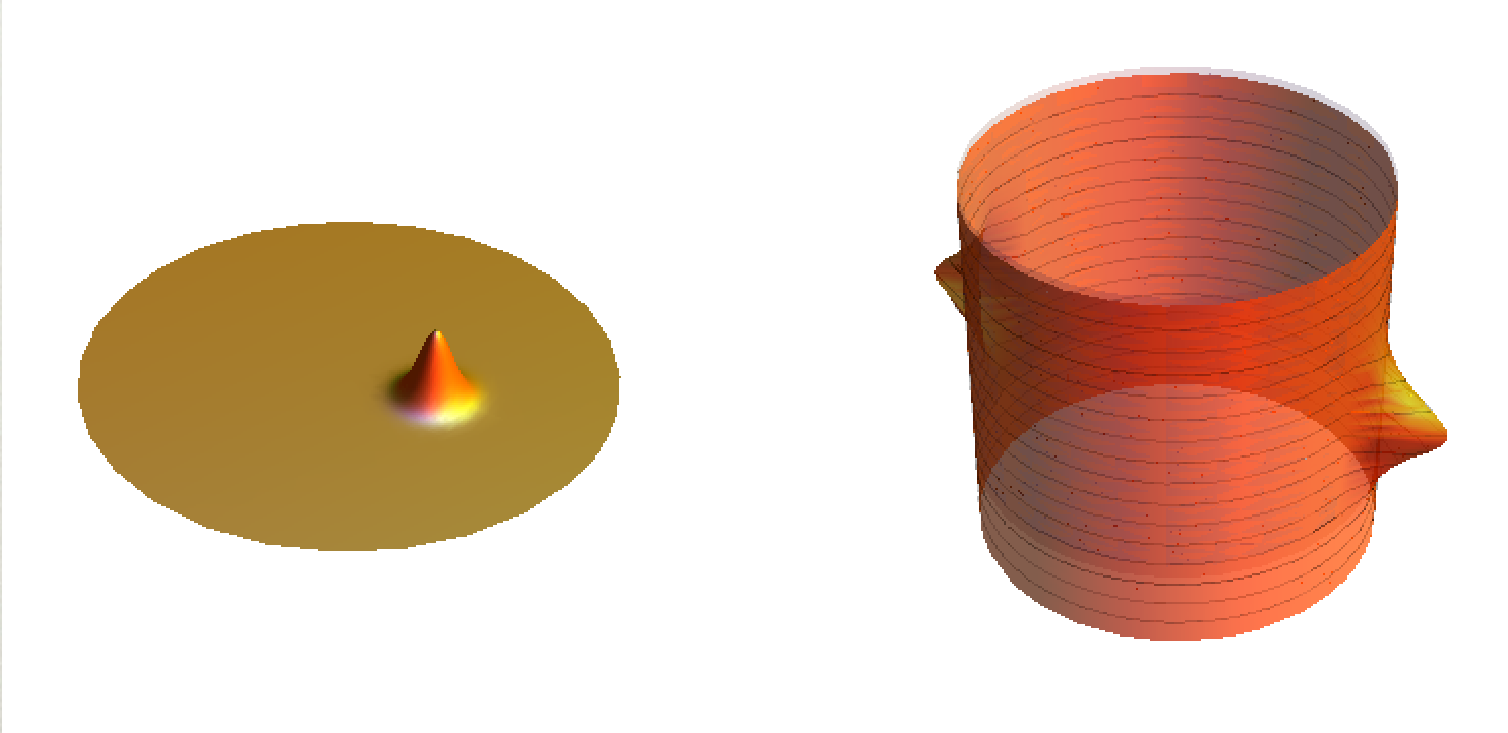
$$|\omega, \hat{v}\rangle = \int_{-\frac{\pi R}{2}-\tau}^{-\frac{\pi R}{2}+\tau} dt e^{i\omega t}\mathcal{O}(t, -\hat{v})|0\rangle$$

Single-trace Operator = Single Particle

Deriving the Flat Space Limit

Point-source at the boundary = plane wave in the center of AdS, energy set by frequency:

$$|\omega, \hat{v}\rangle = \int_{-\frac{\pi R}{2}-\tau}^{-\frac{\pi R}{2}+\tau} dt e^{i\omega t}\mathcal{O}(t, -\hat{v})|0\rangle$$



(an example of a wave packet state)

Deriving the Flat Space Limit

Integrating CFT Correlator against plane waves:

$$T(s_{ij}) = \lim_{\frac{R}{\tau},\tau\to\infty} \int [d\delta] \int_{-\tau\pm\frac{\pi R}{2}}^{\tau\pm\frac{\pi R}{2}} dt_i e^{i(\omega_i-\Delta_i)t_i} M(\delta_{ij}) \prod_{i<j} \left(\cos\left(\frac{t_i - t_j}{R}\right) - \hat{p}_i \cdot \hat{p}_j \right)^{-\delta_{ij}} \Gamma(\delta_{ij})$$

Time differences small: $|t_i - t_j| \ll R$

leading to approximately Gaussian time integrals.

δ_{ij} integrals can be evaluated via stationary phase in the flat space limit of Gamma functions:

$$\int [d\epsilon] M(\delta_{ij}) \exp\left[\sum_{ij} R^2 s_{ij} \left(\frac{1}{\alpha} + \epsilon_{ij}\right) \log\left[R^2 \left(\frac{1}{\alpha} + \epsilon_{ij}\right)\right]\right]$$

Mellin Diagrams to Feynman Diagrams

δ_{ij} variables align with s_{ij}, leaving us with:

$$T(s_{ij}) = \Gamma\left(\Delta_\Sigma - h\right) \lim_{R\to\infty} \int_{-i\infty}^{i\infty} d\alpha\, e^{\alpha} \alpha^{h-\Delta_\Sigma} M\left(\delta_{ij} = \frac{R^2 s_{ij}}{2\alpha}, \Delta_a = R m_a\right)$$

$i\epsilon$ prescription comes from CFT prescription.

We showed that our factorization formula for the Mellin amplitude reduces to factorization of the tree-level scattering amplitudes, and that our Feynman rules reduce to the flat space rules.

Analyticity and the Holographic S-Matrix

Analyticity in the Flat Space Limit

$$T(s_{ij}) = \Gamma\left(\Delta_\Sigma - h\right) \lim_{R\to\infty} \int_{-i\infty}^{i\infty} d\alpha\, e^{\alpha} \alpha^{h-\Delta_\Sigma} M\left(\delta_{ij} = \frac{R^2 s_{ij}}{2\alpha}, \Delta_a = R m_a\right)$$

For finite R, just contour integral of meromorphic function, so obviously analytic.

Flat Space Limit just expands near infinity.
We get branch cuts and imaginary parts from the coalescence of poles.

Locality = Analyticity?

Only precise notion of locality (I'm aware of) is via analyticity and boundedness of S-Matrix.

The Scattering Amplitudes are given by a simple integral transform of the Mellin Amp.

The Mellin Amplitude is a meromorphic function with only simple poles, in any CFT.

Is this how we should think of locality emerging from a CFT!?

Flat Space Limit of a Bulk Exchange

In the flat space limit, a bulk propagator is simply:

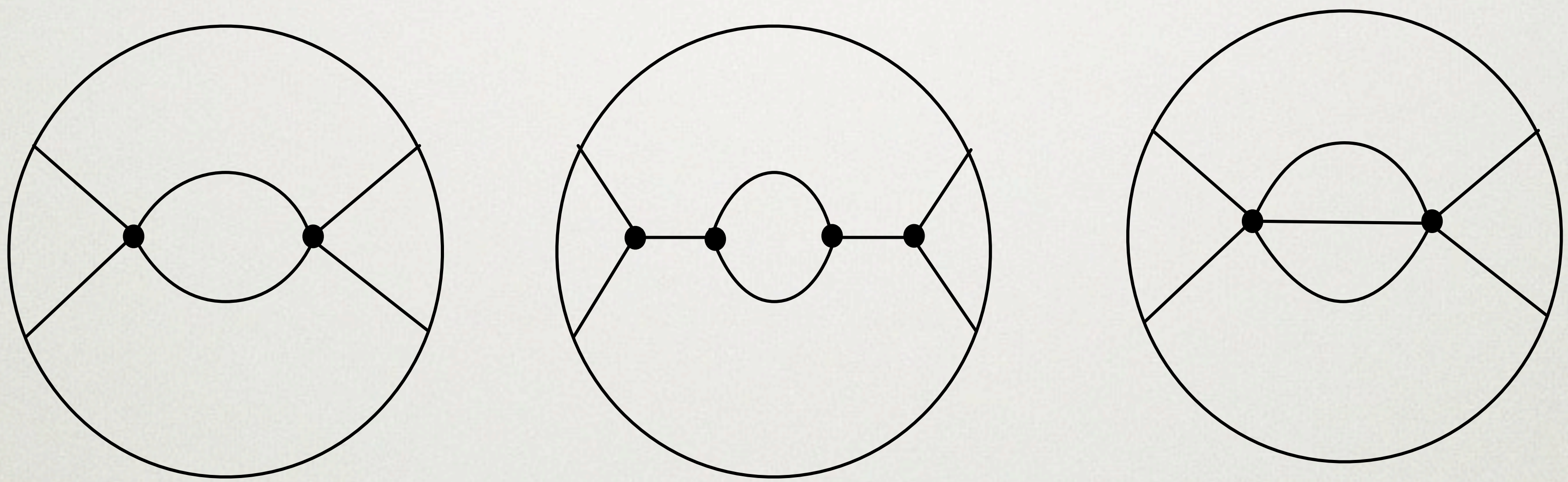
$$\sum_m \frac{R(\Delta, m)}{\delta - (\Delta + m)} \to \frac{1}{s + \Delta^2}$$

The Mellin amplitude is dominated
by poles where $m \approx \Delta^2$,
when we take the flat space limit.

Loops?

Computing Loop Diagrams

We can also compute AdS loop diagrams

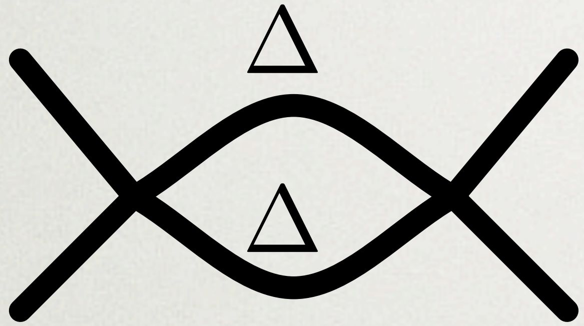


$\lambda\phi^4$ $\mu\phi^2\chi$ $g\phi^5$

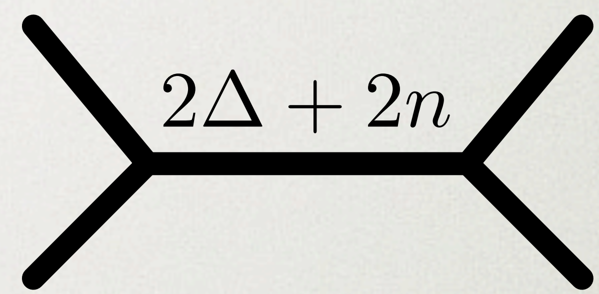
Using an AdS version of Kallen-Lehman, which makes it possible to write 2-point functions of local operators as a positive integral over free propagators.

1-Loop Computations a la Kallen-Lehman

At 1-loop, can write bubble diagram using:



$$= \sum_{n=0}^{\infty} N_\Delta(n)$$



or

$$G_\Delta(X,Y)^2 = \sum_{n=0}^{\infty} N_\Delta(n) G_{2\Delta+2n}(X,Y)$$

This gives a **Kallen-Lehman**-esq Mellin Amplitude:

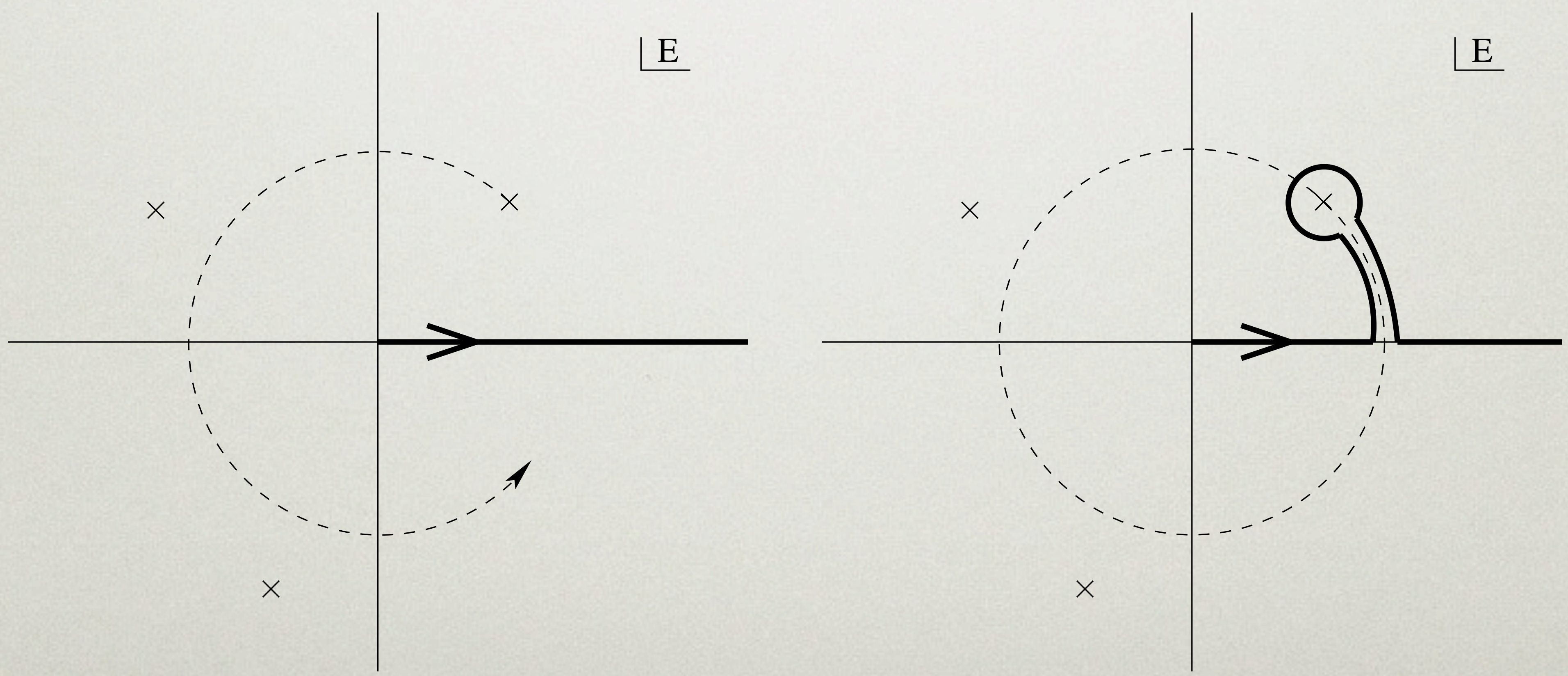
$$M(\delta) = \sum_n N(n) \sum_m \frac{R(2\Delta+2n,m)}{\delta-(2\Delta+2n+m)}$$

Branch Cuts

In the flat space limit, we find the integral:

$$M(\delta) \to \int_0^\infty dn \frac{N(n)}{s + (2\Delta + 2n)^2}$$

Circling in the complex plane gives a branch cut.



Branch Cuts from Mellin Amplitudes

$$M(\delta) \to \int_0^\infty dn \frac{N(n)}{s + (2\Delta + 2n)^2} \quad \text{with} \quad N(n) \propto n^{d-2}$$

for $\lambda\phi^4$ theory. Gives branch cut! Discontinuity:

$$\frac{N(\sqrt{s})}{\sqrt{s}} \propto \sqrt{s}^{d-3}$$

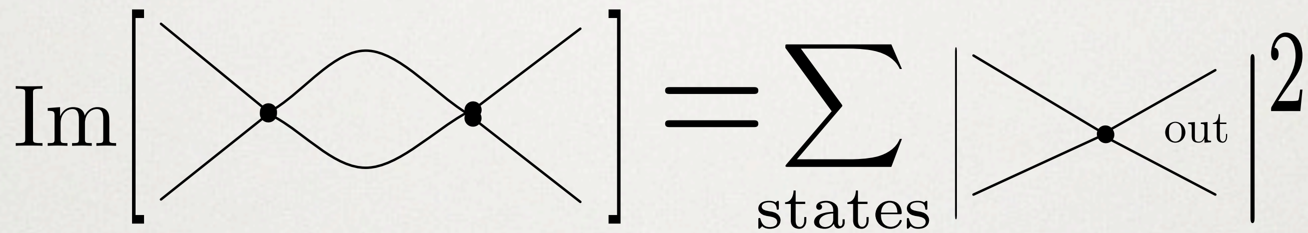
This comes from the exchange of **double-trace primary** states of dimension $\sqrt{s}$!

This is controlled by the OPE, as we will see...

Unitarity of the Holographic S-Matrix

S-Matrix Unitarity from the OPE

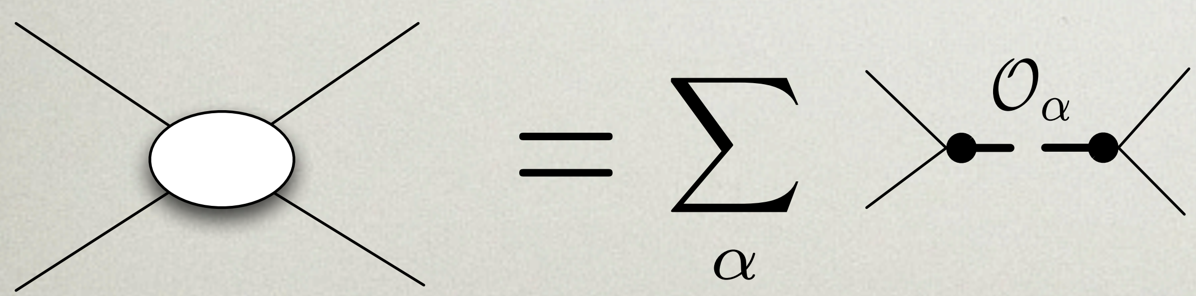
The cutting rules for the discontinuity of a loop amp



or more generally the unitarity relation

$$-i(T - T^{\dagger}) = T^{\dagger}T$$

look reminiscent of the **Conformal Block** decomp:

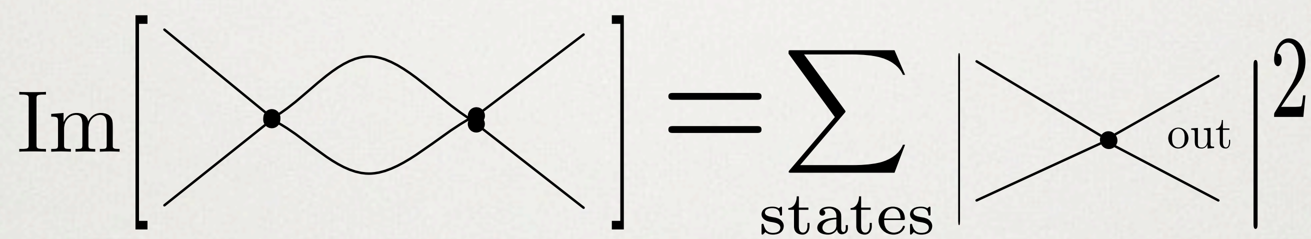


from $\langle \mathcal{O}_1 \mathcal{O}_2 \left(\sum_\alpha |\alpha\rangle\langle\alpha| \right) \mathcal{O}_3 \mathcal{O}_4 \rangle$

especially since operators = states in the CFT.

Let's Check It at 1-Loop

We need to compute both sides from the CFT.



The goal is to see that both are determined by a specific conformal block coefficient.

First let's compute the left side, using the 1-loop result we discussed.

Branch Cut Discontinuity

Recall that at 1-loop, branch cuts came from:

$$M(\delta) \to \int_0^\infty dn \frac{N_W(n)}{s + (2\Delta + 2n)^2} \implies \text{disc} = \frac{N_W(\sqrt{s})}{\sqrt{s}}$$

where we had defined (a la Kellan-Lehman)

$$G_\Delta(X,Y)^2 = \sum_{n=0}^{\infty} N_W(n) G_{2\Delta+2n}(X,Y)$$

But the contribution of bulk exchange implies the exchange of a primary operator in the conformal block decomposition.

Bulk Exchange Leads to Operator Exchange

$$G_\Delta(X,Y)^2 = \sum_{n=0}^{\infty} N_W(n) G_{2\Delta+2n}(X,Y)$$

implies that we must have terms
in the conformal block decomposition:

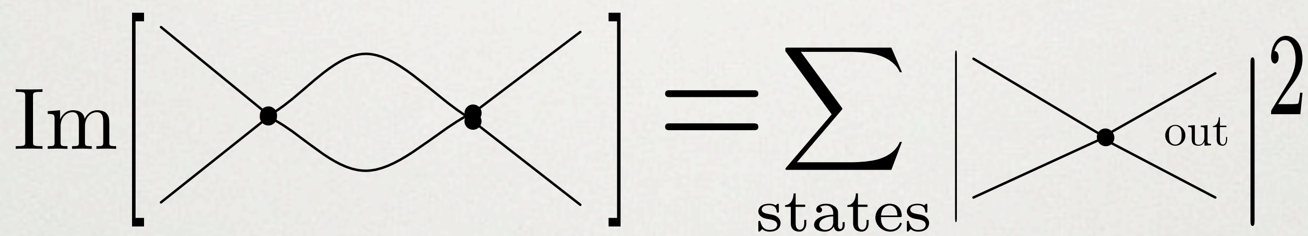
$$N_B(2\Delta + 2n) = N_W(n)$$

where the decomposition is defined by

$$M_4(\delta_{ij}) = \sum_\alpha N_B(\Delta_\alpha) B_{\Delta_\alpha}(\delta_{ij})$$

Conformal Blocks and the Imaginary Piece

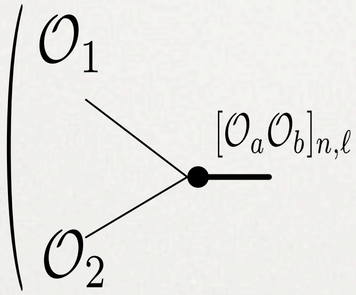
In other words, we see that the conformal block decomposition determines the left side of

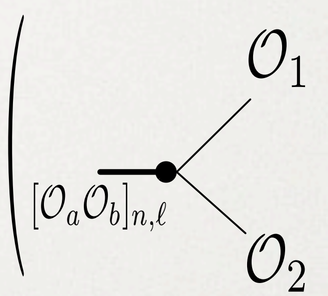


Now we will compute the right side.

Unitarity Relation Determined by Blocks

$$M_4(\delta_{ij}) = \sum_\alpha \left(\cdots \right)\left(\cdots \right) B_{\Delta_\alpha}(\delta_{ij})$$





As with dispersion relations, one order in perturbation theory gives info about the next.

$$M_4(\delta_{ij}) = \sum_\alpha N_B(\Delta_\alpha) B_{\Delta_\alpha}(\delta_{ij})$$

Gives a distinct way to compute coefficients in the conformal block expansion.

Conformal Blocks from 3-pt Correlators

$$M_4(\delta_{ij}) = \sum_{\alpha} N_B(\Delta_\alpha) B_{\Delta_\alpha}(\delta_{ij})$$

Coefficients of each block come from 3-pt correlators

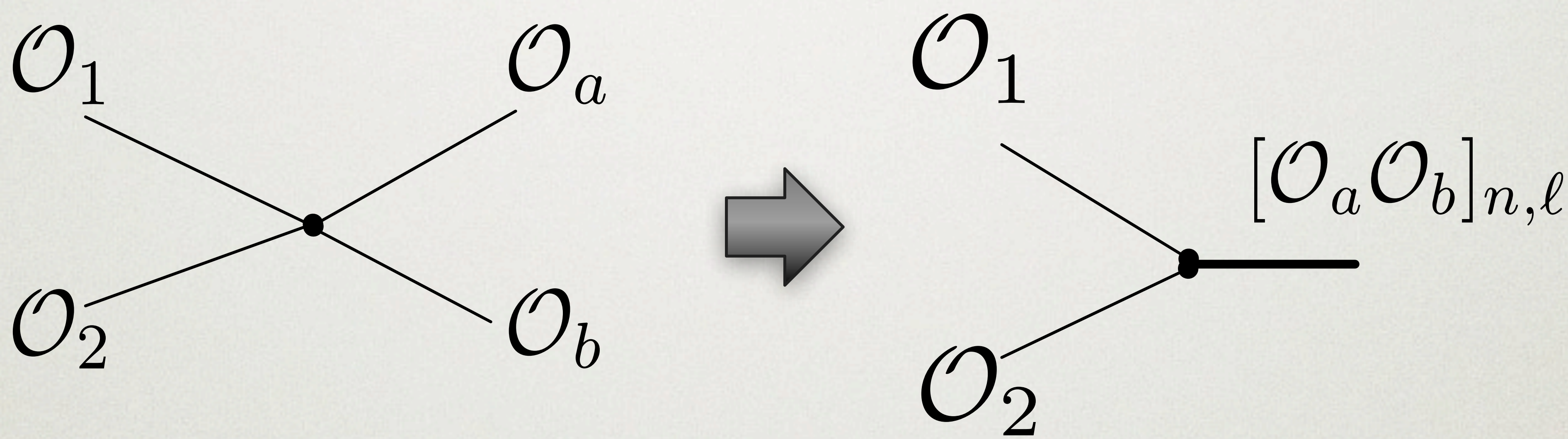
$$N_B(\Delta_\alpha) = \frac{C_3(1,2,\alpha)C_3(\alpha,3,4)}{C_2(\alpha,\alpha)}$$

Where the coefficients multiply universal functions

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_\alpha \rangle = \frac{C_3(1,2,\alpha)}{x_{12}^{\Delta_{12,\alpha}} x_{2\alpha}^{\Delta_{2\alpha,1}} x_{\alpha 1}^{\Delta_{\alpha 1,2}}}$$

Conglomerating Operators

To compute need to **conglomerate** single trace operators into one multi-trace:

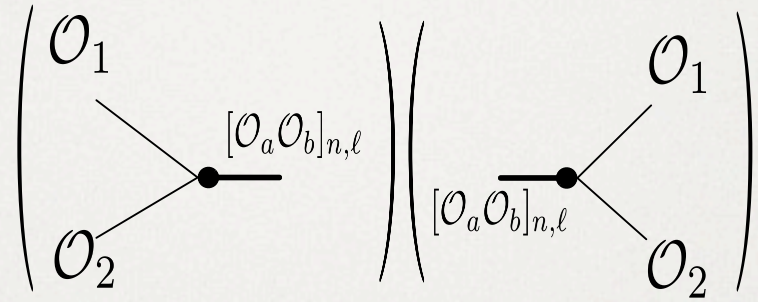


Easy in Mellin space, convolve with wavefunction.

By operator-state correspondence, this picks a state in the CFT (the state appearing in cutting rules!).

Unitarity Relation Determined by Blocks

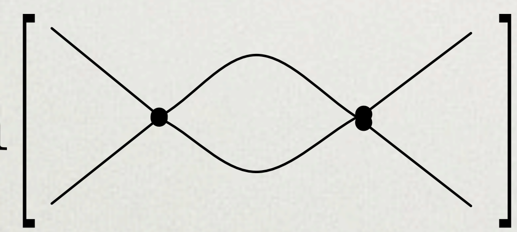
$$M_4(\delta_{ij}) = \sum_{\alpha}$$



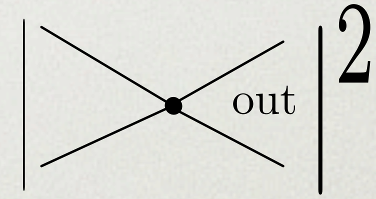
$$B_{\Delta_\alpha}(\delta_{ij})$$

Coeff of Block at a given Dimension/Energy
is the square of 2 --> X amp, summed over states!

Im



$$= \sum_{\text{states}}$$



Sum on CFT states = phase space integral in Flat Limit.

Both sides compute the same Conformal Block Coeff!

Bootstrap Programme => S-Matrix Programme

What is the flat space limit of a conformal block??

$$B_{\Delta_\alpha} \to \delta\left(s - \Delta_\alpha^2\right)$$

"Obvious", since blocks have definite angular momentum and definite dimension = energy.

$$M_4(\delta_{ij}) = \sum_\alpha N_B(\Delta_\alpha) B_{\Delta_\alpha}(\delta_{ij})$$

becomes (when we take the flat space limit)

$$\mathcal{M}(s,t) = N_B(s,t)$$

A Peak at Black Holes

$$\mathcal{M}(s,t) = N_B(s,t)$$

But on very general grounds, expect that

$$\mathcal{M}(s) \sim e^{-S_{BH}} \approx e^{-G_{d+1} s^{\frac{d-1}{2}}}$$

This gives a concrete prediction for the OPE and the conformal block decomposition of any CFT with a gravity dual where effective field theory applies!

Some Future Directions

- Mellin diagrammatic rules for loops, higher spin particles, twistors/spinor-helicity, SUSY, compactifications, dS/CFT, beloved theories...
- bolster recent progress on CFT Bootstrap?
- broken conformal invariance (eg QCD), flows between CFTs??
- sharpen criterion for analyticity = bulk locality?
- do **all** Gravitational S-Matrices come from CFTs??
- Find a CFT description of Hawking Evaporation, or at least see its simple and robust features!?

Conclusion

- Mellin Space = ``Momentum Space for CFTs”, conceptually and computationally
- Mellin Amplitude -> Holographic S-Matrix
- Analyticity follows from Meromorphy
- the OPE implies Unitarity, Cutting Rules
- Expect scattering through BHs is a robust ingredient in CFT dynamics, so we should attempt to understand it!