Hidden symmetry of correlation functions and amplitudes in $\mathcal{N} = 4$ SYM, part II

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Dualities in planar $\mathcal{N} = 4$ SYM

✔ Natural observables in a (conformal) gauge theory:

- ✗ Correlation functions: $G_n(x_i) = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\dots\mathcal{O}(x_n)\rangle$
- ✗ Scattering amplitudes: $A_n(p_i) = \langle p_1, p_2, \dots, p_n|S|0\rangle$

✔ Carry different/supplementary information about gauge theory:

$G_n =$ off-shell (anomalous dimensions, structure constants of OPE)

$A_n =$ on-shell (S-matrix)

✔ They are related to each other in planar $\mathcal{N} = 4$ SYM:

$$\lim_{x^2_{i,i+1}\to 0} \ln G_n(x_i) \sim 2\ln A_n(p_i)\,, \qquad p_i = x_i - x_{i+1}$$

✔ Have a new hidden symmetry (ultimately related to integrability of $\mathcal{N} = 4$ SYM)

✔ Allows us to predict correlators/amplitudes at higher loops without any Feynman graph calculations!

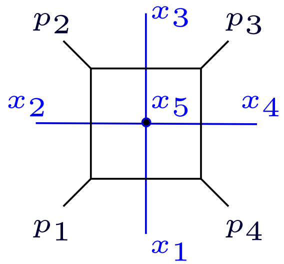
Gluon amplitudes in $\mathcal{N}=4$ SYM

- Four-gluon amplitude in $\mathcal{N}=4$ SYM at weak coupling $a = g^2 N_c/(8\pi^2)$

$$A_4^{++--}/A_4^{(\text{tree})} = 1 + a\, st I^{(1)}(s,t) + O(a^2)\,,$$

Scalar box in the dimensional regularization (for IR divergences) with $D = 4-2\epsilon$

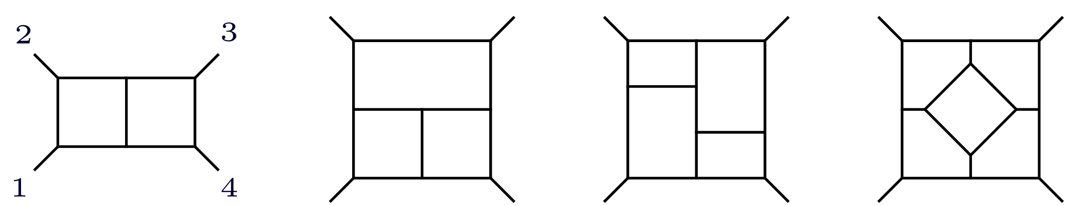
$$I^{(1)}(s,t) \;=$$



$$\sim \int \frac{d^D x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}\,, \qquad (x_{12}^2 = x_{23}^2 = x_{34}^2 = x_{41}^2 = 0)$$

Dual variables $p_i = x_i - x_{i+1}$ with $p_i^2 = x_{i,i+1}^2 = 0$

- (Broken) dual conformal symmetry
- All-loop BDS ansatz / AdS prediction / Wilson loop duality
- Explicit expressions for loop *integrands* up to 5 loops ... and even 7 loops [Marcus talk]
- Seemingly increasing complexity of diagrams at higher loops



$h(1,3;2,4)$ $\quad$ $T(1,3;2,4)$ $\quad$ 4-loops $\quad$ 5-loops

Correlation functions

- ✔ Protected superconformal operators made from six real scalars Φ^I

$$\mathcal{O}(x) = \mathrm{Tr}(ZZ)\,, \qquad \tilde{\mathcal{O}}(x) = \mathrm{Tr}(\bar{Z}\bar{Z})\,, \qquad Z = \Phi^1 + i\Phi^2$$

  - ✗ All-loop scaling dimension = tree level dimension
  - ✗ Two- and three-point correlation functions do not receive quantum corrections

- ✔ Simplest correlation function

$$G_4 = \langle \mathcal{O}(x_1)\tilde{\mathcal{O}}(x_2)\mathcal{O}(x_3)\tilde{\mathcal{O}}(x_4)\rangle = G_4^{(0)}\left[1 + 2a\, x_{13}^2 x_{24}^2 g(1,2,3,4) + O(a^2)\right]$$

One-loop 'cross' integral

$$g(1,2,3,4) = \frac{1}{4\pi^2}\int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}\,, \qquad (x_{12}^2, x_{23}^2, x_{34}^2, x_{41}^2 \neq 0)$$

- ✔ Loop corrections to the amplitude and to the correlator involve *the same* integral $g(1,2,3,4)$ but for *different* kinematics: on-shell $x_{i,i+1}^2 = 0$ for A_4 and off-shell $x_{i,i+1}^2 \neq 0$ for G_4

- ✔ *Amplitude/correlation function duality*

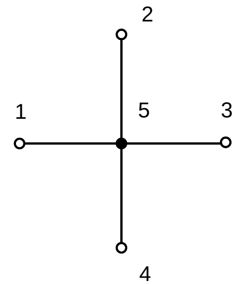
$$\lim_{x_{i,i+1}^2 \to 0} \ln\left(G_4/G_4^{(0)}\right) = \ln\left(A_4/A_4^{(\mathrm{tree})}\right)$$

Understood at the level of integrands in planar $\mathcal{N}=4$ SYM

A hidden symmetry

Examine one-loop correction to the correlator

$$G_4^{(1)} \sim \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \quad =$$



The corresponding integrand

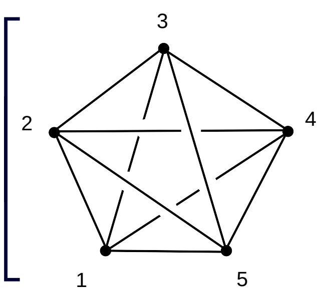
$$[G_4^{(1)}]_{\text{Integrand}} \sim \frac{1}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

The r.h.s. has S_4 permutation symmetry w.r.t. exchange of the external points $1, 2, 3, 4$

Equivalent form of writing

$$[G_4^{(1)}]_{\text{Integrand}} \sim x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \times \left[\prod_{i<j} \frac{1}{x_{ij}^2} \right]$$

$$= x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \times \Big[\quad \Big]$$



The second factor in the r.h.s. has the complete S_5 permutation symmetry!

Two loops

- Explicit two-loop calculation:

$$G^{(2)} = h(1,2;3,4) + h(3,4;1,2) + h(2,3;1,4) + h(1,4;2,3)$$
$$+ h(1,3;2,4) + h(2,4;1,3) + \frac{1}{2}\left(x_{12}^2 x_{34}^2 + x_{13}^2 x_{24}^2 + x_{14}^2 x_{23}^2\right)[g(1,2,3,4)]^2$$

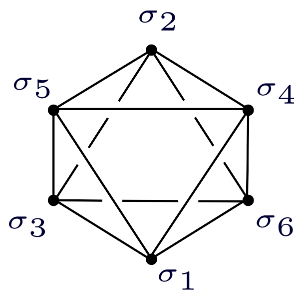
$h(1,2;3,4)-$ 'double' scalar box integral

- Go to a common denominator

$$G_4^{(2)} = x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \int d^4x_5 d^4x_6 \, f^{(2)}(x_1, \ldots, x_6)\,,$$

- 7 integrals in $G_4^{(2)}$ are described by a single $f-$function

$$f^{(2)}(x_1, \ldots, x_6) = \frac{1}{48} \sum_{\sigma \in S_6} \frac{x^2_{\sigma_1\sigma_2} x^2_{\sigma_3\sigma_4} x^2_{\sigma_5\sigma_6}}{\prod_{1 \le i < j \le 6} x^2_{ij}} \quad =$$



Has the complete S_6 permutation symmetry !

- Integrand of *all-loop* correlator has the complete permutation symmetry exchanging the external $1,2,3,4$ and internal, integration points *(no need for the planar limit !)*

All-loop integrand

Loop corrections to 4-point correlator

$$G_4^{(\ell)}(1,2,3,4) = x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \int d^4x_5 \dots d^4x_{4+\ell}\, f^{(\ell)}(x_1,\dots,x_{4+\ell})\,,$$

✔ General form of $f^{(\ell)}$ for arbitrary ℓ:

$$f^{(\ell)}(x_1 \dots, x_{4+\ell}) = \frac{P^{(\ell)}(x_1,\dots x_{4+\ell})}{\prod_{1\le i<j\le 4+\ell} x_{ij}^2}\,,$$

Can be deduced from the OPE analysis of the tree-level correlator

✔ The polynomial $P^{(\ell)}$ should satisfy the conditions:

- ✗ be invariant under $S_{4+\ell}$ permutations of $x_1, ..., x_{4+\ell}$
- ✗ have a uniform conformal weight $(1-\ell)$ at each point, both external and internal

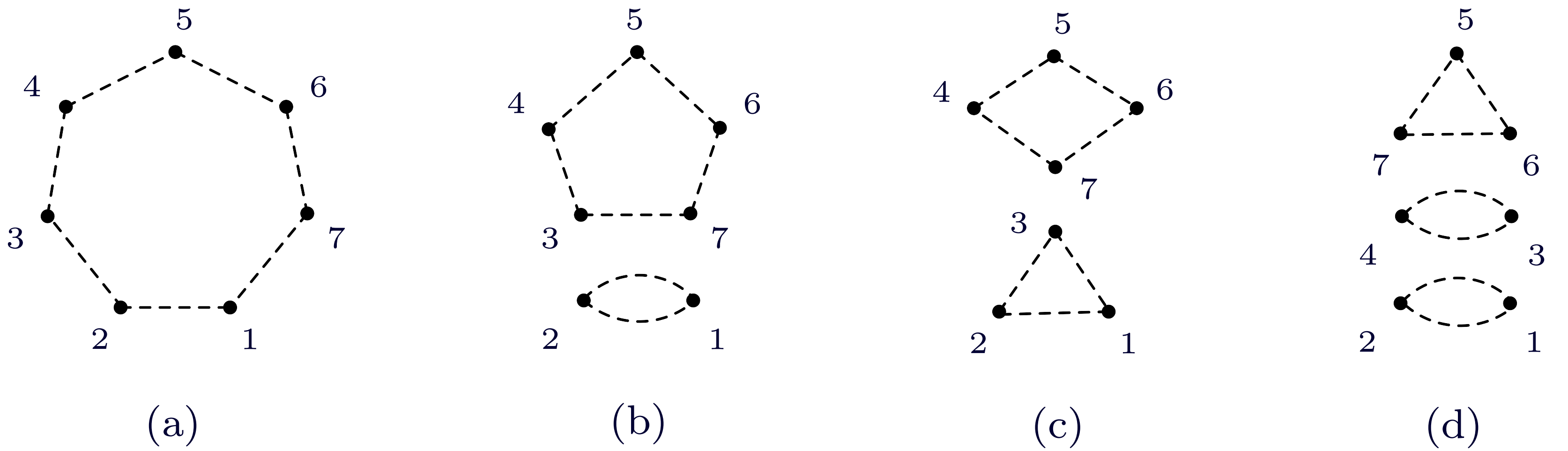
$$P^{(\ell)}(x_i^{-1}) = \prod_{i=1}^{4+\ell} (x_i^2)^{-\ell+1} P^{(\ell)}(x_i)$$

✔ Graph theory solution:

$P^{(\ell)} \mapsto$ ***Multi-graph with*** $(4+\ell)$ ***vertices of degree*** $(\ell-1)$

Three loops

$P^{(3)} \mapsto$ ***Multi-graph with*** 7 ***vertices of degree*** 2



There are only 4 independent possibilities for $P^{(3)}$:

(a) heptagon: $x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{71}^2 \; + \; S_7 \text{ permutations}$

(b) 2-gon $\times$ pentagon: $(x_{12}^4)(x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{73}^2) \; + \; S_7 \text{ permutations}$

(c) triangle $\times$ square: $(x_{12}^2 x_{23}^2 x_{31}^2)(x_{45}^2 x_{56}^2 x_{67}^2 x_{74}^2) \; + \; S_7 \text{ permutations}$

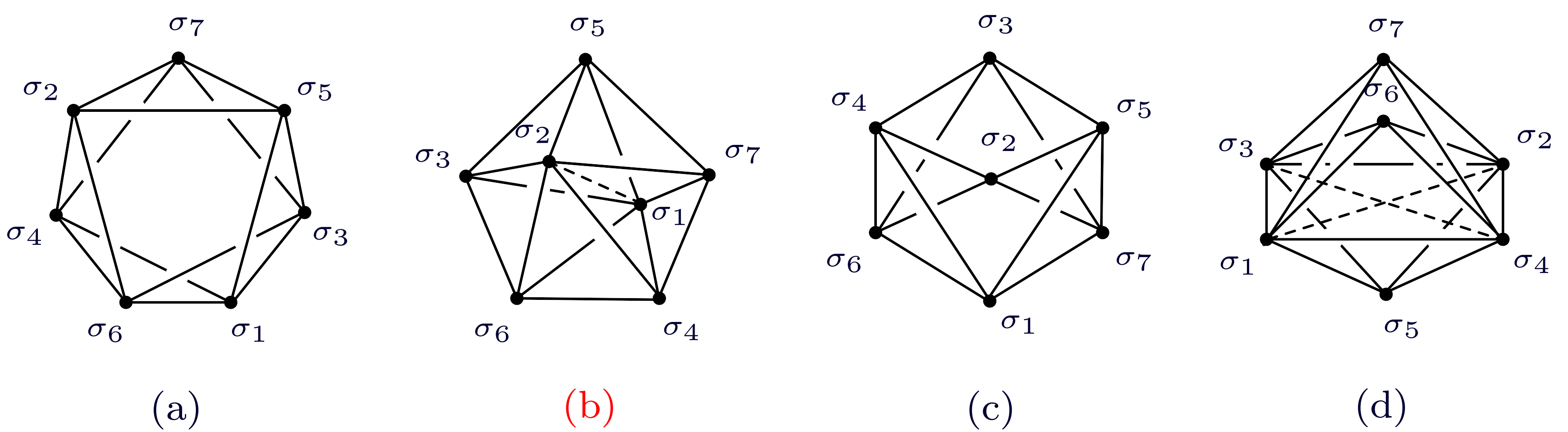
(d) 2-gon $\times$ 2-gon $\times$ triangle: $(x_{12}^4)(x_{34}^4)(x_{56}^2 x_{67}^2 x_{75}^2) \; + \; S_7 \text{ permutations} \; .$

$P^{(3)} =$ **linear combination of four terms with *arbitrary* coefficients**

From polynomial to integrand

$$f^{(3)}(x_1,\ldots,x_7) = \sum_{i=a,b,c,d} C_i \frac{P_i^{(3)}(x_1,\ldots x_7)}{\prod_{1\le i<j\le 7} x_{ij}^2}$$

Contributions corresponding to four graphs for $f^{(3)}$ (solid line $= 1/x^2_{\sigma_i\sigma_j}$, dashed line $= x^2_{\sigma_i\sigma_j}$):



$f^{(3)} =$ **linear combination of four diagrams with *arbitrary* coefficients** C_a, C_b, C_c, C_d

This holds in $\mathcal{N}=4$ SYM for *arbitrary* gauge group $SU(N_c)$

All topologies except (b) are nonplanar, $C_a = C_c = C_d = O(1/N_c^2)$, but how to fix C_b?

Number of coefficients to determine in the planar limit:

3 loops: 1 , 4 loops: 3 , 5 loops: 7 , 6 loops: 36

OPE constraints

- ✔ Correlation function in the like-cone limit $x_{12}^2, x_{23}^2, x_{34}^2, x_{41}^2 \to 0$

$$\ln G_4(1,2,3,4) \sim \Gamma_{\rm cusp}(a) \ln u \ln v\,, \qquad u\,, v \to 0$$

  Conformal cross-ratios $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2}$

- ✔ Examine two-loop integrand

$$\ln G_4 \sim a\, G_4^{(1)} + a^2 \left[G_4^{(2)} - \frac{1}{2} (G_4^{(1)})^2 \right]$$

$$= a \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} + a^2 \frac{x_{13}^2 x_{24}^2 [x_{13}^2 (x_{25}^2 x_{46}^2 + x_{45}^2 x_{26}^2) + x_{24}^2 (x_{36}^2 x_{15}^2 + x_{16}^2 x_{35}^2) - x_{13}^2 x_{24}^2 x_{56}^2]}{2 x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2 x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2 x_{56}^2}$$

  Divergences come from integration over x_5 and x_6 approaching the light-like edges, e.g. $x_5 \to x_1 - \alpha x_{12}$

$$x_{5i}^2 \to \alpha x_{1i}^2 + (1-\alpha) x_{2i}^2\,, \qquad 0 \le \alpha \le 1$$

- ✔ For the integral to have at most double-log asymptotics $\sim \ln u \ln v$ the polynomial in the numerator should vanish in this limit

- ✔ This condition alone fixes all the coefficients C_i. Checked to 2-, 3-, 4-, 5- and 6-loops.

  Permutation symmetry + OPE constraints allow us to construct the integrand of G_4 in the planar limit up to 6 loops!

Four-point correlator at three loops

Our result for 4-point correlation function in planar $\mathcal{N}=4$ SYM

$$G_4(1,2,3,4) = G_4^{(0)} + aG^{(1)} + a^2 G^{(2)} + a^3 G^{(3)} + O(a^4)$$

$G^{(\ell)}$ are given by the sum of scalar Feynman integrals

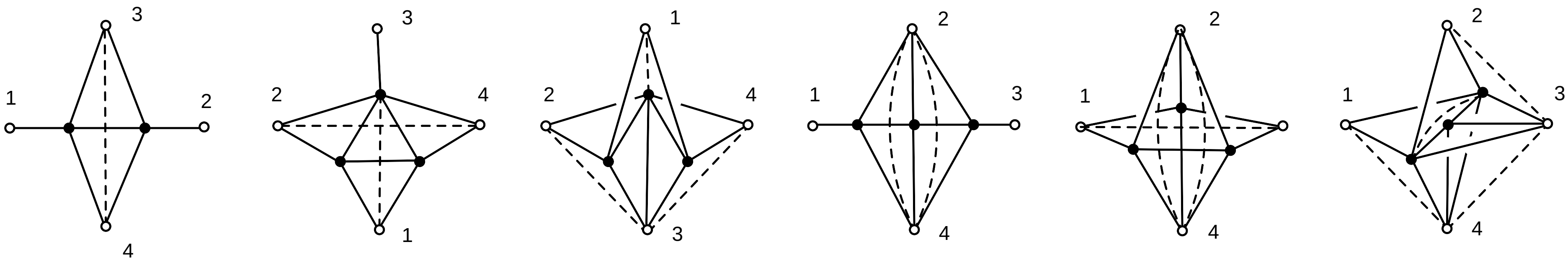
$$G^{(1)} = g(1,2,3,4)\,,$$

$$G^{(2)} = \big[h(1,2;3,4) + 5 \text{ perms}\big] + \tfrac{1}{2}\big[x_{12}^2 x_{34}^2 (g(1,2,3,4))^2 + 3 \text{ perms}\big]\,,$$

$$G^{(3)} = \big[T(1,3;2,4) + 11 \text{ perms}\big] + \big[E(2;1,3;4) + 11 \text{ perms}\big] + \big[L(1,3;2,4) + 5 \text{ perms}\big]$$

$$+ \big[(g\times h)(1,3;2,4) + 5 \text{ perms}\big] + \tfrac{1}{2}\big[H(1,3;2,4) + 11 \text{ perms}\big],$$

2- and 3-loop topologies:



$h(1,2;3,4)$ $\quad T(1,3;2,4)$ $\quad E(1;2,4;3)$ $\quad L(1,3;2,4)$ $\quad g\times h(1,3;2,4)$ $\quad H(1,2;3,4)$

Correlation function/Amplitude duality

$$\lim_{x^2_{i,i+1}\to 0} \ln\Big(G_4(x_i)/G_4^{(0)}(x_i)\Big) = 2\ln\Big(A_4(p_i)/A_4^{(0)}(p_i)\Big)\,, \qquad p_i = x_i - x_{i+1}$$

Is understood at the level of *integrands* (and not in terms of divergent *integrals*)

$$\lim_{x^2_{i,i+1}\to 0} G^{(3)} = M^{(3)} + M^{(1)}M^{(2)}$$

3-loop correlator:

$$\lim_{x^2_{i,i+1}\to 0} G^{(3)} = T(1,3;2,4) + T(1,3;4,2) + T(2,4;1,3) + T(2,4;3,1) + L(1,3;2,4) + L(2,4;1,3) + (g\times h)(1,3;2,4) + (g\times h)(2,4;1,3)$$

Compare with 3-loop 4-gluon amplitude

$$M^{(1)} = g(1,2,3,4)\,,$$

$$M^{(2)} = h(1,3;2,4) + h(2,4;1,3)\,,$$

$$M^{(3)} = T(1,3;2,4) + T(1,3;4,2) + T(2,4;1,3) + T(2,4;3,1) + L(1,3;2,4) + L(2,4;1,3)$$

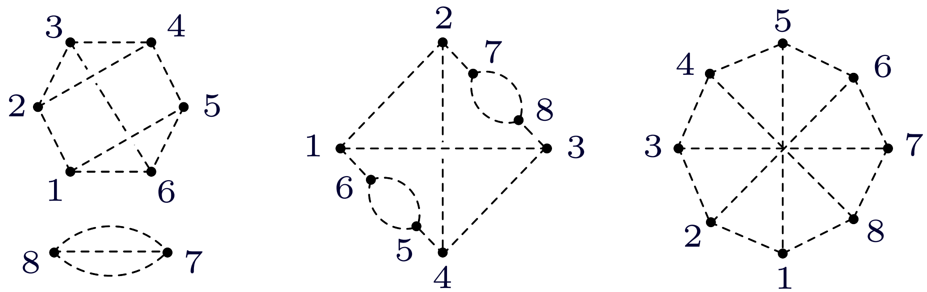
Precise agreement with the amplitude/correlator duality!

Four loops

4-loop 4-point integrand *in the planar limit*

$$f^{(4)}(1,\ldots,8) = \frac{\sum_{\sigma\in S_8}\left[c_A P_A(x_{\sigma(i)}) + c_B P_B(x_{\sigma(i)}) - c_C P_C(x_{\sigma(i)})\right]}{\prod_{1\le i<j\le 8} x_{ij}^2}\,,$$

Multi-graphs with 8 vertices of degree 3



P_A P_B P_C

Conformal polynomials

$$P_A(x_1,\ldots,x_8) = \frac{1}{24} x_{12}^2 x_{13}^2 x_{16}^2 x_{23}^2 x_{25}^2 x_{34}^2 x_{45}^2 x_{46}^2 x_{56}^2 x_{78}^6\,,$$

$$P_B(x_1\ldots,x_8) = \frac{1}{8} x_{12}^2 x_{13}^2 x_{16}^2 x_{24}^2 x_{27}^2 x_{34}^2 x_{38}^2 x_{45}^2 x_{56}^4 x_{78}^4\,,$$

$$P_C(x_1\ldots,x_8) = \frac{1}{16} x_{12}^2 x_{15}^2 x_{18}^2 x_{23}^2 x_{26}^2 x_{34}^2 x_{37}^2 x_{45}^2 x_{48}^2 x_{56}^2 x_{67}^2 x_{78}^2\,.$$

The coefficients $c_A = c_B = c_C = 1$ follow from the OPE constraint for $\ln G_4$ at 4 loops

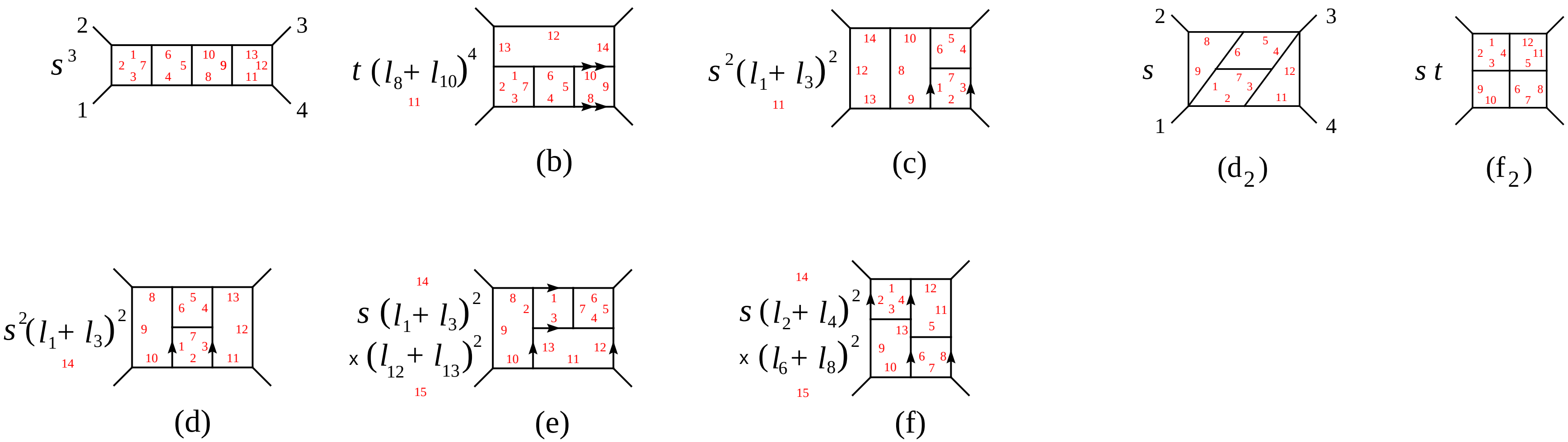
We constructed integrand of 4-point correlator up to 6 loops!

Back to the amplitudes: 4-loop 4-gluons

Amplitude/correlator duality

$$M_4^{(4)} = \lim_{x_{i,i+1}^2 \to 0} \left[\frac{1}{2} G_4^{(4)} - \frac{1}{4} G_4^{(3)} G_4^{(1)} - \frac{1}{8} (G_4^{(2)})^2 + \frac{3}{16} G_4^{(2)} (G_4^{(1)})^2 - \frac{5}{128} (G_4^{(1)})^4 \right]$$

All pseudo-conformal integrals that contribute to four-loop four-point amplitude



Perfect agreement with the known 4-loop result [Bern,Czakon,Dixon,Kosower,Smirnov'06]

$$M_4^{(4)} = \mathcal{I}^{(\mathrm{a})}(s,t) + \mathcal{I}^{(\mathrm{a})}(t,s) + 2\mathcal{I}^{(\mathrm{b})}(s,t) + 2\mathcal{I}^{(\mathrm{b})}(t,s) + 2\mathcal{I}^{(\mathrm{c})}(s,t) + 2\mathcal{I}^{(\mathrm{c})}(t,s) + \mathcal{I}_4^{(\mathrm{d})}(s,t)$$
$$+ \mathcal{I}^{(\mathrm{d})}(t,s) + 4\mathcal{I}^{(\mathrm{e})}(s,t) + 4\mathcal{I}^{(\mathrm{e})}(t,s) + 2\mathcal{I}^{(\mathrm{f})}(s,t) + 2\mathcal{I}^{(\mathrm{f})}(t,s) - 2\mathcal{I}^{(\mathrm{d_2})}(s,t) - 2\mathcal{I}^{(\mathrm{d_2})}(t,s) - \mathcal{I}^{(\mathrm{f_2})}(s,t$$

All 15 relative signs/coefficients follow from $c_A = c_B = c_C = 1$!

Agreement between correlators and amplitudes verified up to 6 loops

Konishi anomalous dimension at 5 loops

- 4-point correlator predicts integrand for 4-gluon amplitude in light-cone limit $x^2_{i,i+1} \to 0$

- ... and it contains a lot of information about anomalous dimension at short distances $x_1 \to x_2$ through the OPE

$$\mathcal{O}(x_1)\,\mathcal{O}(x_2) = \frac{c_{\mathcal{I}}}{x_{12}^4}\,\mathcal{I} + \frac{c_{\mathcal{K}}(a)}{(x_{12}^2)^{1-\frac{1}{2}\gamma_{\mathcal{K}}}}\,\mathcal{K}(x_2) + \dots\,, \qquad \mathcal{K} = \mathrm{tr}[\Phi^I\Phi^I]$$ – Konishi operator

- Asymptotics of 4-point correlator in the short-distance limit $x_1 \to x_2,\ x_3 \to x_4$

$$\ln G_4 \sim \frac{1}{2}\gamma_{\mathcal{K}}(a)\ln(x_{12}^2 x_{34}^2) + \dots$$

- Evaluation of $\gamma_{\mathcal{K}}(a)$ is reduced to extracting single-log part of a 2-point propagator-type integral for $\ln G_4$ – *enormous simplification*

- Using state-of-the-art technology, we have been able to compute $\gamma_{\mathcal{K}}(a)$ up to five loops!

$$\gamma_{\mathcal{K}}(a) = 3\,a - 3\,a^2 + \tfrac{21}{4}a^3 + \left(-\tfrac{39}{4} + \tfrac{9}{4}\zeta_3 - \tfrac{45}{8}\zeta_5\right)a^4$$

$$+\left(\tfrac{27}{4}\zeta(3) - \tfrac{81}{16}\zeta(3)^2 - \tfrac{135}{16}\zeta(5) + \tfrac{945}{32}\zeta(7) + \tfrac{237}{16}\right)a^5 + O(a^6)$$

The result exactly matches the AdS/CFT prediction [Bajnok,Hegedus,Janik,Lukowski'09]

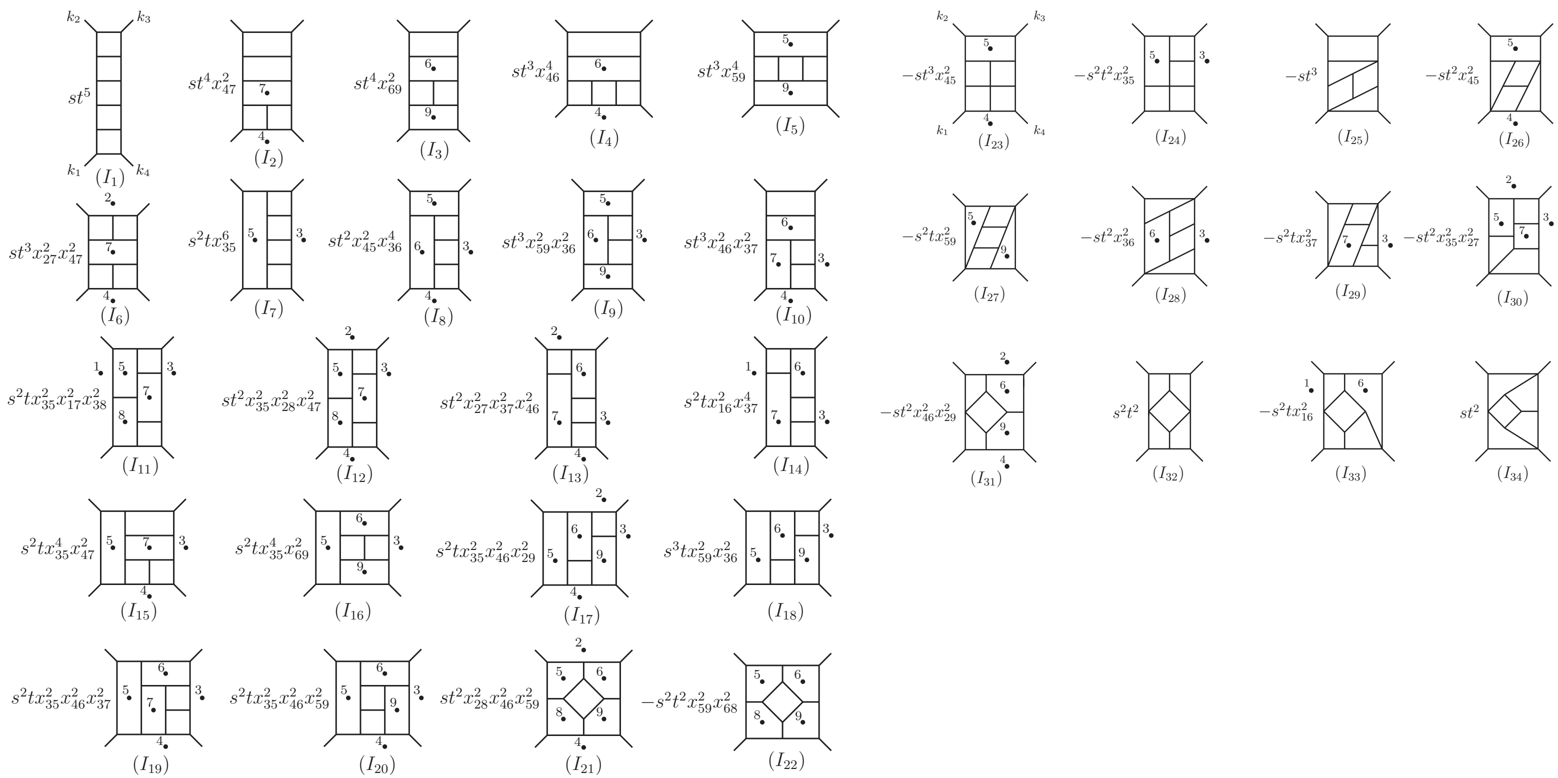
Six loops does not seem impossible . . .

Conclusions

- ✔ The all-loop integrand of 4-point correlator possesses a complete symmetry under the exchange of the four external and all internal (integration) points
- ✔ This symmetry alone + OPE constraints allow us to construct 6-loop integrand of the correlation function in the planar limit (without doing Feynman diagram calculation!)
- ✔ In the light-cone limit, the scattering amplitude/correlator duality predicts the integrand for 4-gluon amplitude
- ✔ In the short-distance limit, the OPE leads to analytical result for the Konishi anomalous dimension at 5 loops
- ✔ Straightforward generalization to higher loops
- ✔ What are hidden symmetries of $\mathcal{N} = 4$ SYM:
  - ✗ Dual (super)conformal symmetry of the amplitudes
  - ✗ permutation symmetry of the correlator
  - ✗ ???

5-loop 4-gluon amplitude

All pseudo-conformal integrals that contribute to five-loop four-point amplitude



The relative signs are determined from unitarity cuts

The complete five-loop four-point MSYM planar amplitude [Bern,Carrasco,Johansson,Kosower,Smirnov'07]

$$M_4^{(5)}(1,2,3,4) = -\frac{1}{32}\Big[\Big(I_1 + 2I_2 + 2I_3 + 2I_4 + I_5 + I_6 + 2I_7 + 4I_8 + 2I_9 + 4I_{10} + 2I_{11} + 4I_{12} + 4I_{13} + 4I_{14} + 4I_{15} + 2I_{16} + 4I_{17} + 4I_{18} + 4I_{19} + 4I_{20} + 2I_{21} + 2I_{23} + 4I_{24} + 4I_{25} + 4I_{26} + 2I_{27} + 4I_{28} + 4I_{29} + 4I_{30} + 2I_{31} + I_{32} + 4I_{33} + 2I_{34} + \{s \leftrightarrow t\}\Big) + I_{22}\Big]$$