Hidden symmetry of four-point correlators and amplitudes in N=4 SYM: Part I

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Outline

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- ✔ Properties of the stress-tensor multiplet in $\mathcal{N} = 4$ SYM
- ✔ Structure of the four-point correlation function
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Introduction

- ✔ Early AdS/CFT correspondence: Duality between scattering amplitudes in AdS$^5 \times S_5$ and correlation functions of gauge invariant operators on the CFT boundary of AdS.
- ✔ Special role played by the half-BPS (or short, or CPO) operators. They:
  - ✗ Are dual to massive Kaluza-Klein modes in the compactification of type IIB supergravity
  - ✗ Have protected scaling dimension and 3-point functions
- ✔ The lowest dimension half-BPS operator, the $\mathcal{N}=4$ stress-tensor multiplet, contains the conserved currents of the theory, as well as the $\mathcal{N}=4$ SYM Lagrangian. Its correlation functions:
  - ✗ Are dual to scattering amplitudes of massless AdS states (gravitons, etc.)
  - ✗ Contain information, via the OPE, about the spectrum of all twist-2 operators, including $\Gamma_{\rm cusp}$
- ✔ Recently, a surprising duality between such correlators in the singular light-cone limit and gauge theory scattering amplitudes was discovered.
- ✔ Here we report on a new (or old but overlooked?) symmetry property of the 4-point correlators which leads to a very efficient procedure for constructing them at high loop orders.
- ✔ As an application, we can compute the Konishi anomalous dimension up to 5 loops, without a single Feynman graph!

$\mathcal{N}=4$ SYM stress-tensor multiplet in analytic superspace

- ✔ $\mathcal{N}=4$ SYM stress-tensor multiplet in ordinary superspace
  - ✗ Half-BPS operator made of 6 scalars Φ^I, $I=1,\dots,6$:

$$\mathcal{O}^{IJ}_{\mathbf{20'}} = \mathrm{tr}(\Phi^I\Phi^J) - 1/6\,\delta^{IJ}\,\mathrm{tr}(\Phi^K\Phi^K)$$

  - ✗ Lowest-weight state of the $\mathcal{N}=4$ stress-tensor supermultiplet:

$$\mathcal{T}(x,\theta^A,\bar\theta_A) = \mathcal{O} + \ldots + (\theta)^4\mathcal{L}_{\mathcal{N}=4} + \ldots + (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)T_{\mu\nu} + \ldots$$

  - ✗ $\mathcal{T}$ is not chiral, but depends on $\theta^A,\bar\theta_A$ ($A=1,2,3,4$) in a restricted half-BPS way
- ✔ $\mathcal{N}=4$ analytic (harmonic) superspace and half-BPS shortening:
  - ✗ Break $SU(4)\to SU(2)\times SU(2)'\times U(1)$ with the help of auxiliary harmonic coordinates $y^a_{a'}$

$$\theta^A_\alpha \;\to\; (\rho^a_\alpha\,,\theta^{a'}_\alpha)\,, \quad \text{with} \;\; \rho^a_\alpha = \theta^a_\alpha + \theta^{a'}_\alpha y^a_{a'}$$

  - ✗ half-BPS = Grassmann analyticity:

$$\mathcal{T} = \mathcal{T}(x^{\dot\alpha\alpha},\,\rho^a_\alpha,\,\bar\rho^{\dot\alpha}_{a'},\,y^a_{a'}) = \mathcal{O}(x,y) + \ldots + (\rho)^4\mathcal{L}_{\mathcal{N}=4}(x) + \ldots + (\rho\sigma^\mu\bar\rho)(\rho\sigma^\nu\bar\rho)T_{\mu\nu}(x) + \ldots$$

$\mathcal{N}=4$ SYM stress-tensor multiplet in analytic superspace II

- Lowest weight state has harmonic dependence

$$\mathcal{O}(x,y) = Y_I\,Y_J\,\mathcal{O}^{IJ}_{\mathbf{20}'}(x) = Y_I\,Y_J\ \mathrm{tr}\left(\Phi^I\Phi^J\right)\ ,$$

  where $Y^I(y)$, $Y^2=0$ are $SO(6)$ null vectors.

- Restrict the odd expansion to the chiral sector by setting $\bar\rho=0$:

$$\mathcal{T}(x,\rho,0,y) = \mathcal{O}(x,y) + \ldots + (\rho)^4\mathcal{L}_{\mathcal{N}=4}(x)$$

- $\mathcal{N}=4$ SYM on-shell action as an integral over $1/4$ superspace:

$$S_{\mathcal{N}=4} = \int d^4x\,\mathcal{L}_{\mathcal{N}=4}(x) = \int d^4x \int d^4\rho\,\mathcal{T}(x,\rho,0,y)$$

Correlation functions of the $\mathcal{N}=4$ stress-tensor multiplet

✔ $n-$point correlation function of analytic supermultiplets $\mathcal{T}(x,\rho,0,y)$

$$G_n = \langle \mathcal{T}(1)\ldots\mathcal{T}(n)\rangle \;=\; \sum_{k=0}^{n-4}\sum_{\ell=0}^{\infty} a^{\ell+k}\, G_{n;k}^{(\ell)}(1,\ldots,n)\,, \qquad a = g^2 N_c/(4\pi^2)$$

The $\ell-$loop correction $G_{n;k}^{(\ell)} \sim (\rho_i)^{4k}$ is a homogeneous polynomial in the odd variables

✔ Consider the four-point case $n=4 \;\Rightarrow\; k=0$: no ρ dependence in the chiral sector. So, we can replace $\mathcal{T}(x,\rho,0,y)$ by just the bosonic 1/2-BPS operator $\mathcal{O}(x,y)$:

$$G_4 = \langle \mathcal{O}(x_1,y_1)\ldots\mathcal{O}(x_4,y_4)\rangle \;=\; \sum_{\ell=0}^{\infty} a^{\ell}\, G_4^{(\ell)}(1,2,3,4)$$

✔ Tree level (with $x_{ij}^2 = (x_i - x_j)^2$, $y_{ij}^2 = (y_i - y_j)^2$)

$$G_4^{(0)}(1,2,3,4) = \frac{N_c^2-1}{(4\pi^2)^4}\left(\frac{y_{12}^2}{x_{12}^2}\frac{y_{23}^2}{x_{23}^2}\frac{y_{34}^2}{x_{34}^2}\frac{y_{41}^2}{x_{41}^2} + \frac{y_{12}^2}{x_{12}^2}\frac{y_{24}^2}{x_{24}^2}\frac{y_{34}^2}{x_{34}^2}\frac{y_{13}^2}{x_{13}^2} + \frac{y_{13}^2}{x_{13}^2}\frac{y_{23}^2}{x_{23}^2}\frac{y_{24}^2}{x_{24}^2}\frac{y_{41}^2}{x_{41}^2}\right) + \text{disconnected}$$

Correlation functions II

- ✔ Loop correction via Lagrangian insertions

$$a\frac{\partial}{\partial a}G_4 = \int d^4x_5 \, \langle \mathcal{O}(x_1,y_1)\ldots\mathcal{O}(x_4,y_4)\mathcal{L}_{\mathcal{N}=4}(x_5)\rangle$$

  - ✗ Repeat ℓ times: the $\ell-$loop 4-point function is given by the tree-level $(4+\ell)-$point function

$$G^{(0)}_{4+\ell;\ell}|_{\rho_1=\ldots=\rho_4=0} = \langle \mathcal{O}(x_1,y_1)\ldots\mathcal{O}(x_4,y_4)\mathcal{L}(x_5)\ldots\mathcal{L}(x_{4+\ell})\rangle^{(0)}(\rho_5)^4\ldots(\rho_{4+\ell})^4$$

    This is a particular component of the super-correlator on $(4+\ell)$ stress-tensor multiplets:

$$\langle \mathcal{T}(1)\ldots\mathcal{T}(4+\ell)\rangle_{\rho_1=\ldots\rho_4=0}$$

- ✔ Integrand of the 4-point function as a tree-level correlator of stress-tensor multiplets

$$G^{(\ell)}_4(1,2,3,4) = \int d^4x_5\ldots d^4x_{4+\ell}\left(\frac{1}{\ell!}\int d^4\rho_5\ldots d^4\rho_{4+\ell}\, G^{(0)}_{4+\ell;\ell}(1,\ldots,4+\ell)\right)$$

  What do we know about this tree-level correlator?

Correlation functions III

- ✔ Examples at one and two loops

$$G^{(0)}_{5;1}(1,2,3,4,5) = \frac{2\,(N_c^2-1)}{(4\pi^2)^5} \times \mathcal{I}_5 \times \frac{1}{\prod_{1\le i<j\le 5} x_{ij}^2}$$

$$G^{(0)}_{6;2}(1,2,3,4,5,6) = \frac{2\,(N_c^2-1)}{(4\,\pi^2)^6} \times \mathcal{I}_6 \times \frac{\frac{1}{48}\sum_{\sigma\in S_6} x^2_{\sigma_1\sigma_2} x^2_{\sigma_3\sigma_4} x^2_{\sigma_5\sigma_6}}{\prod_{1\le i<j\le 6} x_{ij}^2}$$

- ✔ Essential ingredient: nilpotent n-point superconformal invariant

$$\mathcal{I}_n|_{\rho_1=\ldots=\rho_4=0} = (x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2) \times R(1,2,3,4) \times (\rho_5)^4 \ldots (\rho_n)^4$$

$$R(1,2,3,4) = \frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{14}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{14}^2}(x_{13}^2 x_{24}^2 - x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2) + \text{5 other terms}$$

  - ✗ $\mathcal{I}_n$ can be constructed by using the odd part of $PSU(2,2|4)$ to restore $\rho_1,\ldots,\rho_4$.
  - ✗ $\mathcal{I}_n$ has fixed $SU(4)$ and conformal weights
  - ✗ Crucial property: $\mathcal{I}_n(1,\ldots,n)$ is fully permutation invariant.
- ✔ In summary: the $(4+\ell)-$point tree-level correlator has the general form

$$\langle \mathcal{T}(1)\ldots\mathcal{T}(4+\ell)\rangle_{\text{odd level }4\ell} = \mathcal{I}_{4+\ell} \times f^{(\ell)}(x_1,\ldots,x_{4+\ell})$$

$f^{(\ell)}$ is a permutation invariant function of $x_1,\ldots,x_{4+\ell}$ with conformal weight $(+4)$ at each point.

Hidden permutation symmetry of the integrand

- ✔ We predict the form of the four-point correlator at ℓ loops:

$$G_4^{(\ell)}(1,2,3,4) = \frac{2\,(N_c^2-1)}{(4\pi^2)^4} \times R(1,2,3,4) \times F^{(\ell)} \qquad \text{for } \ell \geq 1$$

$$F^{(\ell)}(x_1,x_2,x_3,x_4) = \frac{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2}{\ell!\,(4\pi^2)^\ell} \int d^4x_5 \ldots d^4x_{4+\ell}\, f^{(\ell)}(x_1,\ldots,x_{4+\ell})$$

$$f^{(\ell)}(x_1,\ldots,x_{4+\ell}) = \frac{P^{(\ell)}(x_1,\ldots,x_{4+\ell})}{\prod_{1\leq i<j\leq 4+\ell} x_{ij}^2}$$

  - ✗ The form of the denominator is dictated by the tree-level OPE of

$$\langle \mathcal{O}(1)\ldots\mathcal{O}(4)\mathcal{L}(5)\ldots\mathcal{L}(4+\ell)\rangle^{(0)} \sim R(1,2,3,4)\,(x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2)\, f^{(\ell)}(x)$$

  - ✗ The numerator $P^{(\ell)}$ is a homogeneous polynomial in x_{ij}^2 of conformal weight $(1-\ell)$ at each point, invariant under $S_{4+\ell}$ permutations of x_i.

- ✔ Loop corrections in all $SU(4)$ channels given by single function $F^{(\ell)}$: partial non-renormalization
- ✔ Examples at 1 and 2 loops:

$$P^{(1)}(x_1,\ldots,x_5) = 1$$

$$P^{(2)}(x_1,\ldots,x_6) = \frac{1}{48} \sum_{\sigma\in S_6} x^2_{\sigma(1)\sigma(2)} x^2_{\sigma(3)\sigma(4)} x^2_{\sigma(5)\sigma(6)}$$

- ✔ How do we fix the coefficients? See Gregory's talk...

From correlation functions to MHV amplitudes

- ✔ Wilson loops/Scattering amplitudes duality

$$\ln\left(A_n^{\rm MHV}/A_n^{\rm (tree)}\right) = \ln\left(W[C_n]\right) + O(1/N_c^2)$$

- ✔ Correlation functions/Wilson loops duality

$$\lim_{x_{i,i+1}^2\to 0} \ln\left(G_n/G_n^{\rm (tree)}\right) = 2\ln\left(W[C_n]\right) + O(1/N_c^2)$$

- ✔ Direct link: Correlation functions/Scattering amplitudes duality

$$\lim_{x_{i,i+1}^2\to 0} \ln\left(G_n/G_n^{\rm (tree)}\right) = 2\ln\left(A_n^{\rm MHV}/A_n^{\rm (tree)}\right) + O(1/N_c^2)$$

  - ✗ Amplitudes in terms of loop integrals:

$$\hat{A}_n^{\rm MHV} = 1 + \sum_{\ell=1}^{\infty} a^\ell \int \prod_{i=1}^{\ell} d^{4-2\epsilon}k_i I_\epsilon^{(\ell)}(k_1,\dots,k_\ell;p_1,\dots,p_n) \quad \text{with } \epsilon<0 \text{ and } p_i = x_{i,i+1}$$

  - ✗ Correlators in terms of loop integrals:

$$\lim_{x_{i,i+1}^2\to 0} \frac{G_n}{G_n^{\rm (tree)}} = 1 + \sum_{\ell=1}^{\infty} a^\ell \int \prod_{i=1}^{\ell} d^{4-2\epsilon}x_{0_i} G_\epsilon^{(\ell)}(x_{0_1},\dots,x_{0_\ell};x_1,\dots,x_n) \quad \text{with } \epsilon>0$$

  - ✗ Thus, we can predict the intgerands of all MHV 4-gluon amplitudes.

Conclusions

- ✔ Using only known basic properties of the four-point correlator of $\mathcal{N}=4$ stress-tensor multiplets, we unveiled a hidden, highly symmetric structure.
- ✔ This structure leads to a recursive procedure which, it seems, can predict the off-shell planar integrand of G_4 at any loop level.
- ✔ Two ingredients were essential for this:
  - ✗ $\mathcal{N}=4$ SUSY. It is known that the 2-loop correlator in a generic $\mathcal{N}=2$ conformal theory does not posses the permutation symmetry of the integrand.
  - ✗ The number of point is 4. For $n>4$ the nilpotent superconformal invariant $\mathcal{I}_n$ is not unique, so we have to find a number of functions $F^{(\ell)}$ and the permutation symmetry is lost. Still, we might be able to make some limited predictions in this case.
- ✔ It would be interesting to understand the intimate connection between our construction and the BCFW recursion of the momentum twistor approach. They look completely different, but give the same results, why?
- ✔ The highly predictable structure of G_4 is undoubtedly related to the integrability of $\mathcal{N}=4$ SYM. In particular, the 4-point integrals that we find should have some hidden structure, at the level of their symbols, for example.