On the non-planar beta-deformed N=4 super-Yang-Mills theory

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Based on work with Q. Jin

$\mathcal{N} = 4$ super-Yang-Mills theory has a number of remarkable properties:

- super-conformal invariance
- dual super-conformal invariance at planar level
- integrable planar dilatation operator
- remarkable planar amplitudes / Wilson loops relation

  Alday, Maldacena
- remarkable planar amplitudes / correlation function relation

  Eden, Korchemsky, Sokatchev

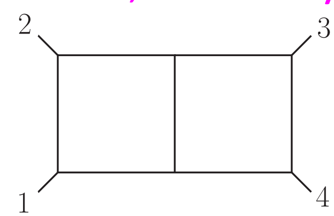
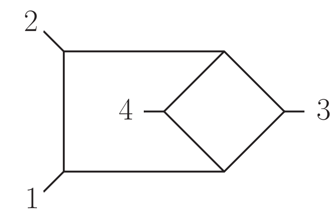
➡ Planar amplitudes have a very constrained structure

Unregularized d=4 integrand determined be symmetries and a small number of unitarity properties Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka

➡ Far fewer constraints for non-planar amplitudes

Yet…

… non-planar amplitudes are simpler than what they could have been

U(1) decoupling:
- 1-loop sub-leading color i.t.o. leading color
  → combination of box integrals — Bern, Kosower
- parts of 2-loop 2-trace related to leading color — Bern, Rozowsky, Yan; Bern, de Freitas, Dixon
  → combination of  and 
- Inversion of higher loop U(1) decouplings — Naculich

Higher loops:
- 3 & 4 loops: 2-trace better in UV than rest — Bern, Carrasco, Dixon, Johansson, RR

$$D_c = 4 + \frac{6}{L} \quad \text{vs} \quad D_c = 4 + \frac{8}{L}$$

Exhibits color-kinematic duality; $c_i + c_j + c_k = 0 \leftrightarrow n_i + n_j + n_k = 0$ — Bern, Carrasco, Johansson

→ (potential) all-order relation between l. and sub-l. color

→ simple and structured expressions

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Question: How much of this structure relies on supersymmetry?
Can there be more structure that is hidden?

Analyze QFT-s which share most of the properties of $\mathcal{N}=4$ sYM

⟶ Deform it in a controlled way

1. orbifolds

Inheritance principle: Bershadsky, Johansen
Bershadsky, Kakushadze, Vafa

$$\varphi_i^I = R^I{}_J\, g^{-1} \varphi_i^J\, g \quad R \in SU(4) \quad g \in SU(4) \subset SU(N)$$

2. the h deformation

Leigh, Strassler

$$W = \mathrm{Tr}[\Phi_1[\Phi_2,\Phi_3]] \;\longrightarrow\; f(h,N)(\mathrm{Tr}[\Phi_1[\Phi_2,\Phi_3]] + h(\mathrm{Tr}[\Phi_1^3] + \mathrm{Tr}[\Phi_2^3] + \mathrm{Tr}[\Phi_3^3]))$$

3. the β deformation

Leigh, Strassler

$$W = \mathrm{Tr}[\Phi_1[\Phi_2,\Phi_3]] \;\longrightarrow\; f(\beta,N)\mathrm{Tr}[\Phi_1(e^{i\beta}\Phi_2\Phi_3 - e^{-i\beta}\Phi_3\Phi_2)]$$

	super-conf.	dual super-conf.	planar integrable	Amp/W.L.
1.	yes; N=2, 1, 0	yes; inherited	yes	quite likely
2.	yes; N=1	not known	sometimes	not clear
3.	yes; N=1, 0	yes	yes	yes

The supersymmetric β -deformed $\mathcal{N}=4$ super-Yang-Mills theory

- the same field content as $\mathcal{N}=4$ sYM
- real β : almost the same properties except for supersymmetry
- a pattern for the deformation:

Lunin, Maldacena

noncommutative deformation: $\varphi_I \varphi_J \mapsto e^{i\hat{\beta}_{ij} q_I^i q_J^j} \varphi_I \varphi_J$

$$\hat{\beta}_{ij} = -\hat{\beta}_{ji} \;;\; \text{if } \hat{\beta}_{12} = \hat{\beta}_{23} = \hat{\beta}_{31} = \beta \mapsto \text{susy}$$

$\mathcal{N}=4$ R-charge vectors

	ϕ^{14}	ϕ^{24}	ϕ^{34}	A_μ	ψ^1	ψ^2	ψ^3	ψ^4	Q^1	Q^2	Q^3	Q^4
J_{12}	1	0	0	0	1/2	-1/2	-1/2	1/2	-1/2	1/2	1/2	-1/2
J_{34}	0	1	0	0	-1/2	1/2	-1/2	1/2	1/2	-1/2	1/2	-1/2
J_{56}	0	0	1	0	-1/2	-1/2	1/2	1/2	1/2	1/2	-1/2	-1/2

Some consequences:

- most non-commutative results survive; planar amplitudes are inherited
  Filk (space-time noncommutativity); Khoze; ...
- vector U(1) factors decouple; chiral superfield U(1) factors are coupled
  $U(N)$ vs $SU(N)$?
- both f_{abc} and d_{abc} couplings

General deformation: $\varphi_I \varphi_J \mapsto e^{i\hat{\beta}_{ij} q_I^i q_J^j} \varphi_I \varphi_J \quad \hat{\beta}_{12} \neq \hat{\beta}_{23} \neq \hat{\beta}_{31}$

Frolov

- Supersymmetry is completely broken;

deformation of the $\mathcal{N}=4$ component Lagrangian

- Renormalizability requires some 2-trace terms

Dymarsky, Klebanov, RR (unpublished)

$$\delta S_{eff} = -\frac{\lambda^2}{16\pi^2} \ln \frac{\Lambda^2}{M^2} \Big[8\,(\cos 4\beta_{ij} - \cos 4\beta_{ik})^2\ \mathrm{Tr}[\phi^i \phi^i]\mathrm{Tr}[\phi^{\bar{\imath}} \phi^{\bar{\imath}}] + 8\,(\cos 4\beta_{ij} - \cos 2(\beta_{ik} + \beta_{jk}))^2\ \mathrm{Tr}[\phi^i \phi^j]\mathrm{Tr}[\phi^{\bar{\imath}} \phi^{\bar{\jmath}}] + 8\,(\cos 4\beta_{ij} - \cos 2(\beta_{ik} - \beta_{jk}))^2\ \mathrm{Tr}[\phi^i \phi^{\bar{\jmath}}]\mathrm{Tr}[\phi^{\bar{\imath}} \phi^j] \Big]\Big|_{i\neq j\neq k}$$

$$\delta S_{tree} = f_{1ij}\,\mathrm{Tr}[\phi^i \phi^i]\mathrm{Tr}[\phi^{\bar{\imath}} \phi^{\bar{\imath}}] + f_{2ij}\,\mathrm{Tr}[\phi^i \phi^j]\mathrm{Tr}[\phi^{\bar{\imath}} \phi^{\bar{\jmath}}] + f_{3ij}\,\mathrm{Tr}[\phi^i \phi^{\bar{\jmath}}]\mathrm{Tr}[\phi^{\bar{\imath}} \phi^j]\Big|_{i\neq j\neq k}$$

- Unstable RG evolution of 2-trace couplings (Landau pole)
- $\mathcal{N}=1$ limit: unique 2-trace coupling with RG fixed point

$$f_{2ij} = \frac{2}{N} |h|^2 \sin^2 2\beta$$

Value required to project out the U(1) auxiliary fields in chiral multiplets

Expectedly, supersymmetry cures instability; does it do anything else?

What is the coefficient of the superpotential and

what is the UV behavior of the theory?

- Leigh/Strassler: not constructive but guarantees that there exists a coefficient $h = f(\beta, N_c, g_{_{YM}})$ that leads to a finite theory

    - 1- and 2-loop expressions are known
    - when/how are they corrected?

Freedman, Gursoy
Penati, Santambrogio, Zanon

- 4-point amplitudes: Manifest susy $\longrightarrow$ formally diverge in

$$D_c = 4 + \frac{2\mathcal{N}}{L} \longrightarrow D_c = 4 + \frac{2}{L}$$

Is this really true?

▶ Try to answer by evaluating 4-point amplitudes

- Single-trace amplitudes:

$$A^{(0)} = \sum_{\rho \in S_n/Z_n} \mathrm{Tr}[T^{a_{\rho(1)}} \ldots T^{a_{\rho(n)}}] A^{(0)}(k_{\rho(1)} \ldots k_{\rho(n)})$$

$$A^{(0)}(k_1 \ldots k_n) \mapsto e^{i\Theta(1,\ldots,n)} A^{(0)}(k_1 \ldots k_n) \, ; \; \Theta(1,\ldots,n) = \sum_{1 \le i < j \le n} q_i \cdot \hat{\beta} \cdot q_j$$

- A simple example -- 3-point amplitudes: $\phi^{ij}\psi^{ikl}\psi^{jkl} \; , \quad i \neq j \neq k = 1,2,3,4$

$$\mathcal{A}^{(0)}(1\phi^{ij}, 2\psi^{ikl}, 3\psi^{jkl}) = \frac{h}{g}\langle 23\rangle e^{i\beta}\mathrm{Tr}_{123} + \frac{h}{g}\langle 32\rangle e^{-i\beta}\mathrm{Tr}_{321}$$

• Single-trace amplitudes:

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- Account for the $\mathcal{O}(1/N^2)$ deformation of the coefficient of the superpotential

Here: focus on double-trace terms; ignore $\mathcal{O}(1/N^2)$ corrections

• non-vanishing tree-level double-trace amplitudes

$$\mathcal{L}_{2\mathrm{tr}} = \frac{1}{2N} |f(\beta, N)^2| \epsilon_{ijk} \epsilon^{ilm} \mathrm{Tr}[[\phi^j, \phi^k]_\beta] \mathrm{Tr}[[\bar{\phi}_l, \bar{\phi}_m]_\beta]$$

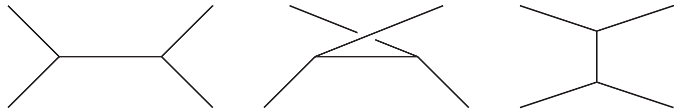
⟶ crucial for finiteness; also $|f(\beta, N)|^2 = \dfrac{g^2_{YM}}{1 - \frac{4}{N^2} \sin^2 \beta}$

With same planar properties, differences appear at subleading color in dim. Reg.

• supersymmetry ➡ more double-trace amplitudes

• structure constant color factors combine 1- and 2-trace terms

- BCJ-like numerator relations



$$\mathcal{A}_4^{\beta,(0)}(1g^+, 2\phi^{23}, 3f^{134}, 4f^{124}) = \frac{n_{12}}{s_{12}} f^{12a} f_\beta{}^{34}{}_a + \frac{n_{23}}{s_{23}} f_\beta{}^{23a} f^{14}{}_a + \frac{n_{13}}{s_{13}} f^{31a} f_\beta{}^{24}{}_a$$

$$\mathcal{A}_4^{\beta,(0)}(1\phi^{23}, 2\phi^{14}, 3\phi^{13}, 4\phi^{24}) = \frac{n_{12}}{s_{12}} f^{12a} f^{34}{}_a + \frac{n_{23}}{s_{23}} f^{23a} f^{14}{}_a + \frac{n_{13}}{s_{13}} f_\beta{}^{31a} f_\beta{}^{24}{}_a$$

$$f_\beta^{abc} = \mathrm{Tr}[T^a[T^b, T^c]_\beta] = e^{i\Phi(a,b,c)}\mathrm{Tr}[T^aT^bT^c] - e^{i\Phi(a,c,b)}\mathrm{Tr}[T^aT^cT^b]$$

Numerator factors -- same as in $\mathcal{N}=4$ sYM:

$$n_{12} + n_{23} + n_{13} = 0$$

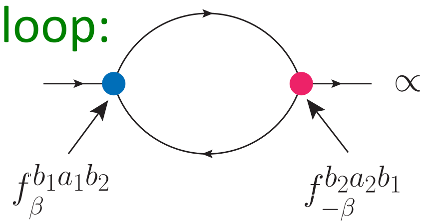
Color factors – different; generically no Jacobi identity involving only d-structure constants:

$$f^{[12}{}_a f^{3]4a} = 0 \qquad f^{[12}{}_a d^{3]4a} = 0$$

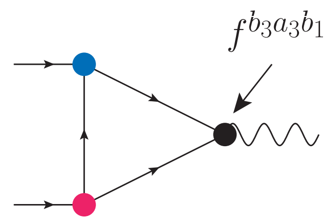
On corrections to the coefficient of the superpotential

Fairly accurate picture from color structure of supergraphs with only scalar vertices

- 1 loop:



$\propto |h|^2\left(1-\frac{1}{N_c^2}|q-q^{-1}|^2\right)\delta^{a_1a_2}$



$\propto |h|^2\left(1-\frac{1}{N_c^2}|q-q^{-1}|^2\right)f^{a_1a_2a_3}$

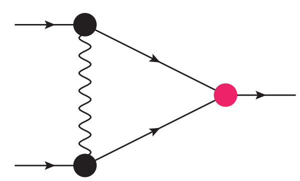
Finiteness as $q\rightarrow 1$ requires they are cancelled by vector multiplet interactions

$$|f(\beta,N_c)|^2\equiv|h|^2=\frac{g_{YM}^2}{1-\frac{1}{N_c^2}|q-q^{-1}|^2}$$

- 2 loops: All 2-point and 3-point graphs contain a triangle

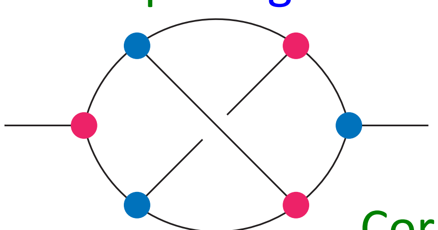
⟶ color structure reduces to 1-loop analysis

⟶ finiteness at $q\rightarrow 1$: same condition as at 1 loop



$\propto f_{\beta}^{a_1a_2a_3}$

- 3 loops: Argument no longer works; there exist graphs with only box subintegrals



$\propto |h|^6\left(N_c-\frac{4}{N_c}\right)|q-q^{-1}|^4\left((q^2+4+q^{-2})+\frac{5}{N_c^2}|q-q^{-1}|^2\right)$

Correction to $f(\beta,N_c)$ expected at this loop order; precise expression of the 2-loop divergence is important.

Some all-order results

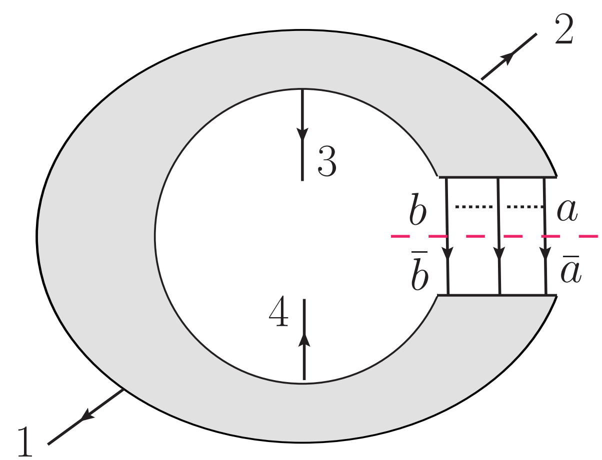
- cusp anomaly is independent of the deformation
  - consequence of integrability; testable at weak and strong coupling
  - ⟶ same leading IR divergences as in the un-deformed theory
- 2-trace β-dependence: iff nontrivial R-charge flow between traces

Generically:

$$q_1 + q_2 + q_3 + q_4 = 0$$

If $q_1 + q_2 = 0 = q_3 + q_4$ all β-dependence drops out

To see this: follow charge flow in generalized unitarity cuts



Charge conservation:

$$\Theta(1,2,a,\dots,b,3,4,\bar{b},\dots,\bar{a}) = \Theta(1,2) + \Theta(3,4) + 2\Theta(1+2,a,\dots,b)$$

Some all-order results

- cusp anomaly is independent of the deformation
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-- 4-gluon double-trace terms: same properties as in $\mathcal{N} = 4$ sYM
- diverge in $D_c = 4 + 8/L$
- same for $\mathrm{Tr}[\phi_i \bar{\phi}^i]\mathrm{Tr}[**]$

-- Generalization to higher-point multi-trace terms

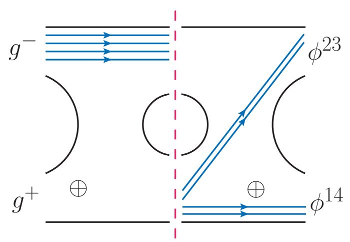
Some explicit examples 4-point loop amplitudes: 1 loop

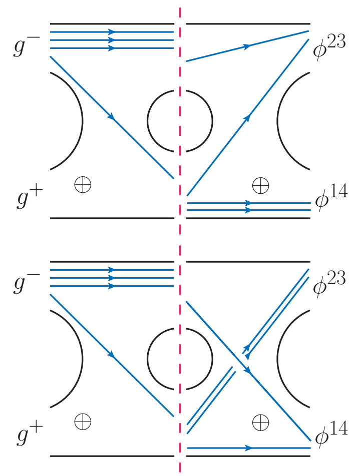
Jin, RR

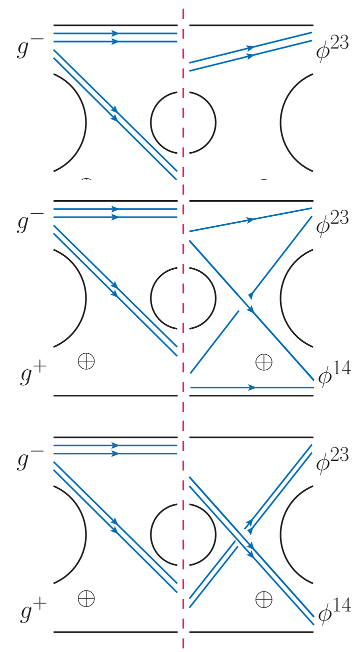
- Construct using generalized unitarity
  - use color-dressed cuts
  - supersums: use pictorial rules dressed with the extra phase factors

Bern, Carrasco, Ita, Johansson, RR

$$\mathcal{A}_4^{\beta,(1)}(1g^+, 2g^-, 3\phi^{23}, 4\phi^{14})$$







$$A = \frac{\langle l_a 2\rangle}{\langle l_b 2\rangle} \qquad B = \frac{\langle l_b 3\rangle}{\langle l_a 3\rangle} \qquad C = \frac{\langle l_b 4\rangle}{\langle l_a 4\rangle}$$

$$\rho^2_{\mathcal{N}=4} = A^2BC - (2AB + 2AC) + (4 + \frac{B}{C} + \frac{C}{B}) - (\frac{2}{AB} + \frac{2}{AC}) + \frac{1}{A^2BC}$$

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Supersum in terms of charge-flow diagrams:

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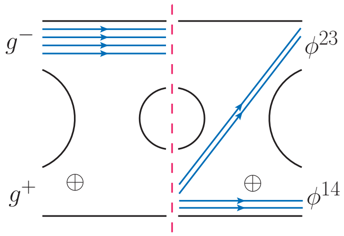
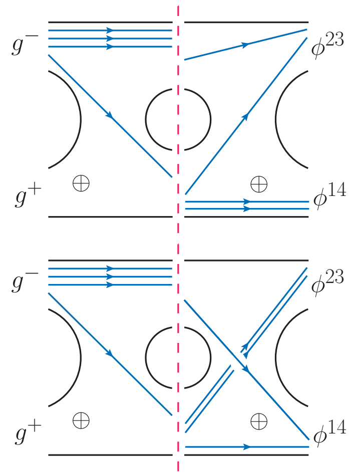
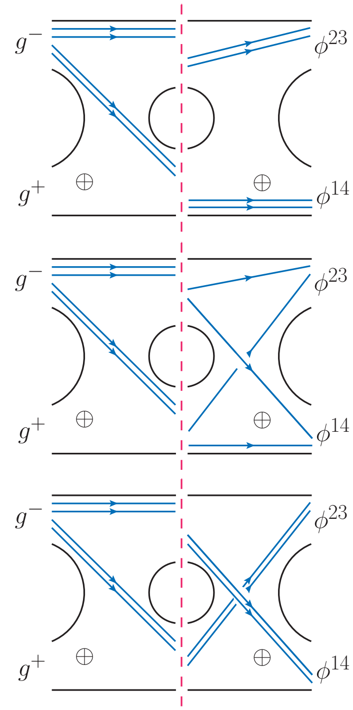


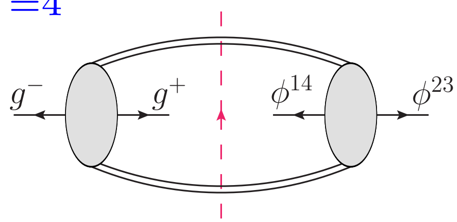




$$\mathcal{A}_4^{\beta,(1)}(1g^+, 2g^-, 3\phi^{23}, 4\phi^{14})$$

- Planar cut topology: same $\rho^2_{\mathcal{N}=4}$

- Non-planar cut topology:



$$\rho_\beta^2 = A^2BC - (2AB + 2\cos(2\beta)AC) + (4\cos(2\beta) + \frac{B}{C} + \frac{C}{B}) - \frac{2}{AB} - \frac{2\cos(2\beta)}{AC} + \frac{1}{A^2BC}$$

$$\mathcal{C} = \rho^2 \, \frac{\langle l_a 2\rangle^2 \langle l_b 2\rangle^2}{\langle 12\rangle\langle 2 l_a\rangle\langle l_a l_b\rangle\langle l_b 1\rangle} \frac{\langle l_b 4\rangle\langle l_b 3\rangle\langle l_a 4\rangle\langle l_a 3\rangle}{\langle 3 l_a\rangle\langle l_a 4\rangle\langle 4 l_b\rangle\langle l_b 3\rangle}$$

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  - supersums: use pictorial rules dressed with the extra phase factors

    Bern, Carrasco, Ita, Johansson, RR
  - focus on 3 terms: $\mathrm{Tr}[T^{a_1}T^{a_i}]\mathrm{Tr}[T^{a_j}T^{a_k}] \quad i \neq j \neq k = 2, 3, 4$

- Classify following the number of vector multiplets
  - 4 vector multiplets: same as in $\mathcal{N} = 4$ sYM
  - 3 vector multiplets + 1 chiral multiplet: vanish identically
  - 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$

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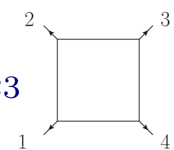
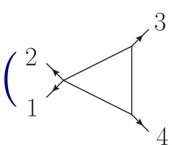
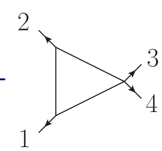
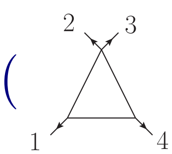
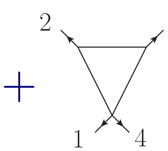
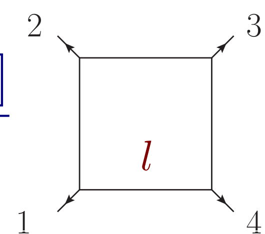
$$\mathcal{A}(1234)_4^{(1)\beta} = \mathcal{A}(1234)_4^{(1)\mathcal{N}=4} - 8\sin^2\beta\,(\mathrm{Tr}_{13}\mathrm{Tr}_{24} + \mathrm{Tr}_{14}\mathrm{Tr}_{23})A(1234)_{4;3}^{(1)\mathrm{extra}}$$

$A(1234)_{4;3}^{(1)\mathrm{extra}} = \dfrac{\langle 23\rangle^2}{\langle 13\rangle^2}\Big[-s_{12}s_{23}$  $+\, s_{12}\big($  $+$  $\big) + s_{23}\big($  $+$  $\big)\Big]$

$= -\dfrac{\langle 23\rangle^2}{\langle 13\rangle^2}\,\dfrac{G[l,1,2,3]}{s_{12}s_{23}}$ 

- IR finite
- UV divergent in 6 dimensions; standard expectation for a conformal $\mathcal{N}=1$ theory

Some explicit examples 4-point loop amplitudes: 1 loop

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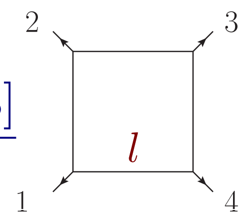
  - 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$

$$\mathcal{A}(1234)_4^{(1)\beta} = \mathcal{A}(1234)_4^{(1)\mathcal{N}=4} + \mathcal{A}(1234)_4^{\text{extra},(1)}$$

$$\mathcal{A}_4^{\text{extra},(1)}(1234) = -8\tilde{h}^2 \sin^2\beta \, \frac{\langle 13\rangle^2}{\langle 23\rangle^2} \frac{G[l,1,2,3]}{s_{12}s_{23}}$$



$$\times \left[\left(1 - \frac{2}{N^2}\right) (\text{Tr}_{13}\text{Tr}_{24} + \text{Tr}_{14}\text{Tr}_{23}) - \frac{1}{N}(\text{Tr}_{1324} + \text{Tr}_{1432}) - \frac{2}{N^2}\text{Tr}_{12}\text{Tr}_{34} \right]$$

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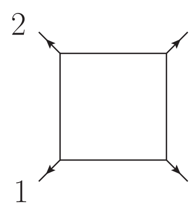
  - 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+},2_{g^-},3_{\phi^{34}},4_{\phi^{12}})$
  - 1 vector multiplet + 3 chiral multiplets: $A(1_{g^-},2_{\phi^{34}},3_{\psi^1},4_{\psi^2})$

$$\mathcal{A}^{(1)2\mathrm{tr}}(1234) = \cos\beta\, A^{(1)2\mathrm{tr}}_{\mathcal{N}=4}(1234)\big(\mathrm{Tr}_{12}\mathrm{Tr}_{34}+\mathrm{Tr}_{13}\mathrm{Tr}_{24}+\mathrm{Tr}_{14}\mathrm{Tr}_{23}\big)$$

$$A^{(1)2\mathrm{tr}}_{\mathcal{N}=4} = -2\, s_{12}s_{23}\,\frac{[23][34]}{[12][13]}\,\Big($$



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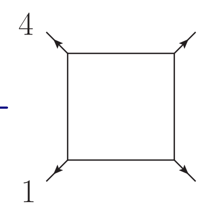

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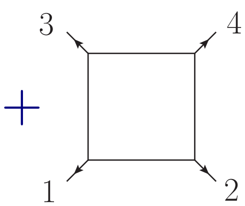


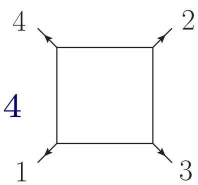
$\Big)$

Some explicit examples 4-point loop amplitudes: 1 loop

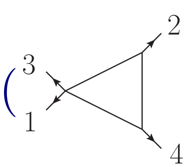
- Classify following the number of vector multiplets
  - 4 vector multiplets: same as in $\mathcal{N}=4$ sYM
  - 3 vector multiplets + 1 chiral multiplet: vanish identically
  - 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$
  - 1 vector multiplet + 3 chiral multiplets: $A(1_{g^-}, 2_{\phi^{34}}, 3_{\psi^1}, 4_{\psi^2})$
  - 4 chiral multiplets: $A(1_{\psi^1}, 2_{\psi^1}, 3_{\psi_1}, 4_{\psi_1})$, $A(1_{\psi^1}, 2_{\psi^2}, 3_{\psi_1}, 4_{\psi_2})$

$$\mathcal{A}(1_{\psi^1}, 2_{\psi^1}, 3_{\psi^{234}}, 4_{\psi^{234}})_4^{(1)\beta} = \mathcal{A}(1_{\psi^1}, 2_{\psi^1}, 3_{\psi^{234}}, 4_{\psi^{234}})_4^{(1)\mathcal{N}=4} - 8\sin^2\beta\, \mathrm{Tr}_{12}\mathrm{Tr}_{34} A(1234)^{(1)\mathrm{extra}}$$

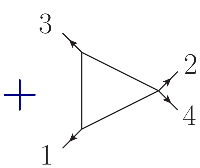
$$\frac{A(1234)^{(1)\mathrm{extra}}}{\cos^2\beta} = \frac{\langle 34\rangle}{\langle 12\rangle}\Big[s_{13}s_{14}$$



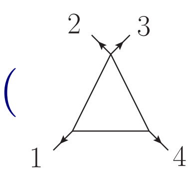
$$- s_{13}\Big($$



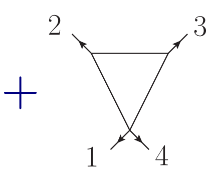
$$+$$



$$\Big) - s_{14}\Big($$

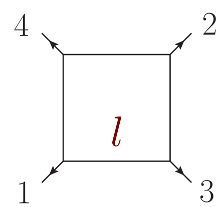


$$+$$



$$\Big)\Big]$$

$$= -\frac{\langle 34\rangle}{\langle 12\rangle}\,\frac{G[l,1,4,2]}{s_{13}s_{14}}$$



Some explicit examples 4-point loop amplitudes: 1 loop

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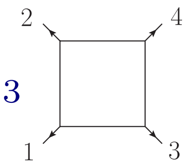
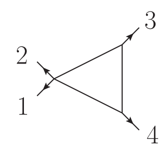
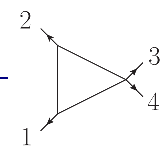
  - 1 vector multiplet + 3 chiral multiplets: $A(1_{g^-}, 2_{\phi^{34}}, 3_{\psi^1}, 4_{\psi^2})$
  - 4 chiral multiplets: $A(1_{\psi^1}, 2_{\psi^1}, 3_{\psi_1}, 4_{\psi_1})$, $A(1_{\psi^1}, 2_{\psi^2}, 3_{\psi_1}, 4_{\psi_2})$

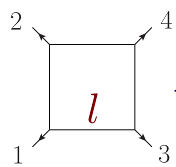
$$\mathcal{A}(1_{\psi^1}, 2_{\psi^2}, 3_{\psi_1}, 4_{\psi_2})_4^{(1)\beta} = \mathcal{A}_4^{(1)\mathcal{N}=4} - 8\sin^2\beta\, \mathrm{Tr}_{12}\mathrm{Tr}_{34} A(1234)^{(1)\mathrm{extra}}_{12;34} - 8\sin^2\beta\, \mathrm{Tr}_{14}\mathrm{Tr}_{23} A(1234)^{(1)\mathrm{extra}}_{14;23}$$

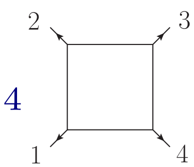
$$A(1234)^{(1)\mathrm{extra}}_{12;34} = \cos^2\beta \frac{\langle 34\rangle}{\langle 12\rangle}\left(\frac{1}{2} s_{12} s_{13}\right.$$


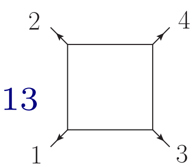
$$- s_{12}\Big($$


$$-$$


$$\left.\Big)\right)$$

$$A(1234)^{(1)\mathrm{extra}}_{14;23} = \frac{\langle 34\rangle}{\langle 12\rangle}\left[\cos^2\beta \frac{G[l,1,2,4]}{2 s_{12} s_{13}}\right.$$


$$+ s_{12} s_{14}$$


$$- s_{12} s_{13}$$


$$\left.\right]$$

Some 1-loop comments:

- results consistent with expected structure of IR divergences
  - -- most corrections are in fact IR-finite; consistent with structure of IR div’s
  - -- only small changes in the soft anomalous dimension matrix
- no real improvement over a finite “garden variety” $\mathcal{N}=1$ theory
  - -- except perhaps absence of incomplete cancellations (of bubbles)
- some details are as if there were more than $\mathcal{N}=1$ susy
  - -- some supersums are perfect squares
  - -- yet, no noticeable effect
- no immediate manifestation of tree-level numerator relations

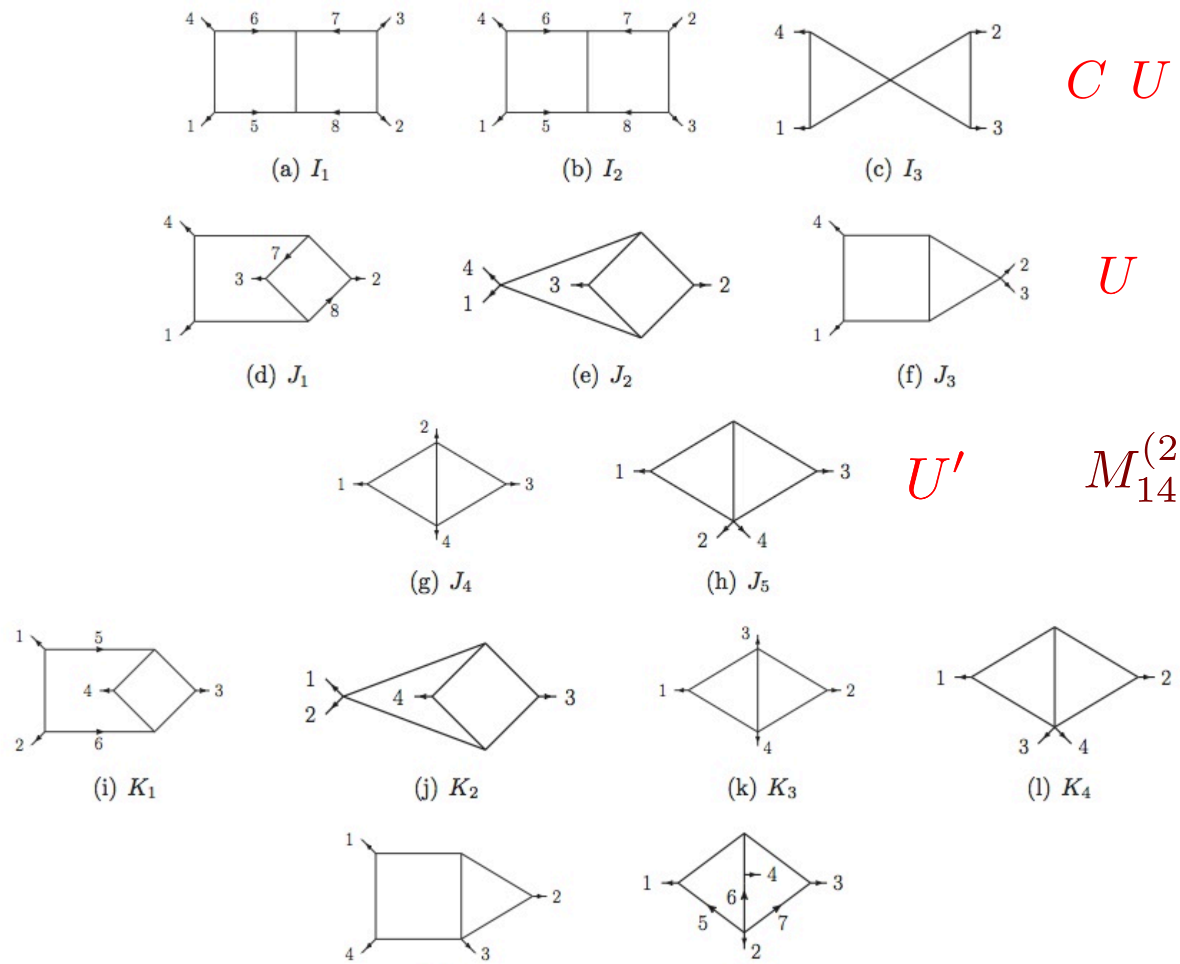
More explicit examples 4-point loop amplitudes: 2 loops

Jin, RR

Same classification:

- 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$

$$A(1234)^{(2)\beta}_{4;\,2\mathrm{tr}} = A(1234)^{(2)\mathcal{N}=4}_{4;\,2\mathrm{tr}} - 8\sin^2\beta\,\mathrm{Tr}_{13}\mathrm{Tr}_{24}A^{(2)\mathrm{extra}}_{13;24} - 8\sin^2\beta\,\mathrm{Tr}_{14}\mathrm{Tr}_{23}A^{(2)\mathrm{extra}}_{14;23}$$



(a) I_1 (b) I_2 (c) I_3

(d) J_1 (e) J_2 (f) J_3

(g) J_4 (h) J_5

(i) K_1 (j) K_2 (k) K_3 (l) K_4

(m) L_1 (n) L_2

C U Symmetries of $\frac{\langle 13\rangle^2}{\langle 23\rangle^2} A^{(2)\mathrm{extra}}_{14;23}$

$C : (1 \leftrightarrow 2, 3 \leftrightarrow 4)$

U $\quad U : (1 \leftrightarrow 4, 2 \leftrightarrow 3)$

U'

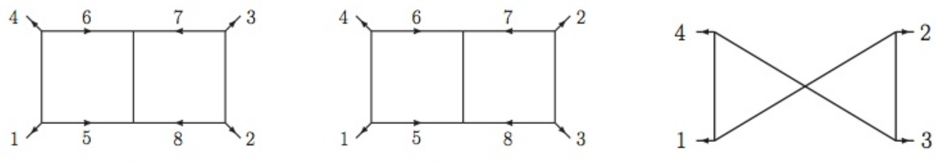
$$M^{(2)\mathrm{extra}}_{14;23} = \sum_i \alpha_i I_i$$

C

$$+ (1 + C)\sum_i \beta_i J_i$$

$$+ (1 + U)\sum_i \gamma_i K_i$$

$$+ (1 + U)(1 + C)\sum_i \delta_i L_i$$



(a) I_1 (b) I_2 (c) I_3

$$\alpha_1 = \tau_{1,4}\left(\tau_{1,8}^2 + \tau_{2,5}^2 + \tau_{4,7}^2 + \tau_{3,6}^2 + \tau_{1,2}(\tau_{1,8} + \tau_{2,5} + \tau_{4,7} + \tau_{3,6} - 2\tau_{1,4}) - \tau_{1,8}\tau_{2,5} - \tau_{4,7}\tau_{3,6}\right)$$

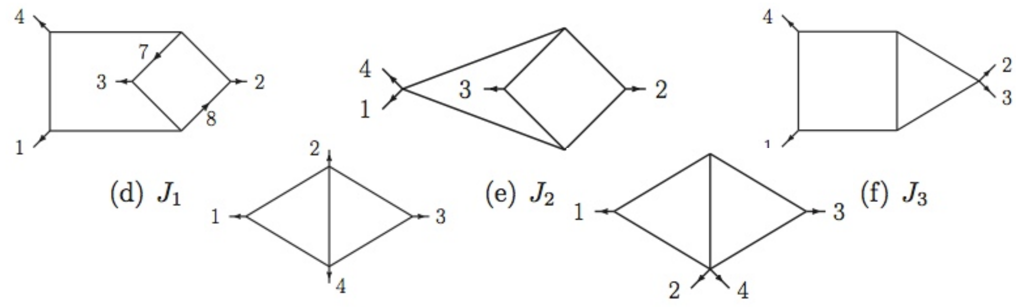
$$\alpha_2 = \tau_{1,4}\left(\tau_{1,8}^2 + \tau_{2,6}^2 + \tau_{4,7}^2 + \tau_{3,5}^2 + \tau_{1,2}(\tau_{1,8} + \tau_{2,6} + \tau_{4,7} + \tau_{3,5}) + \tau_{1,8}\tau_{2,6} + \tau_{4,7}\tau_{3,5}\right)$$

$$\alpha_3 = -4\tau_{1,3}\tau_{1,4}$$

$$\beta_1 = 2\tau_{1,4}\left(\tau_{1,8}^2 + \tau_{4,7}^2 + \tau_{1,3}(\tau_{1,8} + \tau_{4,7})\right)$$

$$\beta_2 = -2\tau_{1,4}^2 \qquad \beta_3 = 2\tau_{1,4}^2$$

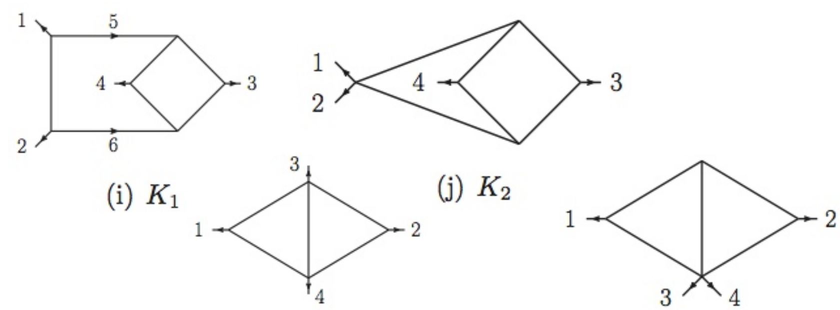
$$\beta_4 = 2\tau_{1,2} \qquad \beta_5 = (\tau_{1,4} - 2\tau_{1,2})$$



(d) J_1 (e) J_2 (f) J_3 (g) J_4 (h) J_5

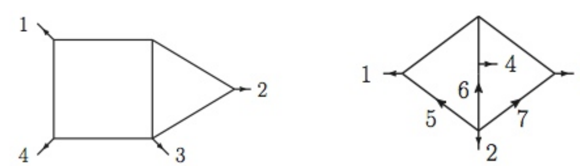
$$\gamma_1 = \tau_{1,2}\left(\tau_{1,5}^2 + \tau_{2,6}^2 + \tau_{1,2}(\tau_{1,5} + \tau_{2,6})\right)$$

$$\gamma_2 = 2\tau_{1,2}^2 \qquad \gamma_3 = 2\tau_{1,2} \qquad \gamma_4 = (\tau_{1,3} - \tau_{1,2})$$



(i) K_1 (j) K_2 (k) K_3 (l) K_4

$$\delta_1 = \tau_{1,4}^2 \qquad \delta_2 = -\tau_{1,4}\tau_{2,5} + \frac{1}{2}\tau_{12}(3\tau_{1,3} + \tau_{2,5} - 2\tau_{3,5} + 2\tau_{2,7})$$



(m) L_1 (n) L_2

FIX FORMULAE AND GRAPHS ON THIS SLIDE!

- The other trace structure $A_{13;24}^{(2)\text{extra}}$: similar structure with $A_{14;23}^{(2)\text{extra}}$ with a few twists

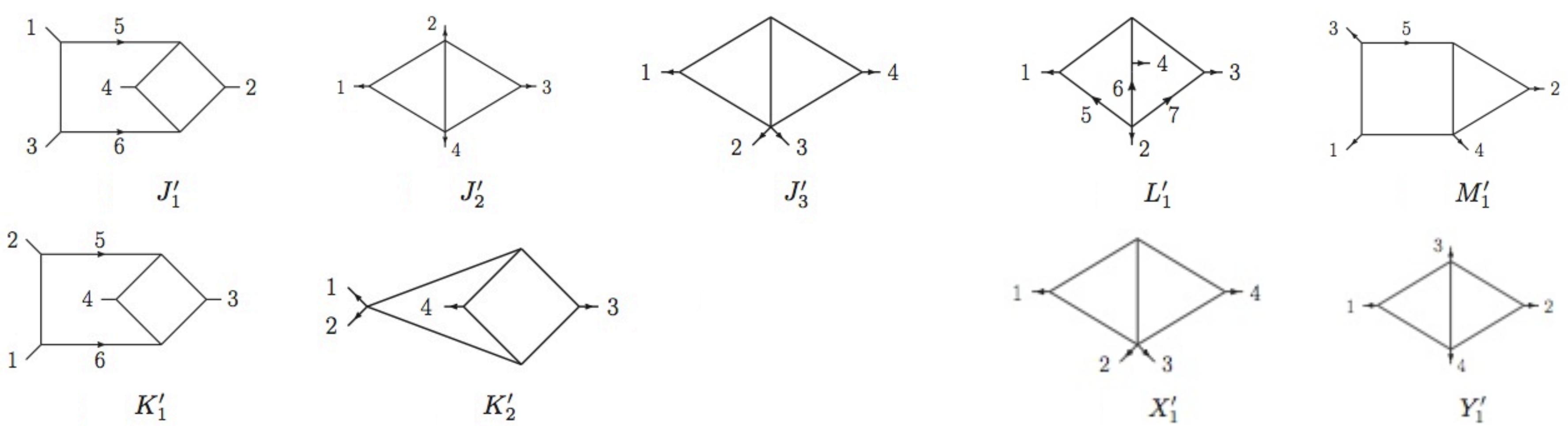
  - planar double-boxes are absent
  - additional symmetries: $C : (1 \leftrightarrow 2, 3 \leftrightarrow 4)$

    $U : (1 \leftrightarrow 3, 2 \leftrightarrow 4)$

    $E : (1 \leftrightarrow 3)$

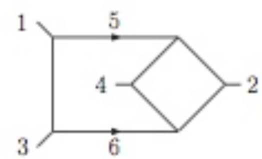
$$A_{*;*}^{(2)\text{extra}} = \frac{\langle 23 \rangle^2}{\langle 13 \rangle^2} M_{*;*}^{(2)\text{extra}}$$

$$M_{13;24}^{(2)\text{extra}} = (1+C) \sum_i \beta_i' J_i' + (1+U) \sum_i \gamma_i' K_i' + (1+U)(1+C)\, \delta_1' L_1'$$
$$+ (1+U')(1+C)(1+E)\, \epsilon_1' M_1' + (1+C)(1+E)\, \rho_1' X_1' + (1+U')(1+E)\, \eta_1' Y_1'$$

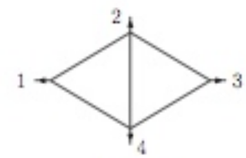


$$M_{13;24}^{(2)\mathrm{extra}} = (1+C)\sum_i \beta_i' J_i' + (1+U)\sum_i \gamma_i' K_i' + (1+U)(1+C)\,\delta_1' L_1'$$
$$+\,(1+U')(1+C)(1+E)\,\epsilon_1' M_1' + (1+C)(1+E)\,\rho_1' X_1' + (1+U')(1+E)\,\eta_1' Y_1'$$

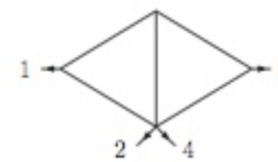
$$\beta_1' = -2\tau_{1,3}\left(\tau_{2,5}\tau_{2,6} + \tau_{4,5}\tau_{4,6}\right)$$
$$\beta_2' = -2\tau_{1,3}$$
$$\beta_3' = 2\tau_{1,3}$$



J_1'

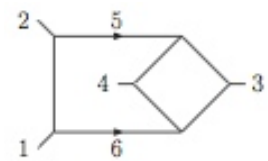


J_2'

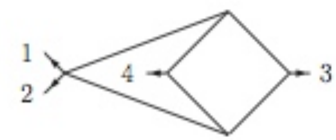


J_3'

$$\gamma_1' = \tau_{1,2}\left(\tau_{4,5}^2 + \tau_{3,6}^2 + \tau_{1,3}(\tau_{4,5} + \tau_{3,6})\right)$$
$$\gamma_2' = \tau_{1,2}^2$$

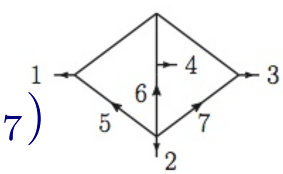


K_1'

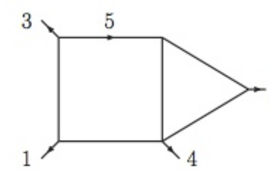


K_2'

$$\delta_1' = -\tfrac{1}{2}\tau_{1,3}\tau_{2,6}$$
$$+\,\tfrac{1}{2}(\tau_{1,4} - \tau_{1,2})(\tau_{2,5} - \tau_{2,7} + 2\tau_{4,5} - 2\tau_{4,7})$$
$$\epsilon_1' = 2\tau_{1,3}\tau_{2,5}$$

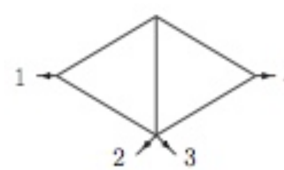


L_1'

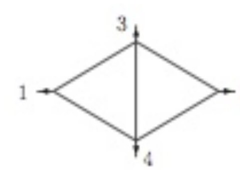


M_1'

$$\rho_1' = 4\tau_{1,3}$$
$$\eta_1' = -4\tau_{1,3}$$



X_1'



Y_1'

The UV behavior of this color structures

Slightly problematic to phrase it in terms of a critical dimension: higher-dimensional theory has a different field content from the four dimensional one

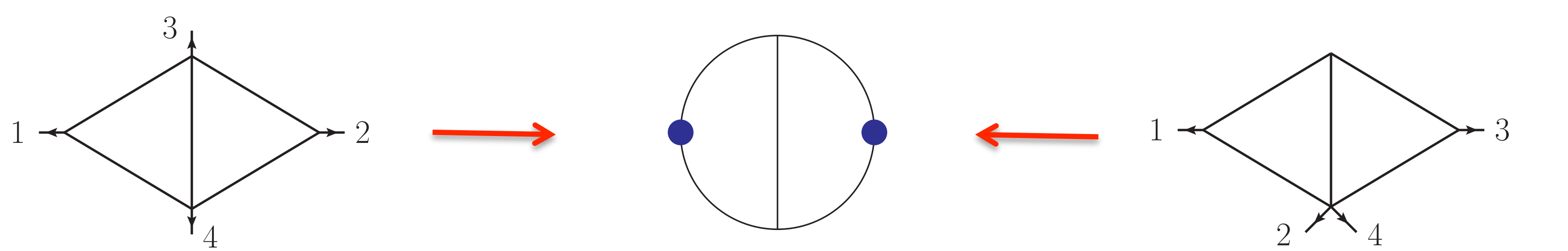
Nevertheless, analytically continuing the result to d>4 is a measure of the degree of divergence of the amplitude; reduces to standard critical dimension as $\beta \to 0$

Startegy: same as for $\mathcal{N} = 4$ sYM and for $\mathcal{N} = 8$ supergravity:

- expand at small external momenta
- use Lorentz-invariance to reorganize tensor integrals

Types of integrals:

- 7-prop. integrals: at most 2 loop mom. num. factors
- 6-prop. integrals: at most 1 loop mom. num. factors
- 5-prop. integrals: no loop mom. num. factors ➡ leading UV



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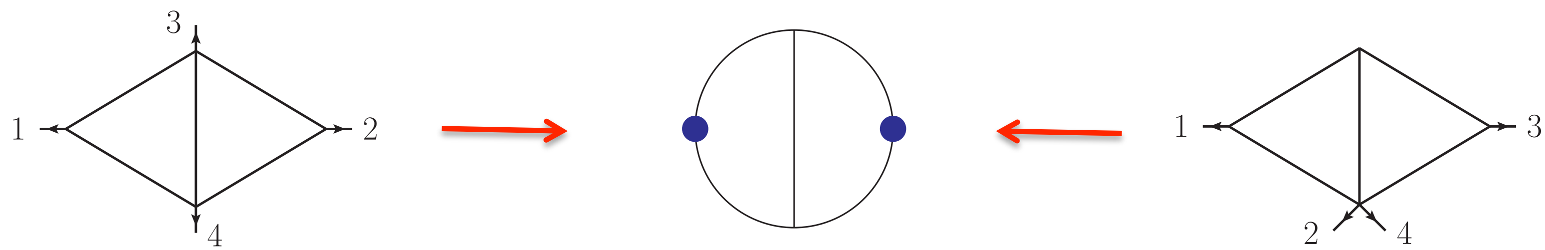

$$M^{\beta,(2)}_{14;23} : 2\beta_4 + 2\beta_5 + 2\gamma_3 + 2\gamma_4 = 0 \qquad M^{\beta,(2)}_{13;24} : 2\beta'_2 + 2\beta'_3 + 4\rho'_1 + 4\eta'_1 = 0$$

- Double-trace part of β-deformed 2-loop amplitude is finite in d=5; diverges only in d=6 – better than manifest $\mathcal{N}=1$ supersymmetry suggests:

$$D_c = 4 + \frac{2\mathcal{N}}{L} \quad \longrightarrow \quad D_c = 4 + \frac{4}{L}$$

- Superficially similar behavior to 2-trace terms at $L \geq 3$ in $\mathcal{N}=4$ sYM

$$D_c = 4 + \frac{6}{L} \quad \text{vs.} \quad D_c^{2\ \text{trace}} = 4 + \frac{8}{L}$$

Summary and some questions

- Despite extensive planar similarity with $\mathcal{N}=4$ sYM, the β -deformed theory is not "simple" at the non-planar level
- certain all-loop $\mathcal{O}(1/N)$ structures inherited from the $\mathcal{N}=4$ theory
- at 1-loop: expected properties of an finite $\mathcal{N}=1$ theory
- at 2-loops: $\mathcal{N}=1$ susy seems more powerful than it should
  -- Better UV convergence properties
  -- Is this an accident or a sign of further structure?
  -- Is "this" present in the $\mathcal{N}=4$ theory but obscured by maximal supersymmetry?
- Do the BCJ-like numerator relations play any role? Is there a generalization of color/kinematic duality to symmetric couplings? Is it possible to break susy with the $\mathcal{N}=4$ sYM field content while preserving conformal and dual conformal inv. and w/o d-str constants?

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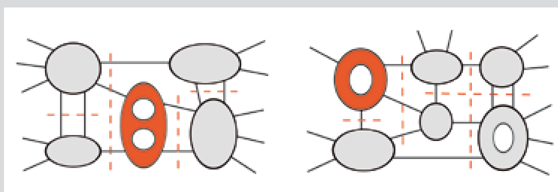
Special issue

Scattering amplitudes in gauge theories: progress and outlook

Guest Editors: R Roiban, M Spradlin and A Volovich



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