# *The Harmony of Superstring Amplitudes: Implications from/to Field-Theory*

Stephan Stieberger, MPP München

Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

Amplitudes 2011
University of Michigan
Nov 10–13, 2011

I will report on my work with:

**Carlos Mafra**, DAMTP Cambridge

**Oliver Schlotterer**, AEI Potsdam

- Complete N-Point Superstring Disk Amplitude I.
  Pure Spinor Computation, `arXiv:1106.2645`

- Complete N-Point Superstring Disk Amplitude II.
  Amplitude and Hypergeometric Function Structure, `arXiv:1106.2646`

- Explicit BCJ Numerators from Pure Spinors, `arXiv:1104.5224`

*Harmony of scattering amplitudes from/in String Theory*

Scattering amplitudes in gauge and gravity theory

have a **remarkably rich yet simple structure**



Cabrillo Highway

**Detour via string theory to arrive at harmony !**

## *Harmony from/in String Theory*

Various **relations within or between** gravity and gauge theory scattering amplitudes suggest a **unification** within or between these theories of the sort **inherent to string theory** !

Tree–level: • Color decomposition of gauge theory amplitudes into sum over partial subamplitudes

• BCJ: relations within gauge theory between color and kinematics

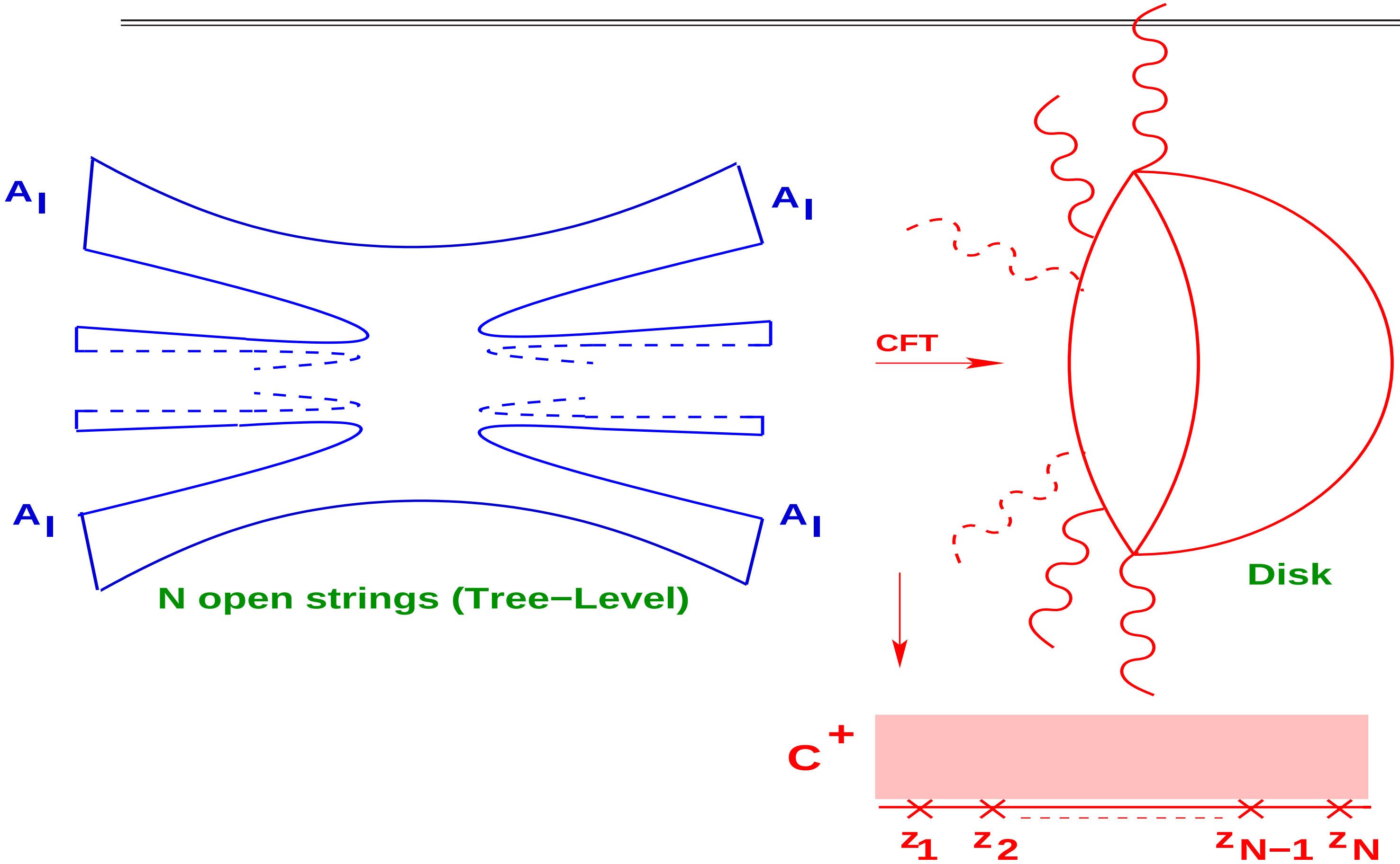
• KLT: relation between gravity and gauge theory

⋮

$\hookrightarrow$ (String) world–sheet derivation of amplitude relations

*(By applying world–sheet string techniques $\Longrightarrow$ new algebraic identities)*

# *Disk scattering of open strings*




$$\mathcal{A}(1,2,\ldots,N) = g_{YM}^{N-2} \sum_{\sigma\in S_N} \mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\ldots T^{a_{\sigma(N)}})\ A(\sigma(1),\sigma(2),\ldots,\sigma(N))$$

$A(1,2,\ldots,N)$ tree–level color–ordered $N$–leg partial amplitude (helicity subamplitude)

## *Disk scattering of open strings*

$N$ open string amplitude given by CFT computation
with gluon vertex operators $V_g(z)$:

$$A(1,\ldots,N;\alpha') \;=\; V_{\mathrm{CKG}}^{-1} \int\limits_{z_1<\ldots<z_N} \left( \prod_{j=1}^{N} dz_j \right) \langle V_g(z_1)\ldots\; V_g(z_N)\rangle$$

$$A(1,\ldots,N;\alpha') = V_{\mathrm{CKG}}^{-1} \int\limits_{z_1<\ldots<z_N} \left( \prod_{j=1}^{N} dz_j \right) \sum_{\mathcal{K}_I} \mathcal{K}_I \prod_{i<j}^{N} |z_i - z_j|^{s_{ij}}\; (z_i - z_j)^{n_{ij}^I}$$

$$s_{ij} = \alpha'(k_i + k_j)^2$$

$\hookrightarrow$ It would be desirable to obtain **simple** and **compact** result !

## *Full $N$–point open superstring amplitude*

Compact and short expression in terms of a *minimal basis* of $(N-3)!$ *building blocks*

$$A(1,2,\ldots,N;\alpha') = \sum_{\sigma\in S_{N-3}} A_{YM}(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N)\ F^{\sigma}_{(1,\ldots,N)}(\alpha')$$

Mafra, Schlotterer, St.St., `arXiv:1106.2645` and `arXiv:1106.2646`

$A_{YM}$ Yang–Mills subamplitudes

$F^{\sigma}(\alpha')$ generalized Euler integral
multiple Gaussian hypergeometric functions

**Pure spinor formalism** allows for remarkable **simplifications** to package kinematics and $\alpha'$–dependence

## *Structure of $N$–point superstring amplitude*

Remarks:

- $N = 4$ : $\begin{cases} A_{YM}(1,2,3,4) \ \rightarrow \ 2g_{YM}^2 \ \dfrac{1}{su} \ t_8(\xi_1,k_1,\xi_2,k_2,\xi_3,k_3,\xi_4,k_4) \\ F_{1234} \ \rightarrow \ \dfrac{\Gamma(1-s)\ \Gamma(1-u)}{\Gamma(1-s-u)} \end{cases}$ Green, Schwarz, 1982

- $D = 4$, MHV

$$A_{YM}(1^-,2^-,3^+,\ldots,N^+) = g_{YM}^{N-2} \ \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle N1\rangle} \ ,$$ Parke, Taylor, 1986; Berends, Giele, 1988

- $(N-3)!$ dimensional minimal basis of hypergeometric functions $F^\sigma$

## *Structure of $N-$point superstring amplitude*

- For any external state of SYM VM (FT = STTH Ward identities)

- Consider (adjacent) Eikonal Regge limit:

  ▷ Keep $2(N-3)$ kinematic invariants $s_{1i},\ s_{2j},\ s_{i,i+1} \neq 0$

  ▷ For the remaining $\frac{1}{2}(N-3)(N-4)$ kinematic invariants $s_{ij} \to 0$

Cheung, O'Connell, Wecht, `arXiv:1002.4674`

$$F^{\sigma}_{(1,\ldots,N)} \longrightarrow 0 \quad , \quad \sigma \neq (1,\ldots,N)$$

$$\Longrightarrow \quad A(1,2,\ldots,N;\alpha') \longrightarrow A_{YM}(1,2,\ldots,N)\ F(\alpha')$$

## Set of $(N-3)!$ basis functions $F^\sigma$

$$F^\sigma_{(1,\ldots,N)}(\alpha') = \int\limits_{z_i<z_{i+1}} \prod_{j=2}^{N-2} dz_j \left(\prod_{i<l} |z_{il}|^{s_{il}}\right) \left\{\prod_{k=2}^{N-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}}\right\} \quad , \quad \sigma \in S_{N-3}$$

$$F^\sigma_{(1,\ldots,N)}(\alpha') = \int\limits_{z_i<z_{i+1}} \prod_{j=2}^{N-2} dz_j \left(\prod_{i<l} |z_{il}|^{s_{il}}\right)$$
$$\times \left(\prod_{k=2}^{[N/2]} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}}\right) \left(\prod_{k=[N/2]+1}^{N-2} \sum_{n=k+1}^{N-1} \frac{s_{kn}}{z_{kn}}\right) \ , \ \sigma \in S_{N-3}$$

<u>$E.g.: N=5$</u>

$$\begin{aligned}
F^{(23)} &= s_{12}\ s_{34} \int_0^1 dx \int_0^1 dy\ x^{s_{45}}\ y^{s_{12}-1}\ (1-x)^{s_{34}-1}\ (1-y)^{s_{23}}\ (1-xy)^{s_{24}} \\
&= 1+\zeta(2)\ (s_1s_3 - s_3s_4 - s_1s_5) \\
&- \zeta(3)\ (s_1^2s_3 + 2s_1s_2s_3 + s_1s_3^2 - s_3^2s_4 - s_3s_4^2 - s_1^2s_5 - s_1s_5^2) + \mathcal{O}(\alpha'^4)\ , \\
F^{(32)} &= s_{13}\ s_{24} \int_0^1 dx \int_0^1 dy x^{s_{45}}\ y^{s_{12}}\ (1-x)^{s_{34}}\ (1-y)^{s_{23}}\ (1-xy)^{s_{24}-1} \\
&= \zeta(2)\ s_{13}\ s_{24} - \zeta(3)\ s_{13}\ s_{24}\ (s_1+s_2+s_3+s_4+s_5) + \mathcal{O}(\alpha'^4)\ .
\end{aligned}$$

with: $s_i \equiv s_{i,i+1} = \alpha'\ (k_i+k_{i+1})^2\ ,\ i+5 \equiv i$

$$B_N[n] = \int_0^1 dx_1 \ldots \int_0^1 dx_{N-3} \prod_{a=1}^{N-3} x_a^{1+a-N+n_a} \prod_{b=a}^{N-3} x_a^{2\alpha' k_{b+3}\left(k_1 + \sum_{j=a+3}^{b+2} k_j\right)}$$

$$\times \left(1 - \prod_{j=a}^{b} x_j\right)^{2\alpha' k_{2+a} k_{3+b} + n_{ab}} , \qquad b \geq a = 1, 2, \ldots, N-3 \ , \quad n_a, n_{ab} = 0, \pm 1$$

$\frac{1}{2}N(N-3)$ Laurent polynomials = number of kinematic invariants

| $N$ | dimension | function $F^\sigma$ | reference |
|---|---|---|---|
| 4 | 1 | ${}_2F_1$ | Green, Schwarz, et al., 1982 |
| 5 | 2 | ${}_3F_2$ | Medina, et al., `hep-th/0208121` |
| 6 | 6 | triple hypergeometric function $F^{(3)}$ | Oprisa, St.St., `hep-th/0509042` |
| 7 | 24 | multiple hypergeometric function | St.St., Taylor, `arXiv:0708.0574` |
| $N$ | $(N-3)!$ | multiple hypergeometric function | Mafra, Schlotterer, St.St. |

Structure of integrals is analyzed:

- ▷ *multiple pole* structure (dual channels)
- ▷ *transcendentality* properties
- ▷ *Gröbner basis* analysis to account for various integral relations

## $\alpha'$–expansion $\Longleftrightarrow$ multiple Euler–Zagier sums

E.g. N=7:

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \ \frac{x^{s_2}\ (1-x)^{s_3}\ y^{t_2}\ (1-y)^{s_4}\ z^{t_6}\ (1-z)^{s_5}\ w^{s_7}\ (1-w)^{s_6}}{(1-xy)\ (1-wz)\ (1-yz)}\ (1-wxyz)^{s_1-t_1+t_4-t_7}$$

$$\times\ (1-xy)^{-s_3-s_4+t_3}\ (1-wz)^{-s_5-s_6+t_5}\ (1-yz)^{-s_4-s_5+t_4}\ (1-wyz)^{s_5+t_1-t_4-t_5}\ (1-xyz)^{s_4-t_3-t_4+t_7}$$

$$= \mathcal{I}_0 + \mathcal{I}_{1a}\ (s_1+s_2+s_3+s_4+s_5+s_6+s_7) + \mathcal{I}_{1b}\ (t_1+t_2+t_3+t_4+t_5+t_6+t_7) + \mathcal{O}(\alpha'^2)$$

with **Multiple Euler–Zagier sums** $\mathcal{I}_0,\ \mathcal{I}_{1a},\ \mathcal{I}_{1b}$:

$$\mathcal{I}_0 = \int \frac{1}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n1,n_2=0\\ n_3=1}}^{\infty} \frac{1}{n_3\ (1+n_1)\ (n_1+n_2+1)\ (n_2+n_3)} = \frac{27}{4}\zeta(4)$$

$$\mathcal{I}_{1a} = \int \frac{\ln w}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n1,n_3=1\\ n_2=0}}^{\infty} \frac{1}{n_1\ n_3^2\ (n_1+n_2)\ (n_2+n_3)} = \frac{7}{2}\zeta(5) - 4\zeta(2)\zeta(3)$$

$$\mathcal{I}_{1b} = \int \frac{\ln y}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n1,n_2=1\\ n_3=0}}^{\infty} \frac{1}{n_1\ n_2\ (n_2+n_3)\ (n_1+n_3)^2} = -\frac{9}{2}\zeta(5) + \zeta(2)\zeta(3)$$

## *I. Color vs. kinematics in string theory*

String amplitude at fixed color ordering $(1,\ldots,N)$
combines two sectors or moduls

$$A(1,\ldots,N) \in \underbrace{\boxed{A_{YM,\pi}}}_{(N-3)!\ \text{basis}} \otimes \underbrace{\boxed{F^{\pi}(\alpha')}}_{\text{dual basis ?}}$$

Can the integrals $F^{\pi}$ be reduced to an $(N-3)!$ basis ?

<u>Or more generally:</u>

| Do we have an analog of<br>Kleiss–Kuijf and BCJ relations<br>in kinematic space ? |
|---|

*Recall:* $(N-3)!$ (independent) building blocks $A_{YM}$, i.e. for any $\Pi$:

$$A_{YM}(1_\Pi,\ldots,N_\Pi)=\sum_{\sigma\in S_{N-3}} K_\Pi^\sigma\ A_{YM,\sigma}$$

- $K_\Pi^\sigma$ can be derived from string theory monodromy relations $\alpha'\to 0$:

$$A(1,2,\ldots,N)\ +e^{i\pi s_{12}}\ A(2,1,3,\ldots,N-1,N)+e^{i\pi(s_{12}+s_{13})}\ A(2,3,1,\ldots,N-1,N)$$
$$+\ldots+e^{i\pi(s_{12}+s_{13}+\ldots+s_{1N-1})}\ A(2,3,\ldots,N-1,1,N)=0$$

(imaginary part) field–theory relations (BCJ relations):

$$s_{12}\ A_{YM}(2,1,3,\ldots,N-1,N)+\ldots+(s_{12}+s_{13}+\ldots+s_{1N-1})\ A(2,3,\ldots,N-1,1,N)=0$$

(real part) field–theory relations (Kleiss–Kuijf relations):

$$A_{YM}(1,2,\ldots,N)+A_{YM}(2,1,3,\ldots,N-1,N)+\ldots+A_{YM}(2,3,\ldots,N-1,1,N)=0$$

St.St., arXiv:0907.2211 & Bjerrum-Bohr, Damgaard, Vanhove, arXiv:0907.1425

- $K_\Pi^\sigma$ can also be derived *directly* from string theory: $$K_\Pi^\sigma=F_\Pi^\sigma(\alpha')\Big|_{\alpha'=0}$$

*Further insights from looking at different representations (basis) for same amplitude:*

$$A(1,\ldots,N)=\sum_{\pi\in S_{N-3}} A_{YM,\pi}\ F^{\pi}_{(1,\ldots,N)}(\alpha')$$

with some basis of $(N-3)!$ independent basis amplitudes $A_{YM,\pi}$ yields:

$$\boxed{F^{\sigma}_{(1,\ldots,N)}=\sum_{\pi\in S_{N-3}} (K^{-1})^{\sigma}_{\pi}\ F^{\pi}_{(1,\ldots,N)}}$$

$\Longrightarrow$ for a given **fixed color ordering** $(1,\ldots,N)$ any function $F^{\sigma}$ (referring to the **kinematics** $\boldsymbol{\sigma}$) may be expressed in terms of a basis of $(N-3)!$ functions $F^{\pi}$ referring to the **kinematics** $\boldsymbol{\pi}$.

## *Color vs. kinematics in string theory*

$\Longrightarrow$ We have found a **dual system of equations** for the functions $F$

( **complementary** to BCJ and KK relations )

reducing the set of functions to a **minimal basis** of dimension $(N-3)!$

| reduction to | $A_{YM,\pi}$ | $F^{\pi}(\alpha')$ |
|---|---|---|
| $(N-2)!$ | Kleiss–Kuijf | partial fraction $\frac{1}{z_{12}z_{23}} - \frac{1}{z_{13}z_{23}} = \frac{1}{z_{12}z_{13}}$ <br> $\Longrightarrow$ Gröbner basis analysis |
| $(N-3)!$ | BCJ | tot. derivatives $0 = \int \partial_k \prod_{i<j} \lvert z_{ij}\rvert^{s_{ij}} \ldots$ |

## II. Effective $D$–brane action ($\alpha'$–expansion)

Series of higher derivative terms ($\alpha'$–corrections to SYM):

$$\mathcal{L}^{Dp}_{\text{effective}} = \text{Tr} \sum_{n\geq 4}^{\infty} \sum_{m=0}^{\infty} \alpha'^{\,n-2+m} \sum'_{\substack{i_r\in\mathbf{N},\, i_1>1 \\ i_1+\ldots+i_d=n-2+m}} \zeta(i_1,\ldots,i_d)\; d_{m,n,\vec{i}}\; (t^{\vec{i}}_{m,n}\, D^{2m} F^n)$$

$$\zeta(i_1,\ldots,i_d) = \sum_{n_1>\ldots>n_d>0} \prod_{r=1}^{d} n_r^{-i_r}, \quad i_r \in \mathbf{N},\; i_1 > 1$$

transcendentality degree

$$\sum_{r=1}^{d} i_r = n-2+m$$

and depth $d$

| | |
|---|---|
| $\alpha'^0$ 1 | $\mathbf{F^2}$ |
| $\alpha'$ 0 | $F^3$   $D^2F^2$ |
| $\alpha'^2$ $\zeta(2)$ | $\mathbf{F^4}$   $D^2F^3$   $D^4F^2$ |
| $\alpha'^3$ $\zeta(3)$ | $\mathbf{F^5}$   $\mathbf{D^2F^4}$   $D^6F^2$ |
| $\alpha'^4$ $\zeta(4)$ | $\mathbf{F^6}$   $\mathbf{D^4F^4}$   $\mathbf{D^2F^5}$ |
| $\alpha'^5$ $\zeta(2)\zeta(3)$, $\zeta(5)$ | $\mathbf{F^7}$   $\mathbf{D^6F^4}$   $\mathbf{D^4F^5}$   $\mathbf{D^2F^6}$ |
| $\vdots$ | ...   ...   ...   ... |

**Degree of transcendentality $\Longleftrightarrow$ order in $\alpha'$–expansion**

## *Arrange higher derivative terms*

String contact terms: Non–trivial relations to field–theory

$$A(1,2,\ldots,N;\alpha') = \sum_{\sigma\in S_{N-3}} \underbrace{A_{YM}(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N)}_{\text{all kinematics}} \ \underbrace{F^{\sigma}_{(1,\ldots,N)}(\alpha')}_{\substack{\text{expansion}\\ \text{coefficients}}}$$

*E.g.:* $N=4$ :

$$\mathcal{A}(1,2,3,4) = A_{YM}(1,2,3,4)\ F$$

$$\Longrightarrow \boxed{A_{F^4}(1,2,3,4) = s\ u\ A_{YM}(1,2,3,4)}$$

# *Arrange higher derivative terms*

E.g.: $N = 5$ : $\mathcal{A}(1,2,3,4,5) = A_{YM}(1,2,3,4,5)\, F_1 + A_{YM}(1,3,2,4,5)\, F_2$

$$\mathcal{A}(1,2,3,4,5) = A_{YM}(1,2,3,4,5)\, C_1 + A_{F^4}(1,2,3,4,5)\, C_2$$

$$\Longrightarrow \boxed{\begin{aligned} A_{F^4}(1,2,3,4,5) &= (s_{12}s_{34} - s_{34}s_{45} - s_{12}s_{51})\; A_{YM}(1,2,3,4,5) \\ &+ \; s_{13}\, s_{24}\; A_{YM}(1,3,2,4,5) \end{aligned}}$$

Higher derivative terms

- *organized* according to the *YM amplitudes* $A_{YM}$
- *constructed* by YM amplitudes $A_{YM}$
  (serve as *building blocks* in the effective action) with the expansion coefficients encoded in the functions $F$.
- *entirely described* in terms of the fundamental *YM three-vertices*
  (as a consequence of BCJ).

## *III. Degree of transcendentality in the $\alpha'$–expansion*

Transcendental power series expansion:

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz\ \frac{I_6(x,y,z)}{xyz} = \frac{1}{s_1 s_5 t_1} - \zeta(2)\ \left(\frac{s_3}{s_1 s_5} + \frac{s_4}{s_1 t_1} + \frac{s_2}{s_5 t_1}\right)$$
$$+\zeta(3)\ \left(\frac{s_3+s_4-t_3}{s_1} + \frac{s_2+s_3-t_2}{s_5} + \frac{s_3^2+s_3t_1}{s_1 s_5} + \frac{s_4^2+s_4s_5}{s_1 t_1} + \frac{s_2^2+s_1s_2}{s_5 t_1}\right) + \mathcal{O}(\alpha').$$

To each power $\alpha'^n$ in $\alpha'$ a Riemann zeta constant of fixed weight $n+3$ (with $n \geq -1$) appears

with: $I_6(x,y,z) = x^{s_5}\, y^{t_1}\, z^{s_1}\, (1-x)^{s_4}\, (1-y)^{s_3}\, (1-z)^{s_2}\, (1-xy)^{s_{35}}\, (1-yz)^{s_{24}}\, (1-xyz)^{s_{25}}$

On the other hand:

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \; \frac{I_6(x,y,z)}{(1-xy)(1-yz)} = 2\;\zeta(2) + [2\;\zeta(2) - 4\;\zeta(3)]\;\;(t_1+t_2+t_3)$$
$$-\,[2\;\zeta(2) - \zeta(3)]\;\;(s_1+s_2+s_3+s_4+s_5+s_6) + \mathcal{O}(\alpha^2)\;,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \; \frac{I_6(x,y,z)}{(1-xyz)^2} = \zeta(2) + \zeta(2)\;\;(s_3+s_6-t_2-t_3)$$
$$-\zeta(3)\;\;(s_1+s_2+2s_3+s_4+s_5+2s_6+t_1-t_2-t_3) + \mathcal{O}(\alpha^2)\;,$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \; \frac{I_6(x,y,z)}{(1-x)(1-y)(1-z)(1-xyz)} = \alpha'^{-2}\;(\ldots) + \alpha'^{-1}\;(\ldots)$$
$$+\alpha'^{0}\;[\;(\ldots) + \zeta(2)\;(\ldots)\;] + \alpha'\;[\;(\ldots) + \zeta(2)\;(\ldots) + \zeta(3)\;(\ldots)\;] + \mathcal{O}(\alpha^2)$$

## *Transcendentality criterion from gluon amplitudes*

$\hookrightarrow$ Regard the rational functions as originating from a CFT computation of a six–gluon amplitude:

$$A(1,\ldots,N) = \langle c(z_1)c(z_{N-1})c(z_N)\rangle \times \left(\prod_{l=2}^{N-2}\int_{z_{l-1}}^{1} dz_l\right) \langle V_g^{(-1)}(z_i)\; V_g^{(-1)}(z_j)\; \prod_{l\neq i,j}^{N} V_g^{(0)}(z_l)\rangle$$

with vertex operators:

$$V_g^{(-1)}(z) = g_A\; T^a\; e^{-\phi}\; \xi_\mu\; \psi^\mu\;\; e^{ik_\rho X^\rho}$$
$$V_g^{(0)}(z) = T^a\, \frac{g_A}{(2\alpha')^{1/2}}\, \xi_\mu \Big[\, i\partial X^\mu + 2\alpha'\; (k_\lambda \psi^\lambda)\; \psi^\mu \,\Big]\; e^{ik_\rho X^\rho}\;,$$

and correlators:

$$\langle \partial X^\mu(z_1) X^\nu(z_2)\rangle = -\frac{2\alpha' \eta^{\mu\nu}}{z_{12}}\;,\; \langle \psi^\mu(z_1)\psi^\nu(z_2)\rangle = \frac{\eta^{\mu\nu}}{z_{12}}$$

## Degree of transcendentality in the $\alpha'$–expansion

More details and structure in $z$–space:

$$\left[\begin{array}{c} z_1{=}0,\ z_2{=}xyz \\ z_3{=}xy,\ z_4{=}x,\ z_5{=}1 \end{array}\right]$$

$$\frac{1}{(1-xyz)^2} \simeq \frac{1}{z_{13}z_{14}z_{25}^2z_{36}z_{46}}\ ,$$

$$\frac{1}{(1-x)(1-y)(1-z)(1-xyz)} \simeq \frac{1}{z_{16}^2z_{23}z_{25}z_{34}z_{45}}$$

$$\int z_{ij}^{s_{ij}-2}\, r(z_{kl}) = \frac{1}{s_{ij}-1}\int r(z_{kl})\, \partial_{z_i} z_{ij}^{s_{ij}-1} = -\frac{1}{s_{ij}-1}\int z_{ij}^{s_{ij}-1}\, \partial_{z_i} r(z_{kl})$$

$$\frac{1}{xyz} \simeq \frac{z_{16}^2}{z_{12}z_{13}z_{14}z_{15}z_{26}z_{36}z_{46}z_{56}} \longrightarrow \frac{1}{z_{12}z_{13}z_{14}z_{15}}\ ,$$

$$\frac{1}{(1-xy)(1-yz)} \simeq \frac{1}{z_{13}z_{15}z_{24}z_{26}z_{35}z_{46}} \longrightarrow \frac{1}{z_{13}z_{15}z_{24}z_{35}}\ ,$$

taking into account: $z_6 = z_\infty = \infty$
and $c$–ghost correlator $\langle c(z_1)c(z_5)c(z_6)\rangle = z_{15}z_\infty^2$

## *Transcendentality criterion from gluon amplitudes*

$$\frac{1}{xyz} \simeq \frac{1}{z_{12}z_{13}z_{14}z_{15}} \simeq (\xi_1\xi_6)(\xi_2 k_1)(\xi_3 k_1)(\xi_4 k_1)(\xi_5 k_1)$$

$$\xi_1^{\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_3}\xi_4^{\mu_4}\xi_5^{\mu_5}\xi_6^{\mu_6}k_1^{\lambda}k_1^{\sigma}k_1^{\rho}k_1^{\tau}\ \langle\psi_1^{\mu_1}\psi_6^{\mu_6}\rangle\langle\partial X_2^{\mu_2}X_1^{\lambda}\rangle\langle\partial X_3^{\mu_3}X_1^{\sigma}\rangle\langle\partial X_4^{\mu_4}X_1^{\rho}\rangle\langle\partial X_5^{\mu_5}X_1^{\tau}\rangle$$

$$\frac{1}{(1-xy)(1-yz)} \simeq \frac{1}{z_{13}z_{15}z_{24}z_{35}} \simeq (\xi_2\xi_6)(\xi_1 k_3)(\xi_3 k_5)(\xi_4 k_2)(\xi_5 k_1)$$

$$\xi_1^{\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_3}\xi_4^{\mu_4}\xi_5^{\mu_5}\xi_6^{\mu_6}k_1^{\lambda_1}k_2^{\lambda_2}k_3^{\lambda_3}k_5^{\lambda_5}\ \langle\psi_2^{\mu_2}\psi_6^{\mu_6}\rangle\underline{\langle\partial X_1^{\mu_1}X_3^{\lambda_3}\rangle\langle\partial X_3^{\mu_3}X_5^{\lambda_5}\rangle}\langle\partial X_4^{\mu_4}X_2^{\lambda_2}\rangle\underline{\langle\partial X_5^{\mu_5}X_1^{\lambda_1}\rangle}$$

$$\xi_1^{\mu_1}\xi_2^{\mu_2}\xi_3^{\mu_3}\xi_4^{\mu_4}\xi_5^{\mu_5}\xi_6^{\mu_6}k_1^{\lambda_1}k_2^{\lambda_2}k_3^{\lambda_3}k_5^{\lambda_5}\ \langle\psi_2^{\mu_2}\psi_6^{\mu_6}\rangle\underline{\langle\psi_1^{\mu_1}\psi_3^{\lambda_3}\rangle\langle\psi_3^{\mu_3}\psi_5^{\lambda_5}\rangle}\langle\partial X_4^{\mu_4}X_2^{\lambda_2}\rangle\underline{\langle\psi_5^{\mu_5}\psi_1^{\lambda_1}\rangle}$$

However:

$$\langle\partial X_1 X_3\rangle\langle\partial X_3 X_5\rangle\langle\partial X_5 X_1\rangle - \langle\psi_1\psi_3\rangle\langle\psi_3\psi_5\rangle\langle\psi_5\psi_1\rangle = 0$$

## *IV. Higher–point closed superstring amplitudes*

(Color ordered) gluon amplitudes give rise to graviton amplitudes in type I or Type II superstring theory (field–theory for $\alpha' \to 0$)

At tree–level: **gravity = gauge theory $\otimes$ gauge theory**

Amplitudes (on–shell $S$–matrix): KLT relations

$$
\begin{aligned}
M_4(1,2,3,4)_{S^2} &= (2\alpha'\pi)^{-1}\ \sin(\pi s_{12})\ \overline{A}_4(1,2,3,4)_{D_2}\ A_4(1,2,4,3)_{D_2} \\
M_5(1,2,3,4,5)_{S^2} &= (2\alpha'\pi)^{-2}\Big\{\ \sin(\pi s_{12})\ \sin(\pi s_{34})\ \overline{A}_5(1,2,3,4,5)_{D_2}\ A_5(2,1,4,3,5)_{D_2} \\
&\qquad + \sin(\pi s_{13})\ \sin(\pi s_{24})\ \overline{A}_5(1,3,2,4,5)_{D_2}\ A_5(3,1,4,2,5)_{D_2}\Big\} \\
&\ \vdots
\end{aligned}
$$

E.g.: $M(1^-,2^-,3^+,4^+) = \left(\frac{\kappa}{2}\right)^2 \frac{\langle 12\rangle^8\ [12]}{N(4)\ \langle 34\rangle} \frac{B(s_{12},s_{14})}{B(-s_{12},-s_{14})} \to \left(\frac{\kappa}{2}\right)^2 \frac{\langle 12\rangle^8\ [12]}{N(4)\ \langle 34\rangle}$

with: and: $\langle ij\rangle[ij] = s_{ij} = \alpha' k_i k_j,\quad N(n) = \prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\langle ij\rangle$

# Tree–level higher order gravitational couplings

For $N=4$, use: $$\frac{B(s,u)}{B(-s,-u)} = -e^{-2\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{2n+1}(s^{2n+1}+t^{2n+1}+u^{2n+1})}$$

| | $N=4$ | $N=5$ | $N=6$ | $N=7$ | $N=8$ |
|---|---|---|---|---|---|
| $\alpha'^3\ \zeta(3)$ | $R^4$ | | | | |
| $\alpha'^4$ ~~$\zeta(4)$~~ | ~~$D^2R^4$~~ | ~~$\boldsymbol{R^5}$~~ | | | |
| $\alpha'^5\ \zeta(5)$ | $D^4R^4$ | $\boldsymbol{D^2R^5}$ | $\boldsymbol{R^6}$ | | |
| $\alpha'^5$ ~~$\zeta(2)\zeta(3)$~~ | ~~$D^4R^4$~~ | ~~$\boldsymbol{D^2R^5}$~~ | ~~$\boldsymbol{R^6}$~~ | | |
| $\alpha'^6\ \zeta(3)^2$ | $D^6R^4$ | $\boldsymbol{D^4R^5}$ | $\boldsymbol{D^2R^6}$ | $R^7$ ? | |
| $\alpha'^6\ \zeta(6)$ | ~~$D^6R^4$~~ | ~~$\boldsymbol{D^4R^5}$~~ | ~~$\boldsymbol{D^2R^6}$~~ | $R^7$ ? | |
| $\alpha'^7\ \zeta(7)$ | $D^8R^4$ | $\boldsymbol{D^6R^5}$ | $\boldsymbol{D^4R^6}$ | $D^2R^7$ ? | $R^8$ ? |
| $\alpha'^7\ \zeta(3)\zeta(4)$ | ~~$D^8R^4$~~ | ~~$\boldsymbol{D^6R^5}$~~ | ~~$\boldsymbol{D^4R^6}$~~ | $D^2R^7$ ? | $R^8$ ? |
| $\alpha'^7\ \zeta(2)\zeta(5)$ | ~~$D^8R^4$~~ | ~~$\boldsymbol{D^6R^5}$~~ | ~~$\boldsymbol{D^4R^6}$~~ | $D^2R^7$ ? | $R^8$ ? |
| $\alpha'^8\ \zeta(3)\zeta(5)$ | $D^{10}R^4$ | $\boldsymbol{D^8R^5}$ | $\boldsymbol{D^6R^6}$ | $D^4R^7$ ? | $D^2R^8$ ? |
| $\alpha'^8\ \zeta(8)$ | ~~$D^{10}R^4$~~ | ~~$\boldsymbol{D^8R^5}$~~ | ~~$\boldsymbol{D^6R^6}$~~ | $D^4R^7$ ? | $D^2R^8$ ? |
| $\alpha'^8\ \zeta(2)\zeta(3)^2$ | ~~$D^{10}R^4$~~ | ~~$\boldsymbol{D^8R^5}$~~ | ~~$\boldsymbol{D^6R^6}$~~ | $D^4R^7$ ? | $D^2R^8$ ? |
| $\alpha'^8\ \zeta(5,3)$ | ~~$D^{10}R^4$~~ | ~~$\boldsymbol{D^8R^5}$~~ | ~~$\boldsymbol{D^6R^6}$~~ | $D^4R^7$ ? | $D^2R^8$ ? |

St.St. arXiv:0910.0180

## *Relevance and implications to field–theory*

$\Longrightarrow$ Constraints on higher order gravitational couplings:
very restricted sets of MZVs appear

Results constrain candidate counter terms in N=8 SUGRA:
serve as unique candidate counter terms for $D = 4, N = 8$
SUGRA for loop levels $L \leq 6$

[ Terms $D^4R^4$ and $D^6R^4$ represent the only local
supersymmetric and $SU(8)$–symmetric operators
invariant under $D = 4, N = 8$ SUGRA
(matrix elements fulfill SUSY Ward identities) ]

$D^4R^4$, $D^6R^4$ have non–vanishing single–soft scalar limits
$\Longrightarrow$ operators violate continous $E_{7(7)}$–symmetry
$\Longrightarrow$ no counter terms at 5– and 6–loop

Beisert, Elvang, Freedman, Kiermaier, Morales, St. St. `arXiv:1009.1643`

# Multi zeta values (MZVs)

Many relations over $\mathbf{Q}$, e.g.:

$$\zeta(2,1) = 2\,\zeta(3) \quad , \quad \zeta(4,1) = 2\,\zeta(5) - \zeta(2)\,\zeta(3)$$

$$\zeta(5,3) = -\frac{5}{2}\,\zeta(6,2) - \frac{21}{25}\,\zeta(2)^4 + 5\,\zeta(3)\,\zeta(5)$$

$$\vdots$$

The set of integral linear combinations of MZVs is a ring

e.g.: $\zeta(m)\,\zeta(n) = \zeta(m,n) + \zeta(n,m) + \zeta(m+n)$

Zagier: For a given weight $w \in \mathbf{N}$ the dimension $d_w$ of the space spanned by MZVs: $\boldsymbol{d_w = d_{w-2} + d_{w-3}}$, $d_0, d_1 = 0$,

| $w$ | $d_w$ | basis |
|---|---|---|
| 2 | 1 | $\zeta(2)$ |
| 3 | 1 | $\zeta(3)$ |
| 4 | 1 | $\zeta(2)^2$ |
| 5 | 2 | $\zeta(5)$, $\zeta(2)\zeta(3)$ |
| 6 | 2 | $\zeta(2)^3$, $\zeta(3)^2$ |
| 7 | 3 | $\zeta(7)$, $\zeta(2)\zeta(5)$, $\zeta(3)\zeta(2)^2$ |
| 8 | 4 | $\zeta(2)^4$, $\zeta(2)\zeta(3)^2$, $\zeta(3)\zeta(5)$, $\zeta(5,3)$ |
| 9 | 5 | $\zeta(2)^3\zeta(3)$, $\zeta(3)^3$, $\zeta(2)^2\zeta(5)$, $\zeta(2)\zeta(7)$, $\zeta(9)$ |
| 10 | 7 | $\zeta(2)^5$, $\zeta(2)^2\zeta(3)^2$, $\zeta(2)\zeta(3)\zeta(5)$, $\zeta(5)^2$, $\zeta(3)\zeta(7)$, $\zeta(2)\zeta(5,3)$, $\zeta(7,3)$ |

# *Gravitational amplitudes in superstring theory*

*Question:* Can we also cast the gravity amplitude $M$ into compact form with simple building blocks ?

KLT:

$$M(1,\ldots,N)_{S^2} \sim \sum_{\sigma,\rho} e^{i\pi\phi(\sigma,\rho)}\ \overline{A}_N(\rho)_{D_2}\ A_N(\rho)_{D_2}$$

sum over $\frac{1}{2}(N-1)! \times \frac{1}{2}(N-1)!$ open string amplitudes

$\hookrightarrow$ With our gluon results we can cast $M(1,2,\ldots,N)$ in terms of a basis of $(N-3)! \times (N-3)!$ building blocks $\overline{A}_{YM,\sigma} \times A_{YM,\pi}$:

$$M(1,\ldots,N)_{S^2} = \sum_{\sigma,\rho\in S_{N-3}} \mathcal{M}(\rho,\sigma)\ \overline{A}_{YM}(\rho)\ A_{YM}(\sigma)$$

## *Gravitational amplitudes in superstring theory*

For $N = 4$:

$$M(1,2,3,4) = \frac{B(s_{12},s_{14})}{B(-s_{12},-s_{14})} \; \frac{s_{12}\; s_{23}}{s_{13}} \; |A_{YM}(1,2,3,4)|^2$$

For $N = 5$:

$$\begin{aligned} M(1,2,3,4,5)_{FT} \;&=\; (s_1 s_2 s_4 + s_1 s_3 s_4 + s_1 s_3 s_5 + s_2 s_3 s_5 + s_2 s_4 s_5)^{-1} \\ &\times \; \{ \; s_{12}\; s_{23}\; s_{34}\; s_{45}\; s_{51}\; |A_{YM}(1,2,3,4,5)|^2 \\ &\quad -s_{13}\; s_{14}\; s_{24}\; s_{25}\; s_{35}\; |A_{YM}(1,3,5,2,4)|^2 \; \} \end{aligned}$$

$\hookrightarrow$ more investigation on world–sheet symmetries

C. Mafra, O. Schlotterer, S. Stieberger, work in progress.