Four-graviton amplitudes in string theory

Pierre Vanhove

Amplitude 2011
Ann Arbor, Novembre 10-13, 2011
based on work 1111.2983
Michael B. Green, Stephen D. Miller

String theory S-matrix

We are interested in the properties of BPS protected interactions in string theory

String theory interactions are invariant under the action of the extended supersymmetries and duality symmetries

These symmetries put strong constraints on

- Non-perturbative contributions: black hole contributions
- the ultraviolet behaviour of the low-energy limit

String theory interactions I

- At low-energy the four-graviton of type II string theory compactified on tori, the S-matrix can be separated into an analytic and non-analytic terms

$$A_D(s,t,u) = A_D^{\text{analytic}}(s,t,u) + A_D^{\text{non-analytic}}(s,t,u)$$

- Due to the maximal supersymmetries the amplitudes factorize the dimension eight operators $\mathcal{R}^4$

- The analytic part $A_D^{\text{analytic}}(s,t,u) = T_D(s,t,u)\,\mathcal{R}^4$ is organized by increasing powers of $\sigma_n := \ell_D^{2n}\,(s^n + t^n + u^n)$ and $\sigma_1 = 0$

$$T_D(\varphi, \sigma_{2,3}) = \frac{\mathcal{E}^{(D)}_{(0,-1)}(\varphi)}{\sigma_3} + \sum_{p,q\geqslant 0} \mathcal{E}^{(D)}_{(p,q)}(\varphi)\,\sigma_2^p \sigma_3^q$$

String theory interactions II

- $\mathcal{E}^{(D)}_{(0,-1)}(\varphi) = 1$: Einstein-Hilbert term for classical gravity
- The coefficient are functions of the moduli φ in the coset $E_{d+1}(\mathbb{R})/K_{d+1}$ parametrizing the vacuum of the theory
- The string theory is invariant under a discrete subgroup $E_{d+1}(\mathbb{Z})$

$$\mathcal{E}^{(D)}_{(p,q)}(\gamma \cdot \vec{\varphi}) = \mathcal{E}^{(D)}_{(p,q)}(\vec{\varphi}); \qquad \gamma \in E_{d+1}(\mathbb{Z})$$

Constraints from supersymmetry I

- The first 3 corrections $\mathcal{E}^{(D)}_{(0,0)}\,\mathcal{R}^4$, $\mathcal{E}^{(D)}_{(1,0)}\,\partial^4\mathcal{R}^4$ and $\mathcal{E}^{(D)}_{(0,1)}\,\partial^6\mathcal{R}^4$ are BPS protected

- Supersymmetry implies they satisfy simple differential equation

$$\left(\Delta - \frac{(k+3)(8+2k-D)(D(k+3)-k-33)}{(k+3)(D-2)}\right)\mathcal{E}^{(D)}_{(p,q)} = S^{(D)}_{(p,q)};\; k = 2p+3q$$

- Boundary conditions from string/M-theory allow to determine the solution [Green, Russo, Miller, Vanhove]

Constraints from supersymmetry II

- The zero instanton sector (the constant term in a Fourier expansion) displays non-renormalisation theorems

$$\mathcal{E}^{(D)}_{(0,0)}\Big|_{\text{pert}} = g_D^{-2\frac{8-D}{D-2}} \left(\frac{a_{\text{tree}}}{g_D^2} + I_{1-\text{loop}} \right)$$

$$\mathcal{E}^{(D)}_{(1,0)}\Big|_{\text{pert}} = g_D^{-4\frac{7-D}{D-2}} \left(\frac{a_{\text{tree}}}{g_D^4} + \frac{1}{g_D^2} I_{1-\text{loop}} + I_{2-\text{loop}} \right)$$

$$\mathcal{E}^{(D)}_{(0,1)}\Big|_{\text{pert}} = g_D^{-6\frac{6-D}{D-2}} \left(\frac{a_{\text{tree}}}{g_D^6} + \frac{1}{g_D^4} I_{1-\text{loop}} + \frac{1}{g_D^2} I_{2-\text{loop}} + I_{3-\text{loop}} + O(e^{-\frac{1}{g_D}}) \right)$$

- Only the invariance under the *continuous* symmetry group $E_{d+1}(\mathbb{R})$ was needed for these non-renormalisation theorems

- This was enough to rule the existence of possible UV divergences for $L \leqslant 6$ in $D = 4$ for $\mathcal{N} = 8$ supergravity [Green et al.; Elvang et al.; Bossard et al.]

Constraints from supersymmetry III

- Beside the perturbative sector these interactions receive non-perturbative contributions
- They are instantons from various supersymmetric BPS configurations. They are necessary for the non-perturbative consistency of the theory
- In this case we will need to make consider the *discrete* duality group $E_{d+1}(\mathbb{Z})$
- The BPS protected operators are expected to have very particular Fourier coefficients.

In this talk we will analyze how supersymmetry restricts the Fourier modes

Duality symmetries

[Hull,Townsend]

D	$E_{11-D(11-D)}(\mathbb{R})$	K_D	$E_{11-D(11-D)}(\mathbb{Z})$
10A	$\mathbb{R}^+$	1	1
10B	$Sl(2,\mathbb{R})$	$SO(2)$	$Sl(2,\mathbb{Z})$
9	$Sl(2,\mathbb{R})\times\mathbb{R}^+$	$SO(2)$	$Sl(2,\mathbb{Z})$
8	$Sl(3,\mathbb{R})\times Sl(2,\mathbb{R})$	$SO(3)\times SO(2)$	$Sl(3,\mathbb{Z})\times Sl(2,\mathbb{Z})$
7	$Sl(5,\mathbb{R})$	$SO(5)$	$Sl(5,\mathbb{Z})$
6	$SO(5,5,\mathbb{R})$	$SO(5)\times SO(5)$	$SO(5,5,\mathbb{Z})$
5	$E_{6(6)}(\mathbb{R})$	$USp(8)$	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_{8(8)}(\mathbb{Z})$

- $E_{11-D(11-D)}$ real split forms, K_D maximal compact subgroup.
- The vacuum of the theory is the scalar manifold $\vec{\varphi}\in\mathcal{M}_D=G_D/K_D$

The $\frac{1}{2}$ and $\frac{1}{4}$ BPS interactions

$E_{d+1}(\mathbb{Z})$	$\mathcal{E}^{(D)}_{(0,0)}$	$\mathcal{E}^{(D)}_{(1,0)}$
$E_{8(8)}(\mathbb{Z})$	$\mathbf{E}^{E_8}_{[1\,0^7];\frac{3}{2}}$	$\frac{1}{2}\,\mathbf{E}^{E_8}_{[1\,0^7];\frac{5}{2}}$
$E_{7(7)}(\mathbb{Z})$	$\mathbf{E}^{E_7}_{[1\,0^6];\frac{3}{2}}$	$\frac{1}{2}\,\mathbf{E}^{E_7}_{[1\,0^6];\frac{5}{2}}$
$E_{6(6)}(\mathbb{Z})$	$\mathbf{E}^{E_6}_{[1\,0^5];\frac{3}{2}}$	$\frac{1}{2}\,\mathbf{E}^{E_6}_{[1\,0^5];\frac{5}{2}}$
$SO(5,5,\mathbb{Z})$	$\mathbf{E}^{SO(5,5)}_{[10000];\frac{3}{2}}$	$\frac{1}{2}\,\widehat{\mathbf{E}}^{SO(5,5)}_{[10000];\frac{5}{2}} + \frac{4}{45}\,\widehat{\mathbf{E}}^{SO(5,5)}_{[00001];3}$
$SL(5,\mathbb{Z})$	$\mathbf{E}^{SL(5)}_{[1000];\frac{3}{2}}$	$\frac{1}{2}\,\widehat{\mathbf{E}}^{SL(5)}_{[1000];\frac{5}{2}} + \frac{3}{\pi^3}\,\widehat{\mathbf{E}}^{SL(5)}_{[0010];\frac{5}{2}}$
$SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$	$\widehat{\mathbf{E}}^{SL(3)}_{[10];\frac{3}{2}} + 2\widehat{\mathbf{E}}_1(U)$	$\frac{1}{2}\,\mathbf{E}^{SL(3)}_{[10];\frac{5}{2}} - 4\,\mathbf{E}^{SL(3)}_{[10];-\frac{1}{2}}\,\mathbf{E}_2(U)$
$SL(2,\mathbb{Z})$	$\mathbf{E}_{\frac{3}{2}}(\Omega)\,\nu_1^{-\frac{3}{7}} + 4\zeta(2)\,\nu_1^{\frac{4}{7}}$	$\frac{1}{2}\,\nu_1^{-\frac{5}{7}}\,\mathbf{E}_{\frac{5}{2}}(\Omega) + \frac{2\zeta(2)}{15}\,\nu_1^{\frac{9}{7}}$
$SL(2,\mathbb{Z})$	$\mathbf{E}_{\frac{3}{2}}(\Omega)$	$\frac{1}{2}\,\mathbf{E}_{\frac{5}{2}}(\Omega)$

Part I

Fourier expansion

Fourier expansion I

- We are interested in type II string compactified on tori T^d
- The instantonic configurations are obtained by wrapping the world-volume of D-brane and NS-brane (or metric, M2-, M5- and Kaluza-Klein monopole from a M-theory point of view)
- If $\theta_i \in U \cong \mathbb{R}^m$ is a set of continuous commuting charges of non-perturbative contributions we define *the fourier modes*

$$\mathcal{F}^{(D)}_{(p,q)}[Q,\varphi] := \int_{[0,1]^m} d\theta\, \mathcal{E}^{(D)}_{(p,q)}\, e^{2i\pi Q\cdot\theta}$$

- In low dimensions, the set of charges is *not* always commutative (when there are NS5-brane, and Kaluza-Klein monople contributions) but it is the integral over the abelianization of U, ie the commutator $[U,U]$, which has a Fourier expansion

Fourier expansion II

- We can examine instanton corrections to the perturbative regime, the strong coupling (M-theory) regime and decompactification regime

- In this talk we will be concerned with the decompactification limit where the connection with the BPS states in higher dimension is visible

- For a compactification of the theory in dimensions $D+1$ to a theory in dimension $D = 10 - d$ on circle of radius $r^2 = r_d/\ell_{D+1}$ the symmetry group becomes

$$P_{\alpha_{d+1}} = GL(1) \times E_d \times U_{\alpha_{d+1}} \subset E_{d+1}$$

Fourier expansion III

- The Fourier coefficient is a sum over integer charges $Q \in E_{d+1}(\mathbb{Z}) \cap U_{\alpha_{d+1}}$ that are conjugated to the angular variables $\theta \in U_{\alpha_{d+1}}$

$$\mathcal{F}^{(D)}_{(p,q)}[Q,\varphi] := \int_{[0,1]^m} d\theta\, \mathcal{E}^{(D)}_{(p,q)}\, e^{2i\pi Q\cdot\theta}$$

- The fourier coefficients are function of the moduli of the symmetry group $\varphi \in E_d/K_d$

Character variety orbits I

- The action of the symmetry group E_d on the angular variabes θ is irreducible if and only if U_α is abelian. This is the case of duality group until E_8 where the charge system is non-abelian (due to the Kaluza-Klein Monopole contribution)

E_{d+1}	$M_{\alpha_{d+1}}$	V_{α_d}
E_8	E_7	q^i : **56**, q : **1**
E_7	E_6	q^i : **27**
E_6	$SO(5,5)$	S_α : **16**
$SO(5,5)$	$SL(5)$	$v_{[ij]}$: **10**
$SL(5)$	$SL(3) \times SL(2)$	v_{ia} : **3** $\times$ **2**
$SL(3) \times SL(2)$	$SL(2) \times \mathbb{R}^+$	vv_a : **2**

Character variety orbits II

- The (complexified) action of the symmetry group $GL(1) \times E_d$ acts on the angular variables θ according irreducible representations and therefore fills out *character variety orbits* $o \in E_d/H_d$ where H_d is the stability group
- The characterity orbits are given by conditions on the representative of the action of the symmetry group E_d given in the previous table
- The relevant continuous BPS orbits have been classified by [Ferrara, Maldacena; Pope, Lu, Stelle]
- One important orbits is the $\frac{1}{2}$-BPS orbits in dimension D

$$\mathcal{O}_{\frac{1}{2}-BPS} = \frac{E_{d+1}}{E_d \ltimes \mathbb{R}^{n_d}}$$

- Where n_d is the number of BPS charges in dimension $D-1$
- The minimal orbit is the $\frac{1}{2}$-BPS orbits, and the next-to-minimal (NTM) orbit is the $\frac{1}{4}$-BPS orbit

Character variety orbits III

$M_{\alpha_{d+1}} = E_d$	BPS	BPS condition	Orbit	Dim.
$SL(2)$	$\frac{1}{2}$	-	1	0
$SL(2)\times\mathbb{R}^+$	$\frac{1}{2}$	$v\,v_\alpha = 0$	$\frac{\mathbb{R}^*\times SL(2,\mathbb{R})}{SL(2,\mathbb{R})}$	1
	$\frac{1}{4}$	$v\,v_\alpha \neq 0$	$\mathbb{R}^* \times \frac{SL(2,\mathbb{R})}{SO(2,\mathbb{R})}$	3
$SL(3)\times SL(2)$	$\frac{1}{2}$	$\epsilon^{ab}\, v_{ia}v_{jb} = 0$	$\frac{SL(3,\mathbb{R})\times SL(2,\mathbb{R})}{(\mathbb{R}^+\times SL(2,\mathbb{R}))\ltimes\mathbb{R}^3}$	5
	$\frac{1}{4}$	$\epsilon^{ab}\, v_{ia}v_{jb} \neq 0$	$\frac{SL(3,\mathbb{R})\times SL(2,\mathbb{R})}{SL(2,\mathbb{R})\ltimes\mathbb{R}^2}$	6
$SL(5)$	$\frac{1}{2}$	$\epsilon^{ijklm}\, v_{ij}\, v_{kl} = 0$	$\frac{SL(5,\mathbb{R})}{(SL(3,\mathbb{R})\times SL(2,\mathbb{R}))\ltimes\mathbb{R}^6}$	7
	$\frac{1}{4}$	$\epsilon^{ijklm}\, v_{ij}\, v_{kl} \neq 0$	$\frac{SL(5,\mathbb{R})}{O(2,3)\ltimes\mathbb{R}^4}$	10
$SO(5,5)$	$\frac{1}{2}$	$(S\Gamma^m S)=0$	$\frac{SO(5,5,\mathbb{R})}{SL(5,\mathbb{R})\ltimes\mathbb{R}^{10}}$	11
	$\frac{1}{4}$	$(S\Gamma^m S)\neq 0$	$\frac{SO(5,5,\mathbb{R})}{O(3,4)\ltimes\mathbb{R}^8}$	16

Character variety orbits IV

$M_{\alpha_{d+1}} = E_d$	BPS	BPS condition	Orbit	Dim.
E_6	$\frac{1}{2}$	$I_3 = \frac{\partial I_3}{\partial q^i} = 0,$ and $\frac{\partial^2 I_3}{\partial q^i \partial q^j} \neq 0.$	$\frac{E_6}{O(5,5) \ltimes \mathbb{R}^{16}}$	17
	$\frac{1}{4}$	$I_3 = 0,\ \frac{\partial I_3}{\partial q^i} \neq 0$	$\frac{E_6}{O(4,5) \ltimes \mathbb{R}^{16}}$	26
	$\frac{1}{8}$	$I_3 \neq 0$	$\mathbb{R}^* \times \frac{E_6}{F_{4(4)}}$	27
E_7	$\frac{1}{2}$	$I_4 = \left.\frac{\partial^2 I_4}{\partial q^i \partial q^j}\right\|_{Adj_{E_7}} = 0,$ and $\frac{\partial^3 I_4}{\partial q^i \partial q^j \partial q^k} \neq 0.$	$\frac{E_7}{E_6 \ltimes \mathbb{R}^{27}}$	28
	$\frac{1}{4}$	$I_4 = \frac{\partial I_4}{\partial q^i} = 0,$ and $\left.\frac{\partial^2 I_4}{\partial q^i \partial q^j}\right\|_{Adj_{E_7}} \neq 0.$	$\frac{E_7}{(O(5,6) \ltimes \mathbb{R}^{32}) \times \mathbb{R}}$	45
	$\frac{1}{8}$	$I_4 = 0,\ \frac{\partial I_4}{\partial q^i} \neq 0$	$\frac{E_{7(7)}}{F_{4(4)} \ltimes \mathbb{R}^{26}}$	55
	$\frac{1}{8}$	$I_4 > 0$	$\mathbb{R}^+ \times \frac{E_7}{E_{6(2)}}$	56

Support of the Fourier modes I

- The Fourier coefficients are non-vanishing when the angular variables is supported on an orbit o
- These orbits have a partial ordering with respect to the closure
- Therefore we can have the following property

Lemma (wavefront set)

If ϕ *is an automorphic representation the* wavefront set $\mathcal{O}_G(\phi)$ *is the orbit* o *such that if* $o' > o$ *then* ϕ *are vanishing Fourier coefficients and has a non-zero Fourier for* o.

Support of the Fourier modes II

- An irreducible representation is related to coadjoint nilpotent orbits through its *wavefront set*
- It is a theorem that this set is always the closure of a unique coadjoint nilpotent orbit.
- Thus a coadjoint nilpotent orbit is attached to every irreducible automorphic representation.

The wavefront set of $\frac{1}{2}$- and $\frac{1}{4}$-BPS couplings

Theorem (Green-Miller-Vanhove)

For all the duality group E_{d+1} in dimension $4 \leqslant D \leqslant 10$ we have

- $\mathcal{R}^4$ *The wavefront set of the automorphic representation attached to the $\mathcal{E}^{(D)}_{(0,0)}$ interaction is the closure of the minimal orbit.*
- $\partial^4\mathcal{R}^4$ *The wavefront set of the automorphic representation $\mathcal{E}^{(D)}_{(1,0)}$ interaction attached to the is the closure of the next-to-minimal (NTM) orbit.*

- The closure of the minimal orbit is simply the union of the minimal orbit and the trivial orbit
- the closure of the next-to-minimal orbit is the union of itself, the minimal orbit, and the trivial orbit.

The wavefront set of $\frac{1}{2}$- and $\frac{1}{4}$-BPS couplings

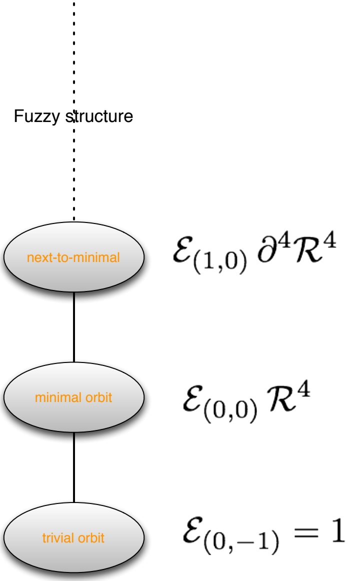
Theorem (Green-Miller-Vanhove)

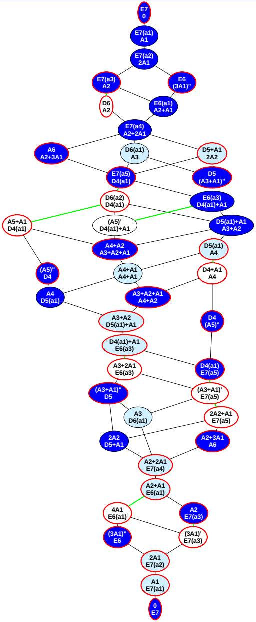
For all the duality group E_{d+1} in dimension $4 \leqslant D \leqslant 10$ we have

- $\mathcal{R}^4$ *The wavefront set of the automorphic representation attached to the $\mathcal{E}^{(D)}_{(0,0)}$ interaction is the closure of the minimal orbit.*
- $\partial^4\mathcal{R}^4$ *The wavefront set of the automorphic representation $\mathcal{E}^{(D)}_{(1,0)}$ interaction attached to the is the closure of the next-to-minimal (NTM) orbit.*

- Therefore the $\mathcal{R}^4$ interaction has non-zero Fourier coefficients only for charge configuration satisfying the $\frac{1}{2}$-BPS condition
- Therefore the $\partial^4\mathcal{R}^4$ interaction has non-zero Fourier coefficients only for charge configurations satisfying the $\frac{1}{4}$-BPS or the $\frac{1}{2}$-BPS conditions

The closure diagram of an Eisenstein series




E7
0
E7(a1)
A1
E7(a2)
2A1
E7(a3)
A2
E6
(3A1)"
D6
A2
E6(a1)
A2+A1
E7(a4)
A2+2A1
A6
A2+3A1
D6(a1)
A3
D5+A1
2A2
E7(a5)
D4(a1)
D5
(A3+A1)"
D6(a2)
D4(a1)
E6(a3)
D4(a1)+A1
A5+A1
D4(a1)
(A5)'
D4(a1)+A1
D5(a1)+A1
A3+A2
A4+A2
A3+A2+A1
D5(a1)
A4
(A5)"
D4
A4+A1
A4+A1
D4+A1
A4
A4
D5(a1)
A3+A2+A1
A4+A2
A3+A2
D5(a1)+A1
D4
(A5)"
D4(a1)+A1
E6(a3)
A3+2A1
E6(a3)
D4(a1)
E7(a5)
(A3+A1)"
D5
(A3+A1)'
E7(a5)
A3
D6(a1)
2A2+A1
E7(a5)
2A2
D5+A1
A2+3A1
A6
A2+2A1
E7(a4)
A2+A1
E6(a1)
4A1
E6(a1)
A2
E7(a3)
(3A1)"
E6
(3A1)'
E7(a3)
2A1
E7(a2)
A1
E7(a1)
0
E7

Part II

Detailed properties of the Fourier modes

Fourier coeffcients and BPS states I

- We have established that the Fourier modes of the $\frac{1}{2}$- and $\frac{1}{4}$-BPS interactions vanish except when the support corresponds to the BPS conditions

- We want to say something about the properties of Fourier modes

- The decompactification limit the symmetry group E_{d+1} is broken into $GL(1) \times E_d \times U_{\alpha_{d+1}}$

- We want to analyze the asymptotic behaviour of the Fourier modes when the $GL(1)$ parameter $r^2 = r_d/\ell_{D+1} \to \infty$

Fourier coeffcients and BPS states II

- In the decompactification limit $r_d \gg \ell_{D+1}$ the Fourier modes take the form

$$\mathcal{F}^{(D)}_{(p,q)}[Q,\varphi] \sim \sum_{Q\in\mathbb{Z}^n} d_Q f_Q(\varphi)\, e^{-2\pi r_d\, m_Q}$$

- The Fourier transform induces a condition on the discrete charges Q in the charge lattice $E_{d+1}(\mathbb{Z}) \cap U_{\alpha_{d+1}}$ which lies in *discrete orbits*
- m_Q is the mass of the BPS particle state in dimension $D+1$ that lead to an instanton once wrapped on the (euclidean) circle of radius r_d
- d_Q is a number theoretic function counting the instanton configurations
- $f_Q(\varphi)$, due to the quantum fluctuations, is a function of the moduli invariant under the symmetry group E_d in dimension $D+1$

BPS states and instanton charges

$D = 10 - d$	$M_{\alpha_{d+1}} = E_d$	# point charges (dim $U_{\alpha_{d+1}}$)	# instanton charges = # +ve roots of E_d
10A	1	1	0
10B	$SL(2)$	0	1
9	$SL(2) \times \mathbb{R}^+$	3	1
8	$SL(3) \times SL(2)$	6	4
7	$SL(5)$	10	10
6	$SO(5,5)$	16	20
5	E_6	27	36
4	E_7	56 (57)	63
3	E_8	120	120

- The dimensions of the spaces spanned by the BPS point-like charges and BPS instantons of maximal supergravity in various dimensions

BPS states and instanton charges

$D =$ $10-d$	$M_{\alpha_{d+1}} = E_d$	# point charges (dim $U_{\alpha_{d+1}}$)	# instanton charges = # +ve roots of E_d
10A	1	1	0
10B	$SL(2)$	0	1
9	$SL(2) \times \mathbb{R}^+$	3	1
8	$SL(3) \times SL(2)$	6	4
7	$SL(5)$	10	10
6	$SO(5,5)$	16	20
5	E_6	27	36
4	E_7	56 (57)	63
3	E_8	120	120

- Notice that number of instantons in dimensions $D+1$ is the sum of instantons and BPS states charge in dimensions D
- Except for $D = 3$ where a KKM instanton does not decompactify to a state

Fourier modes of $\frac{1}{2}$-BPS interactions

- The $\frac{1}{2}$-BPS $\mathcal{R}^4$ interaction has the following asymptotic behaviour

$$\lim_{r_d \gg 1} \mathcal{F}^{(D)}_{(0,0)}[Q_{\frac{1}{2}-BPS}, \varphi] \sim \left(\frac{r_d}{\ell_{D+1}}\right)^{n_D} \sigma_{7-D}(|k|) \, \frac{e^{-2\pi |k| r_d m_{\frac{1}{2}}}}{(2\pi |k| r_d m_{\frac{1}{2}})^{\frac{8-D}{2}}}$$

- k is the number of times the $\frac{1}{2}$-BPS bound states wraps the circle of radius r_d
- Quantum corrections are given by the fluctuations from open string attached to the instantons
- This formula is consistent with the interpretation of the number of bosonic and fermionic zero modes associated with this instanton
- the measure factor $\sigma_{7-D}(k) = \sum_{d|k} d^{7-D}$ has a matrix model derivation for $D \leqslant 8$ [Green,Gutperle; Moore, Nekrasov, Shatashvili; Sugino, Vanhove]

Fourier modes of $\frac{1}{4}$-BPS interactions

- The $\partial^4\mathcal{R}^4$ interaction receives $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS contributions where fourier modes have an asymptotic behaviour of the form

$$\lim_{r_d\gg 1} \mathcal{F}^{(D)}_{(1,0)}[Q_{\frac{1}{2}-BPS},\varphi] \sim \left(\frac{r_d}{\ell_{D+1}}\right)^{\alpha_D} d^{(1)}(|k|)\, f_k(Q_{\frac{1}{2}-BPS},\varphi)\, e^{-2\pi|k|r_d m_{\frac{1}{2}}}$$

- and

$$\lim_{r_d\gg 1} \mathcal{F}^{(D)}_{(1,0)}[Q_{\frac{1}{4}-BPS},\varphi] \sim \left(\frac{r_d}{\ell_{D+1}}\right)^{\beta_D} d^{(2)}(|k|)\, g_k(Q_{\frac{1}{4}-BPS},\varphi)\, e^{-2\pi|k|r_d m_{\frac{1}{4}}}$$

- The degeneracy factors $d^{(i)}(|k|)$ have a polynomial growth and they differ from the black hole degeneracies

Conclusion and outlook I

- The Fourier coefficients of BPS protected are supported on the (character variety) orbits familiar from the BPS state U-duality conditions
- The counting factors for $\frac{1}{2}$- and $\frac{1}{4}$-BPS states have a polynomial growth and differ from the black hole degeneracies
- The counting factor for the $\frac{1}{2}$-BPS interactions seem consistent with a matrix model evaluation
- The $\frac{1}{8}$-BPS couplings $\mathcal{E}^{(D)}_{(0,1)}$ and higher derivative interaction involve automorphic forms that are not Eisenstein series
- It is a challenges for physicists and mathematician to understand the characterisation of their Fourier coefficients and associate BPS coupling the orbits in the Hasse closure diagram for the orbits
- The fine details of the Fourier modes of the BPS saturated coupling depends on the integral structure of the character variety orbits known in some case in the works by [Bhargava; Krutelevich]