Causal Holographic Information

Mukund Rangamani

Durham University

RG Flows, Entanglement & Holography Workshop @MCTP
September 21, 2012

Algorithmic holography: reconstructing spacetime

Mukund Rangamani

Durham University

RG Flows, Entanglement & Holography Workshop @MCTP
September 21, 2021

Bulk surfaces & reduced density matrices

Mukund Rangamani

Durham University

RG Flows, Entanglement & Holography Workshop @MCTP
September 21, 2012

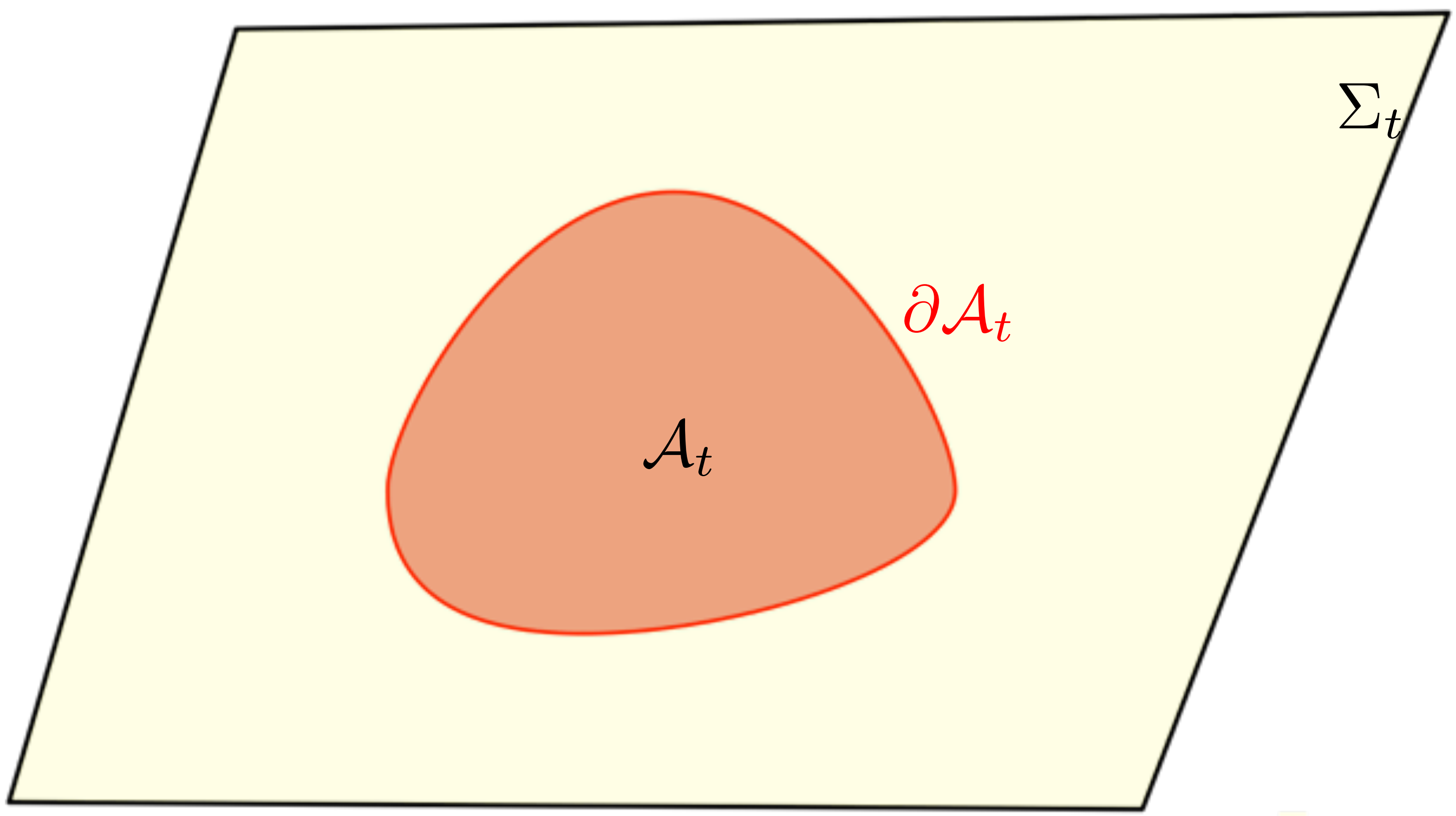
Holographic Surfaces

- ★ Motivation
- ★ Entanglement viewed geometrically
- ★ Causal constructions
- ★ Entanglement vs causal constructions
- ★ Properties of χ
- ★ Discussion

Motivation

- Consider a QFT in a pure state or more generally in a density matrix, living on a background $\mathcal{M}_d$ which is globally hyperbolic with a nice time foliation (Cauchy slices Σ_t).

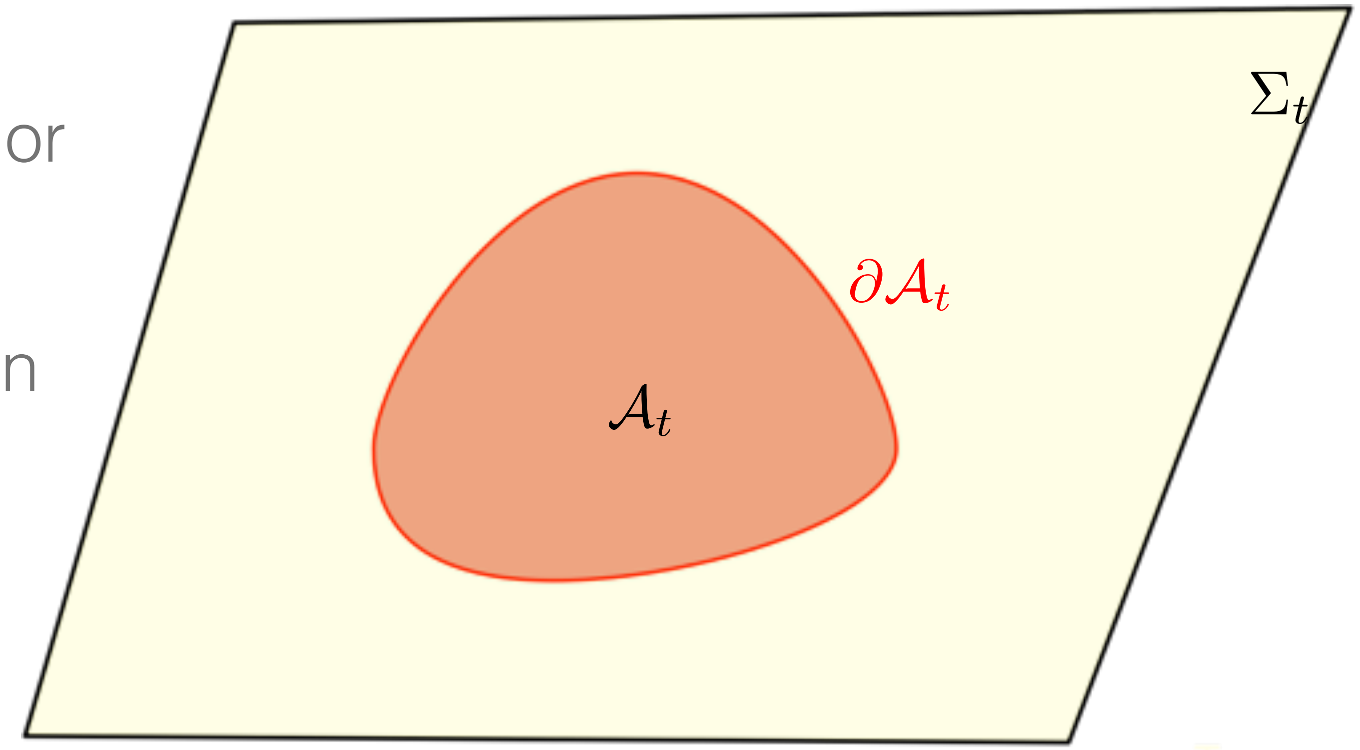
- $\mathcal{A}_t$ is a subregion of the Cauchy slice, with an “entangling surface” $\partial\mathcal{A}_t$.



Motivation I: Regional observables

- What are the observables that one can associate with this region?
  - e.g., spectral information of the reduced density matrix (Entanglement).
- Are there other observables that could be regarded as `natural'?

- Class of potential observables:
  - sensitivity to distribution of matter or charges.
  - ability to characterize distinctions in phase structure.
  - sensitivity to underlying causal structure.
  - cognizance of holography/entropy bounds.



$$\mathcal{F}\left[\mathcal{A}_t, \rho_{\Sigma_t}\right]$$

Motivation II: Locality of the holographic map

- The holographic map between strongly coupled planar QFTs and classical gravity is remarkable & mysterious.
- Various questions: emergence of spacetime locality, bulk causality, etc..
- Is there a quantitative characterization of the degree of non-locality in the holographic map?
- Given access to part of the field theory how much of the bulk can we reconstruct?
- To be precise, assume we know the reduced density matrix $\rho_{\mathcal{A}}$ associated with a spatial region on the boundary: what part of the bulk can be reconstructed from it?
- Aim: to quantify the amount of information in the holographic map contained in the data $(\mathcal{A}, \rho_{\mathcal{A}})$.

Motivation II: Locality of the holographic map

- Finer distinctions of holographic map given $(\mathcal{A}, \rho_{\mathcal{A}})$
  - in what region of the bulk spacetime does the geometry get determined from this data?
  - in what region of the bulk spacetime are we sensitive to the bulk geometry?
- Note that these are a-priori distinct questions and the resulting regions whilst overlapping might end-up being distinct.
- Also, we are going to focus attention to the semi-classical limit, assuming that notions of geometry, causal structure etc., are well defined.

- <u>Criterion:</u> Naturalness. Minimal assumptions about the holographic map

A geometric view on entanglement

- Ryu-Takayanagi (RT) have provided us with a natural geometric construction to the data $(\mathcal{A}, \rho_{\mathcal{A}})$: minimal surfaces ending on the entangling surface on the boundary.

- More generally, in generic non-static situations, we are required to find an extremal surface $\mathfrak{E}_{\mathcal{A}}$ which is anchored at the boundary on the entangling surface $\partial\mathcal{A}_t$.

Hubeny, MR, Takayanagi (2007)

$$S_{\mathcal{A}} = \frac{\text{Area}(\mathfrak{E}_{\mathcal{A}})}{4\,G_N}$$

- The extremal surface is such that the light-sheets emanating from it towards the boundary of the spacetime have zero expansion

  - natural candidate from viewpoint of covariant entropy bounds.

Naturalness & pre-geometric construct

- Are the extremal surfaces $\mathfrak{E}_{\mathcal{A}}$ the most natural construct given $(\mathcal{A}, \rho_{\mathcal{A}})$?

- Naturalness criterion: the minimal requirement for the holographic map is consistency of bulk & boundary causality.

- Minimalism: use the bulk causal structure, eschewing use of geometry a-priori, to associate a bulk spacetime region to $(\mathcal{A}, \rho_{\mathcal{A}})$.

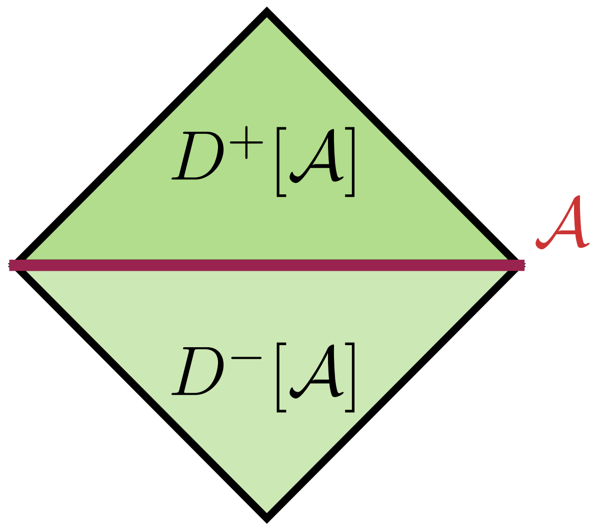
- Claim: The unique minimal construction gives the bulk causal wedge $\blacklozenge_{\mathcal{A}}$ associated with the boundary region $\mathcal{A}_t$.

- Further use of geometry (metric data) allows us to associate a number, $\chi_{\mathcal{A}}$ to $(\mathcal{A}, \rho_{\mathcal{A}})$. We'll call this causal holographic information.

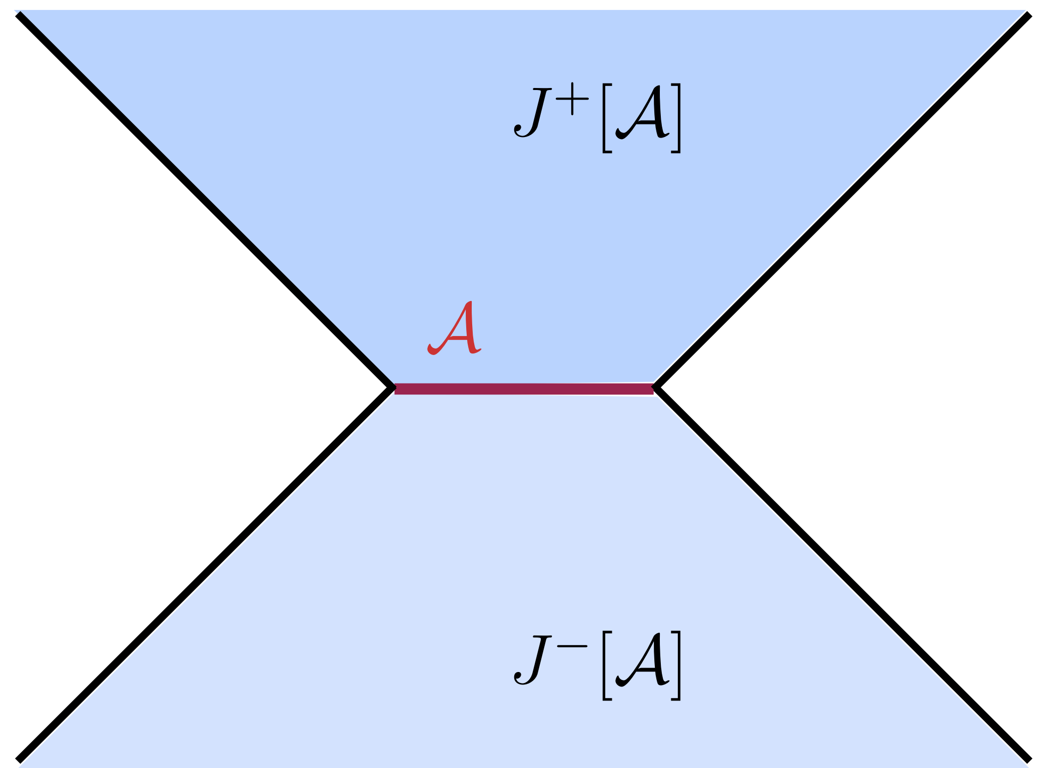
$$\chi_{\mathcal{A}} = \frac{\text{Area}(\Xi_{\mathcal{A}})}{4\, G_N}$$

Hubeny, MR (2012)

The causal construction I: boundary



$$\Diamond_{\mathcal{A}} \equiv D^+[\mathcal{A}] \cup D^-[\mathcal{A}]$$



- Domain of *dependence*: the region of the boundary spacetime that **must** influence or be influenced by events in $\mathcal{A}_t$.
- Domain of *influence*: the region of the boundary spacetime that **can** influence or be influenced by events in $\mathcal{A}_t$.

- Causality implies that $(\mathcal{A}, \rho_{\mathcal{A}})$ determines all observables in $\Diamond_{\mathcal{A}}$.

Causal construction II: into the bulk

- Bulk causal wedge $\blacklozenge_{\mathcal{A}}$

$$\blacklozenge_{\mathcal{A}} \equiv J^-[\Diamond_{\mathcal{A}}] \cap J^+[\Diamond_{\mathcal{A}}]$$

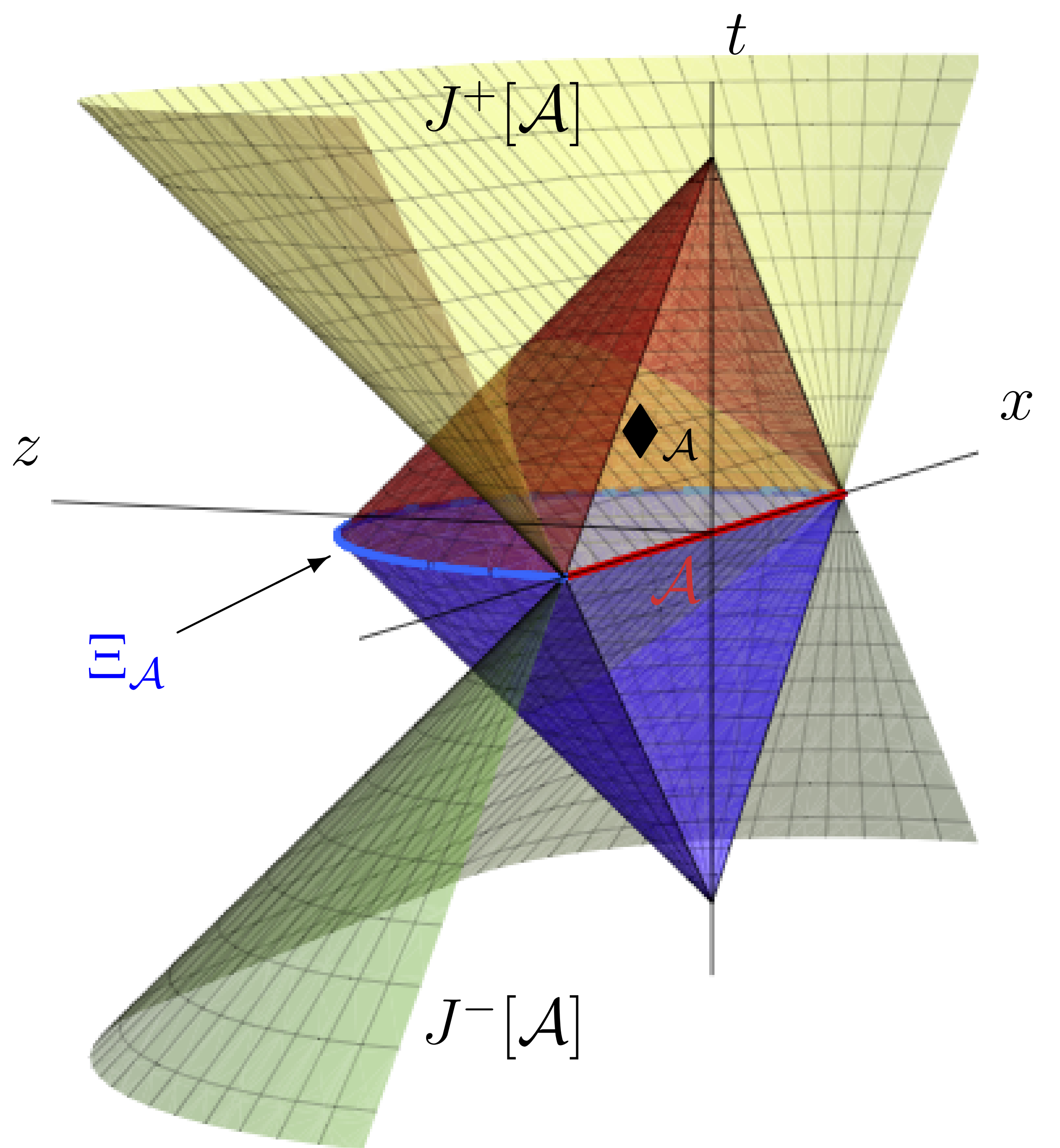
= { bulk causal curves which begin and end on $\Diamond_{\mathcal{A}}$ }

- Causal information surface

$$\Xi_{\mathcal{A}} \equiv \partial_+(\blacklozenge_{\mathcal{A}}) \cap \partial_-(\blacklozenge_{\mathcal{A}})$$

- Causal holographic information $\chi_{\mathcal{A}}$

$$\chi_{\mathcal{A}} \equiv \frac{\text{Area}(\Xi_{\mathcal{A}})}{4\, G_N}$$



Open question

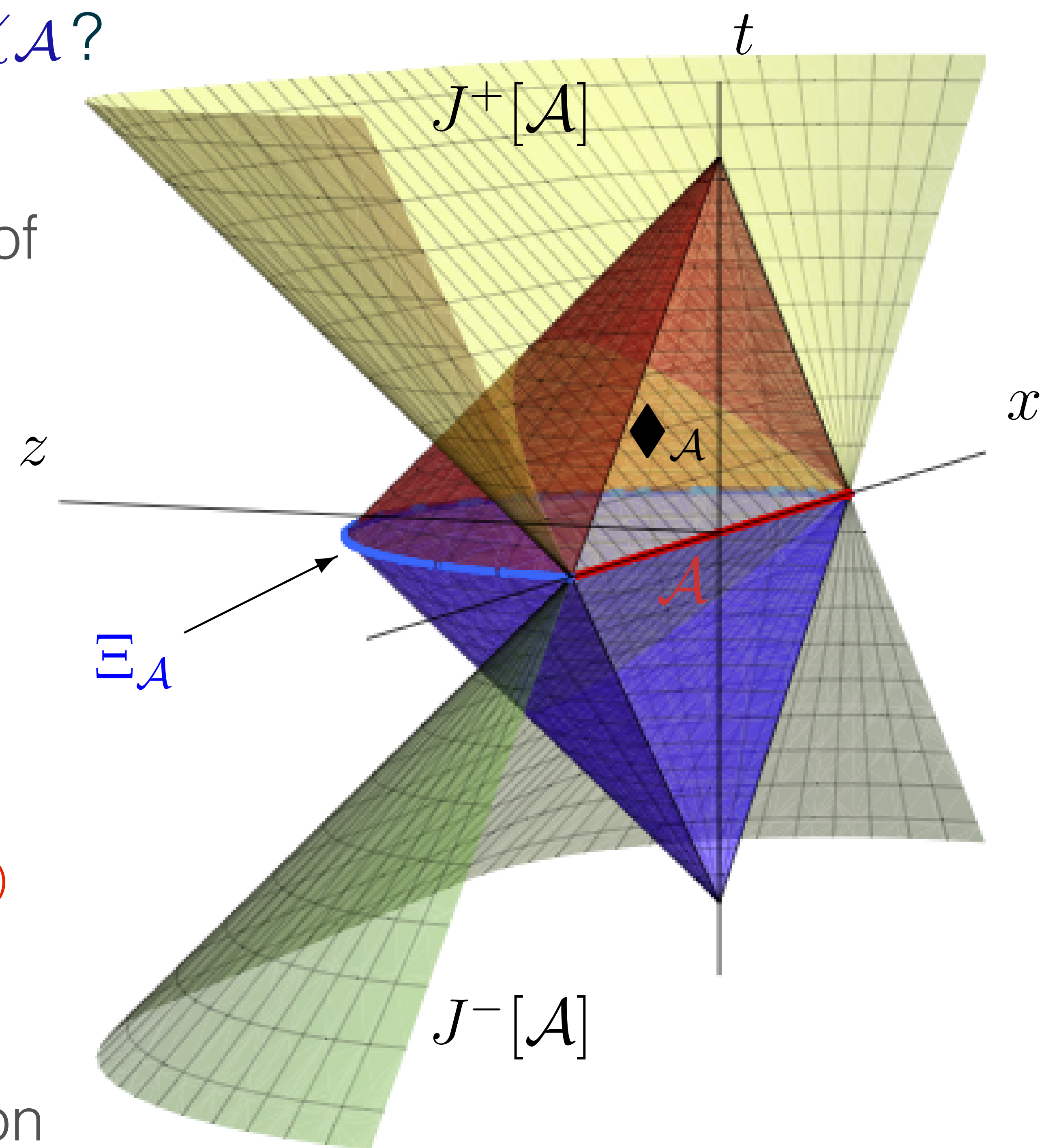
✴ CFT interpretation of $\Xi_{\mathcal{A}}$ and $\chi_{\mathcal{A}}$?

✴ Do they satisfy our requirements of naturalness in the field theory?

◉ Correlation functions of local observables can be computed within the causal wedge as a natural consequence of causality.

Marolf (2005)

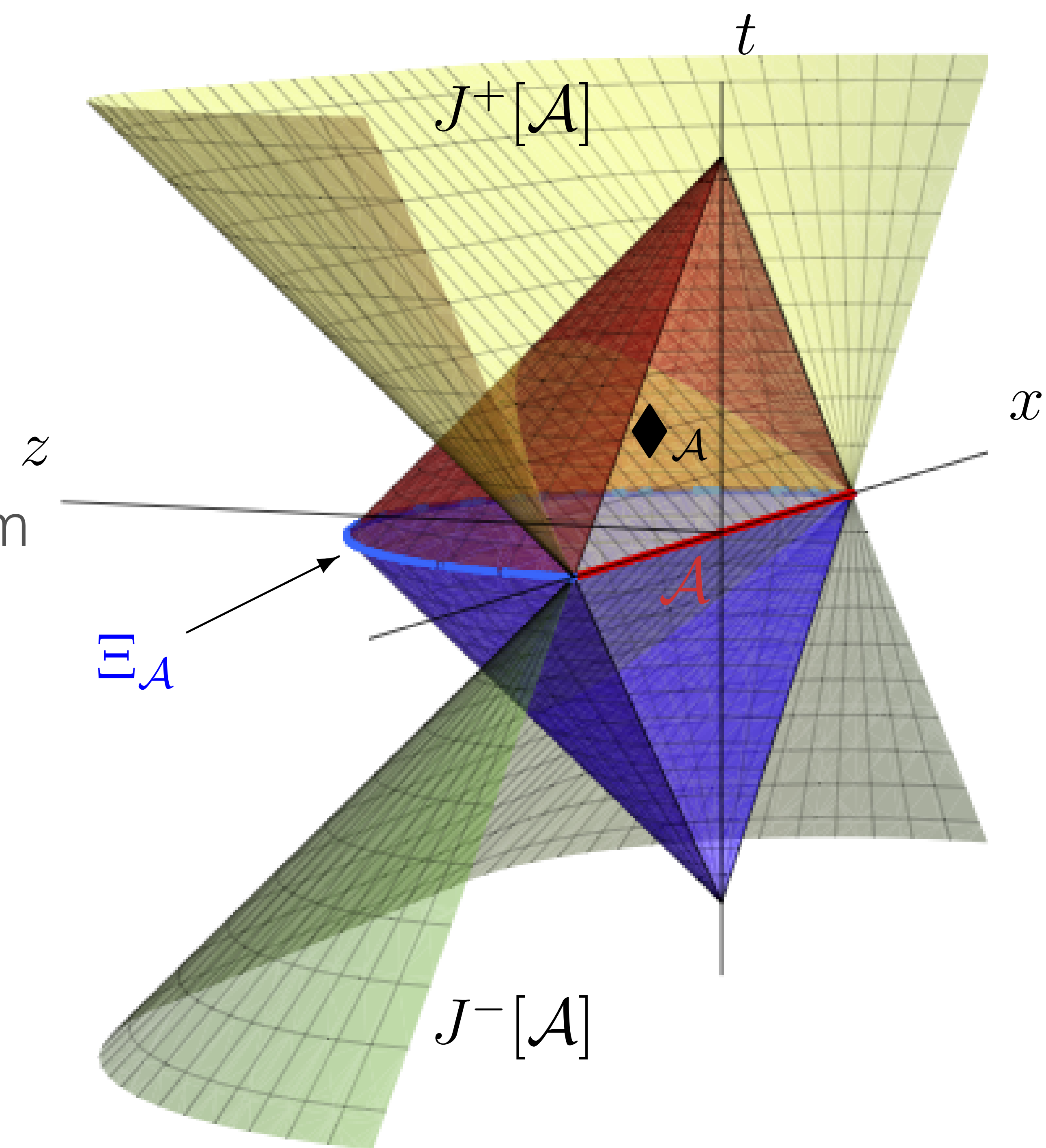
◉ Explore features of the construction to gather data....



Basic features of the causal surface

★ Causal information surface $\Xi_{\mathcal{A}}$ is a *d-1* dimensional spacelike bulk surface which:

- ❖ is anchored on $\partial\mathcal{A}$
- ❖ lies within (on boundary of) $\blacklozenge_{\mathcal{A}}$
- ❖ reaches deepest into the bulk from among surfaces in $\blacklozenge_{\mathcal{A}}$
- ❖ is a minimal-area surface among surfaces on $\partial(\blacklozenge_{\mathcal{A}})$ anchored on the entangling surface



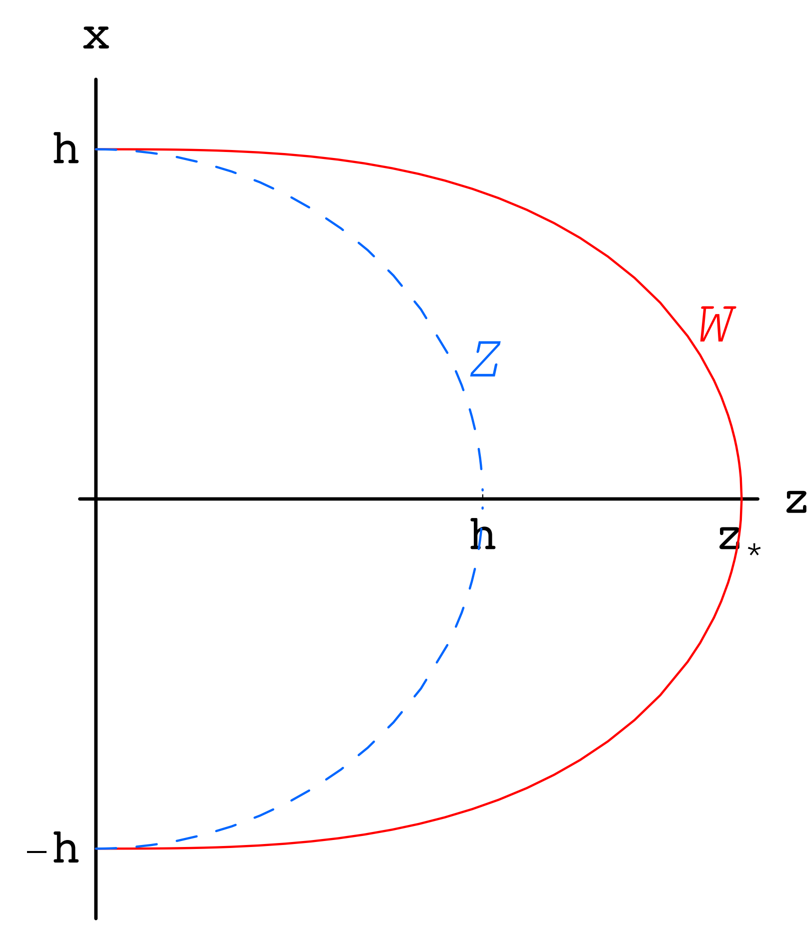
➡ However, $\Xi_{\mathcal{A}}$ is in general not an extremal surface $\mathfrak{E}_{\mathcal{A}}$ in the bulk.

General properties of $\Xi_{\mathcal{A}}$

- In general $\Xi_{\mathcal{A}}$ does not penetrate as far into the bulk as the bulk extremal surface $\mathfrak{E}_{\mathcal{A}}$ associated with $(\mathcal{A}, \rho_{\mathcal{A}})$

  ★ Justification 1: explicit construction in a specific example. The the region to be an infinite strip in $d > 2$ dimensions.

$$z^*_{\Xi} = h \ , \qquad z^*_{\mathfrak{E}} = \frac{\Gamma\left(\frac{1}{2(d-1)}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d}{2(d-1)}\right)}\, h$$



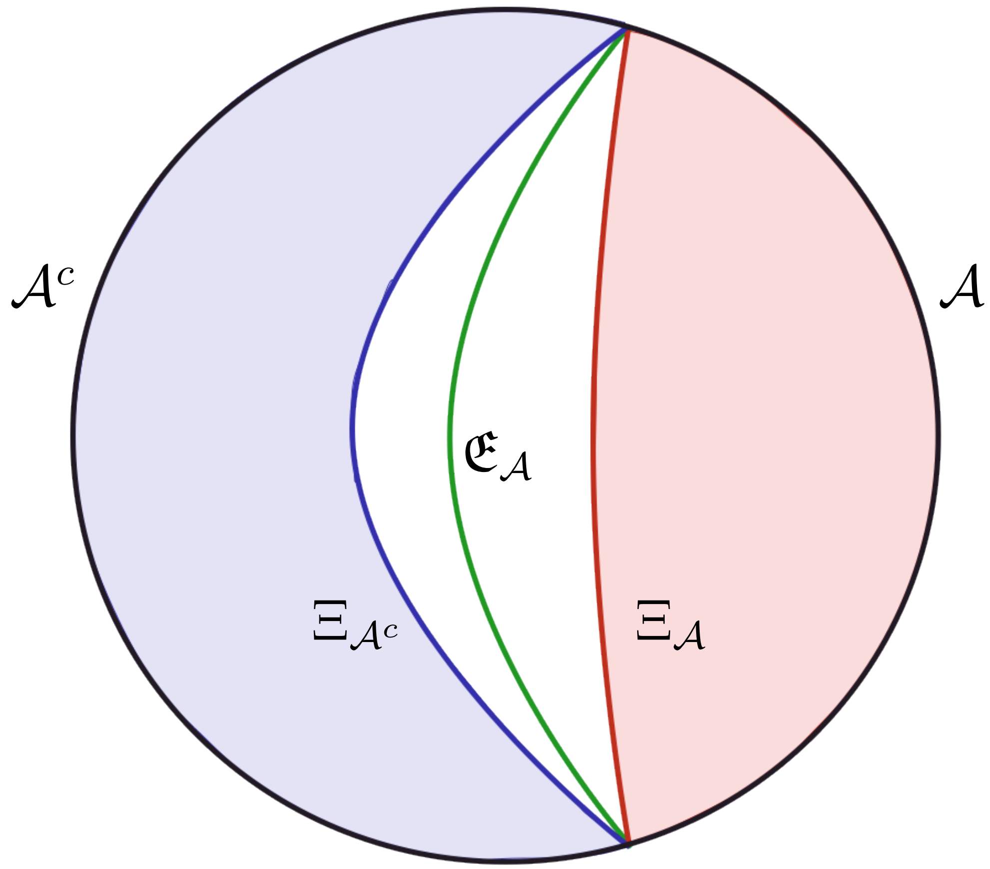
General properties of $\Xi_{\mathcal{A}}$

- In general $\Xi_{\mathcal{A}}$ does not penetrate as far into the bulk as the bulk extremal surface $\mathfrak{E}_{\mathcal{A}}$ associated with $(\mathcal{A}, \rho_{\mathcal{A}})$

  ★ Justification 2: general argument based on the features of the causal wedge for a region and its complement with a pure state $(\mathcal{A}, |\,\Psi\rangle \to \rho_{\mathcal{A}})$

  Require that $S_{\mathcal{A}} = S_{\mathcal{A}^c}$

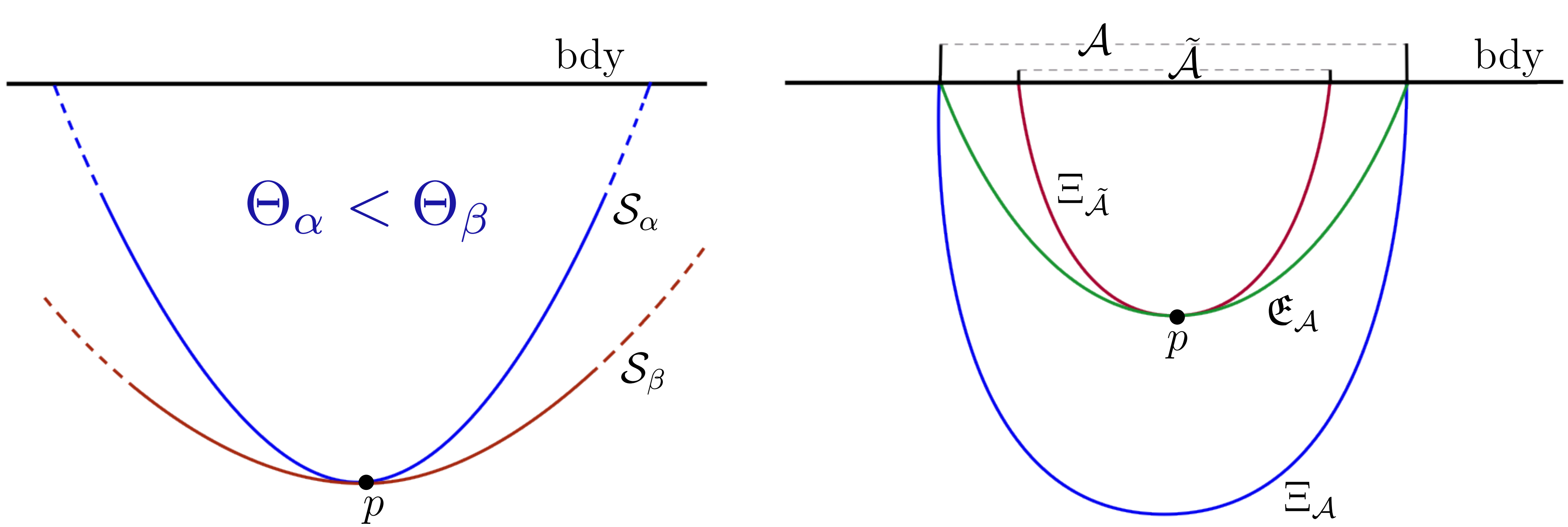
  ★ however, causal wedge differs for $\mathcal{A}$ and $\mathcal{A}^c$. The surface Ξ reach furthest in pure AdS (vacuum),but in general recedes closer to the boundary.



Gao, Wald (2000)

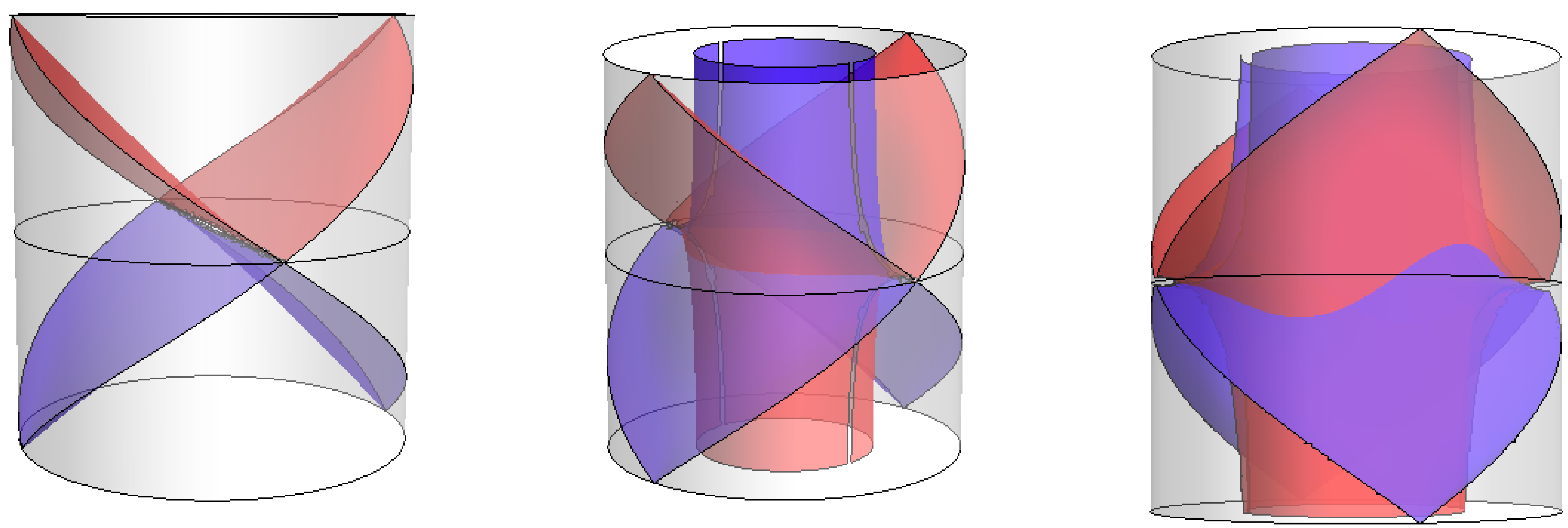
General properties of $\Xi_{\mathcal{A}}$

- In general $\Xi_{\mathcal{A}}$ does not penetrate as far into the bulk as the bulk extremal surface $\mathfrak{E}_{\mathcal{A}}$ associated with $(\mathcal{A}, \rho_{\mathcal{A}})$
  - Justification 3: general argument based on expansion of null generators: By construction, $\Theta_{\Xi} \geq 0$ while $\Theta_{\mathfrak{E}} = 0$
  - Proof by contradiction: suppose $\mathfrak{E}_{\mathcal{A}}$ lay closer to bdy than $\Xi_{\mathcal{A}}$. Then tangent to $\mathfrak{E}_{\mathcal{A}}$, there is a surface $\Xi_{\tilde{\mathcal{A}}}$ for some smaller region $\tilde{\mathcal{A}}$. But for such configuration, $\Theta_{\Xi_{\tilde{\mathcal{A}}}} < 0$, which is a contradiction.



Concordances: when $\Xi_{\mathcal{A}}$ & $\mathfrak{E}_{\mathcal{A}}$ coincide

bdy:	CFT vacuum:	thermal density matrix:	grand canonical density matrix:
bulk:	pure AdS:	static BTZ:	rotating BTZ:



(a) (b) (c)

$$\text{(a).} \qquad S_{\mathcal{A}} = \chi_{\mathcal{A}} = \frac{c_{\text{eff}}}{3} \log\left(\frac{2\varphi_0}{\varepsilon}\right)$$

$$\text{(b).} \qquad S_{\mathcal{A}} = \chi_{\mathcal{A}} = \frac{c_{\text{eff}}}{3} \log\left[\frac{\beta}{\pi\,\varepsilon} \sinh\left(\frac{2\pi\,\varphi_0}{\beta}\right)\right]$$

$$\text{(c).} \qquad S_{\mathcal{A}} = \chi_{\mathcal{A}} = \frac{c_{\text{eff}}}{6} \log\left[\frac{\beta_+\,\beta_-}{\pi^2\,\varepsilon^2} \sinh\left(\frac{2\pi\,\varphi_0}{\beta_+}\right) \sinh\left(\frac{2\pi\,\varphi_0}{\beta_-}\right)\right]$$

Concordances: when $\Xi_{\mathcal{A}}$ & $\mathfrak{E}_{\mathcal{A}}$ coincide

- What is special about these examples?

- Situations where we have been able to understand & derive the RT formula directly from field theory + holographic map. Casini, Heurta, Myers (2011)

- Logic: apply a unitary transformation to convert the reduced density matrix to a thermal density matrix. Converts computation of EE to a partition function computation.

- Lesson: The agreement between $\chi_{\mathcal{A}}$ and $S_{\mathcal{A}}$ occurs whenever the degrees of freedom in $\mathcal{A}$ are "maximally entangled" with those in $\mathcal{A}^c$.

- Conjecture: The quantity $\chi_{\mathcal{A}}$ provides a lower bound on the *holographic information* contained in the boundary region $\mathcal{A}$.

Detour: Bulk reconstruction

- What is the gravity dual of the density matrix? Given the data $(\mathcal{A}, \rho_{\mathcal{A}})$ what portion of the bulk spacetime should we be able to reconstruct?

- Answer 1: The bulk causal wedge and nothing more.

- Justification: argue that the boundary of the bulk causal wedge $\blacklozenge_{\mathcal{A}}$ is the surface obtained by taking the union of ingoing light-sheets from

Bousso, Leichenauer, Rosenhaus (2012)

- Answer 2: The bulk domain of dependence associated with the extremal surface $\mathfrak{E}_{\mathcal{A}}$.

- Justification: Entanglement computations imply that we can probe at least as deep as the extremal surface $\lozenge_{\mathcal{A}}$(+other justifications based on reasonable assumptions).

Czech, Karczmareck, Nogueira, Van Raamsdonk (2012)

Back to $\chi_{\mathcal{A}}$: summary of explorations

- The Causal Holographic Information $\chi_{\mathcal{A}}$
  - in special (maximally entangled) cases, coincides with $\mathcal{S}_{\mathcal{A}}$

$$\chi_{\mathcal{A}} \equiv \frac{\text{Area}(\Xi_{\mathcal{A}})}{4\,G_N} \quad = \quad \mathcal{S}_{\mathcal{A}} \equiv -\text{Tr}\,(\rho_{\mathcal{A}}\,\log\rho_{\mathcal{A}}) = \frac{\text{Area}(\mathfrak{E}_{\mathcal{A}})}{4\,G_N}$$

  - but in general diverges more strongly than entanglement entropy

    e.g. for d=4, $\mathcal{A}$ = strip of width w, w/ IR regulator L & UV regulator ε ,

$$S_{\mathcal{A}} = c_{\text{eff}}\,L^2\left(\frac{1}{\varepsilon^2} - \frac{0.32}{w^2}\right)\,, \qquad \chi_{\mathcal{A}} = c_{\text{eff}}\,L^2\left(\frac{1}{\varepsilon^2} - \frac{2}{w^2} + \frac{4}{w^2}\,\log\left(\frac{w}{\varepsilon}\right)\right)$$

  - hence provides a bound on entanglement entropy
  - unlike entanglement entropy, always varies continuously with size of the region $\mathcal{A}$ under consideration.

General properties of $\chi_{\mathcal{A}}$

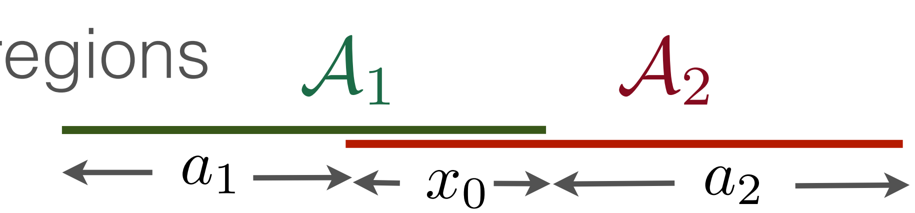
- The Causal Holographic Information unlike entanglement entropy, does NOT satisfy strong subadditivity

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$
$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \backslash \mathcal{A}_2} + S_{\mathcal{A}_2 \backslash \mathcal{A}_1}$$

- We know that the RT formula crucially satisfies strong subadditivity, and there is now evidence that perhaps the covariant proposal also does.

Headrick, Takayanagi (2007); Callan, He, Headrick (2012)

- There are easy counter-examples for $\chi_{\mathcal{A}}$: strip-regions



SS requires

$$F(a_1 + x_0) + F(a_2 + x_0) - F(a_1 + a_2 + x_0) - F(x_0) > 0\,, \qquad F(x) = \frac{1}{x^2} \log\left(\frac{x}{\tilde{\varepsilon}}\right)$$

but this can be violated - e.g. by $x_0 = a_1 = a_2$

Dynamical situations: toy model

Vaidya-AdS spacetime, describing a null shell in AdS:

$$ds^2 = -f(r,v)\,dv^2 + 2\,dv\,dr + r^2\,d\Omega^2$$

$$f(r,v) = r^2 + 1 - \vartheta(v)\,m(r)$$

with $$m(r) = \begin{cases} r_+^2 + 1 & , \quad \text{in AdS}_3 \\ \frac{r_+^2}{r^2}\,(r_+^2+1) & , \quad \text{in AdS}_5 \end{cases}$$

and $$\vartheta(v) = \begin{cases} 0 & , \quad \text{for } v < 0 \\ 1 & , \quad \text{for } v \geq 0 \end{cases}$$

$v < 0$ → pure AdS

$v \geq 0$ → Schw-AdS (or BTZ)

we can think of this as $\delta \to 0$ limit of smooth shell with thickness δ :

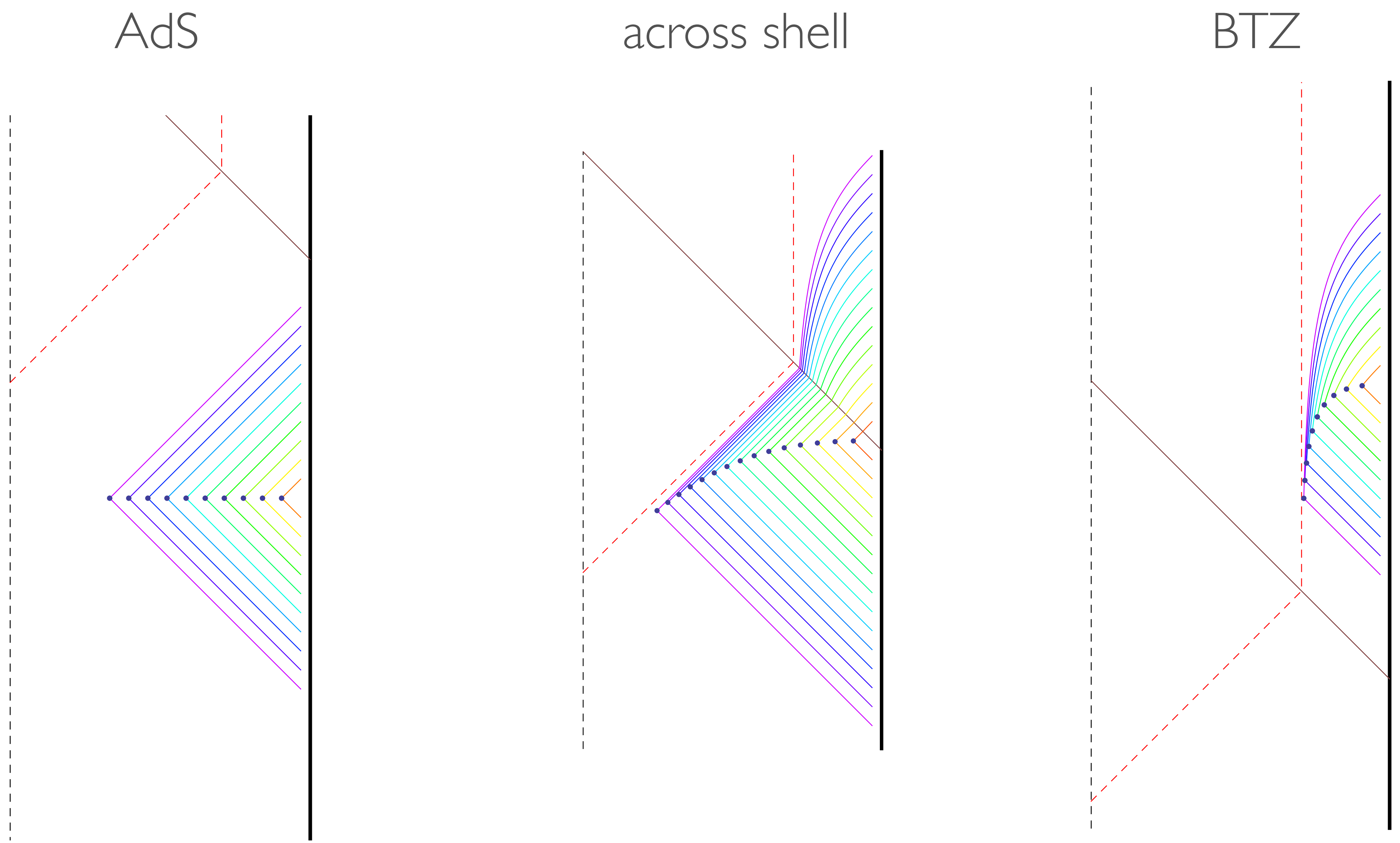
$$\vartheta(v) = \frac{1}{2}\left(\tanh\frac{v}{\delta} + 1\right)$$

Hubeny, MR, Takayanagi (2007)
Hubeny, MR, Tonni (wip)

holographic quench literature....

Profile of the causal wedge in Vaidya AdS

For fixed size of $\mathcal{A}$, causal wedge profile changes in time:



Quasi-teleological nature of $\chi_{\mathcal{A}}$

For fixed size of $\mathcal{A}$, deepest reach of $\Xi_{\mathcal{A}}$ monotonically increases from AdS value to BTZ value:

$r_\Xi^{\min}$

$t_{\mathcal{A}}$

Similarly for $\chi_{\mathcal{A}}$: Note that it starts increasing before $t_{\mathcal{A}} = t_{\text{shell}}$

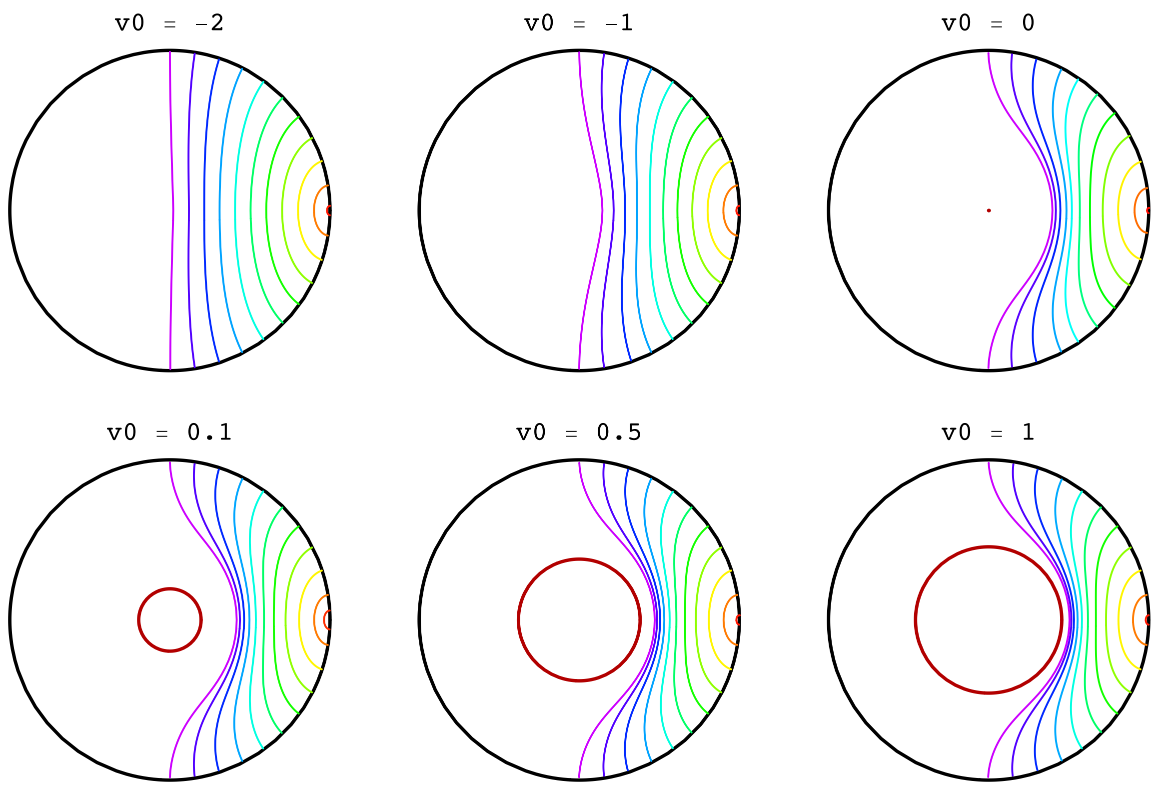
Time dependence: contrast $\chi_{\mathcal{A}}$ & $S_{\mathcal{A}}$

- Unlike $\Xi_{\mathcal{A}}$, the extremal surface $\mathfrak{E}_{\mathcal{A}}$ depends only on spatial information.
- Temporally we see local behaviour: $S_{\mathcal{A}}$ starts increasing only after the perturbation has come into play.

$\Xi_{\mathcal{A}}$

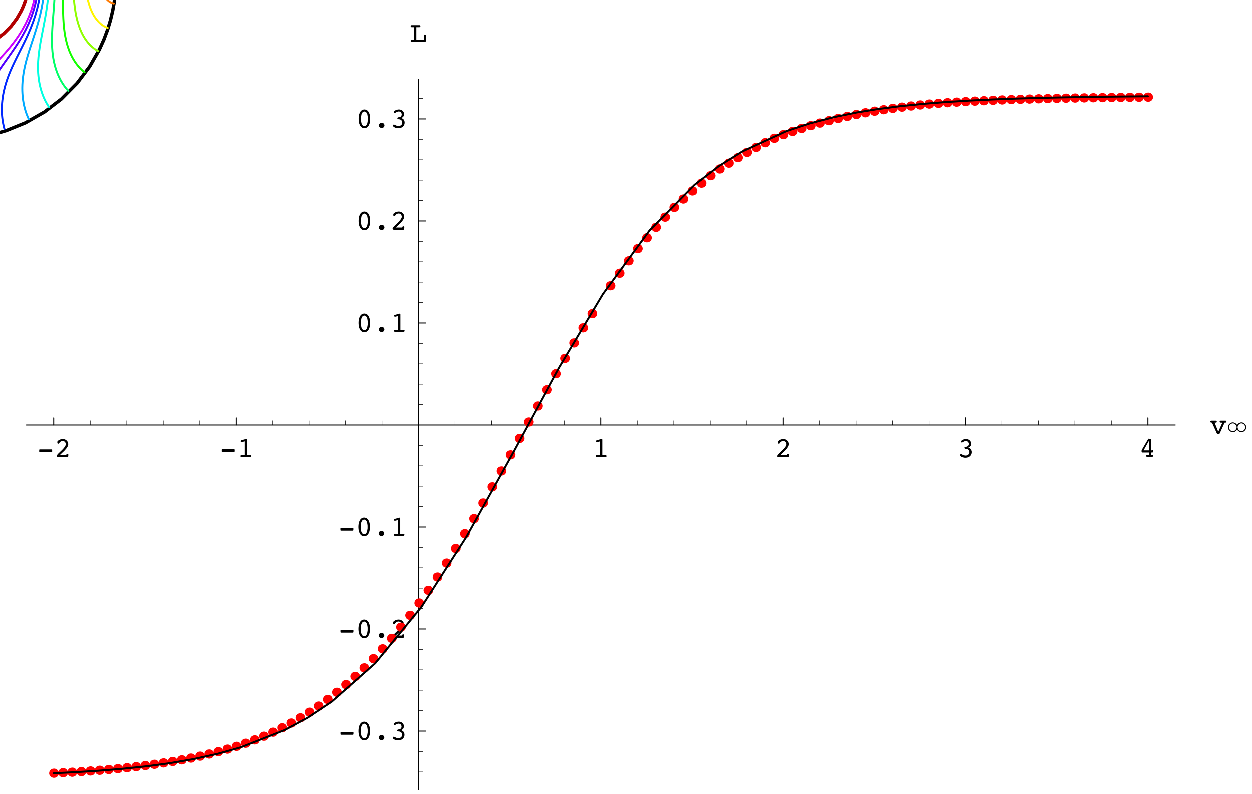
$\mathfrak{E}_{\mathcal{A}}$

Time dependence of $S_{\mathcal{A}}$



❖ Time-sequence of black hole formation in the bulk modeled by a null shell collapse.

❖ The temporal evolution of entanglement entropy.



Hubeny, MR, Takayanagi (2007)

Summary

- Conjecture that $\chi_{\mathcal{A}}$ is a field theoretic quantity that
  - provides a bound on the holographic information associated with $(\mathcal{A}, \rho_{\mathcal{A}})$
  - has entropy-like behaviour, without quite being a von Neumann entropy (violates strong subadditivity)
  - It bounds the entanglement entropy from above.
  - coincides with the entanglement entropy for special choice of $(\mathcal{A}, \rho_{\mathcal{A}})$
  - has intriguing quasi-teleological properties

- The bulk causal wedge $\blacklozenge_{\mathcal{A}}$ is a natural region that can be associated with the region of interest:
  - it is the minimal region that is related to & be reconstructable from $(\mathcal{A}, \rho_{\mathcal{A}})$.

Discussion

- Field theory interpretation of $\chi_{\mathcal{A}}$ and the causal wedge $\blacklozenge_{\mathcal{A}}$?
- Utility in setting up a reconstruction algorithm? With knowledge of $\chi_{\mathcal{A}}$ for various sub-regions can we recover all of the bulk geometry in $\blacklozenge_{\mathcal{A}}$?
- Bulk surfaces that are sensitive to field theory phases?
  - Flux sensitive surfaces that can distinguish between fractionalized and cohesive phases. Hartnoll, Radicevic (2012)
- Surfaces that can probe details of matter distribution in the bulk?
- Other causal constructions: complements of unions of various causal sets?
- Formulation of bulk locality & causality more directly from field theory?