Holographic Calculations of Renyi Entropy

(with H. Casini, M. Huerta, J. Hung, M. Smolkin & A. Yale)

(arXiv:1102.0440, arXiv:1110.1084)

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Renyi Entropy:

- generalization of entanglement entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$

$$S_\alpha = \frac{1}{1-\alpha} \log Tr\left[\,\rho_A^\alpha\,\right]$$

- recover entanglement entropy as a limit: $S_{EE} = \lim_{\alpha\to 1} S_\alpha$
- latter is now part of "standard" approach to calculating S_{EE} (powers easier than logarithm)
- other interesting limits:

$$S_\infty = \lim_{\alpha\to\infty} S_\alpha = -\log\lambda_1$$ where λ_1 is largest eigenvalue

$$S_0 = \lim_{\alpha\to 0} S_\alpha = \log\left[\,\mathcal{D}\right]$$

where $\mathcal{D} =$ number of nonvanishing eigenvalues

Renyi Entropy:

- generalization of entanglement entropy: $S_{EE} = -Tr\left[\rho_A \, \log \rho_A\right]$

$$S_\alpha = \frac{1}{1-\alpha} \log \, Tr\left[\, \rho_A^\alpha \,\right]$$

(Calabrese & Cardy)

- simple universal result for interval of length ℓ in d=2 CFT:

$$S_n = \frac{c}{6}\left(1+\frac{1}{n}\right) \log\left(\ell/\delta\right) \quad \left[\, S_{EE} = S_1 = \frac{c}{3} \log\left(\ell/\delta\right) \,\right]$$

(Calabrese, Cardy & Tonni)

- two intervals (in d=2 CFT): S_n considerably more complicated

⟶ involves entire spectrum; continuation to n=1 unknown

- for d > 2: growing number of examples (analytic and numerical)

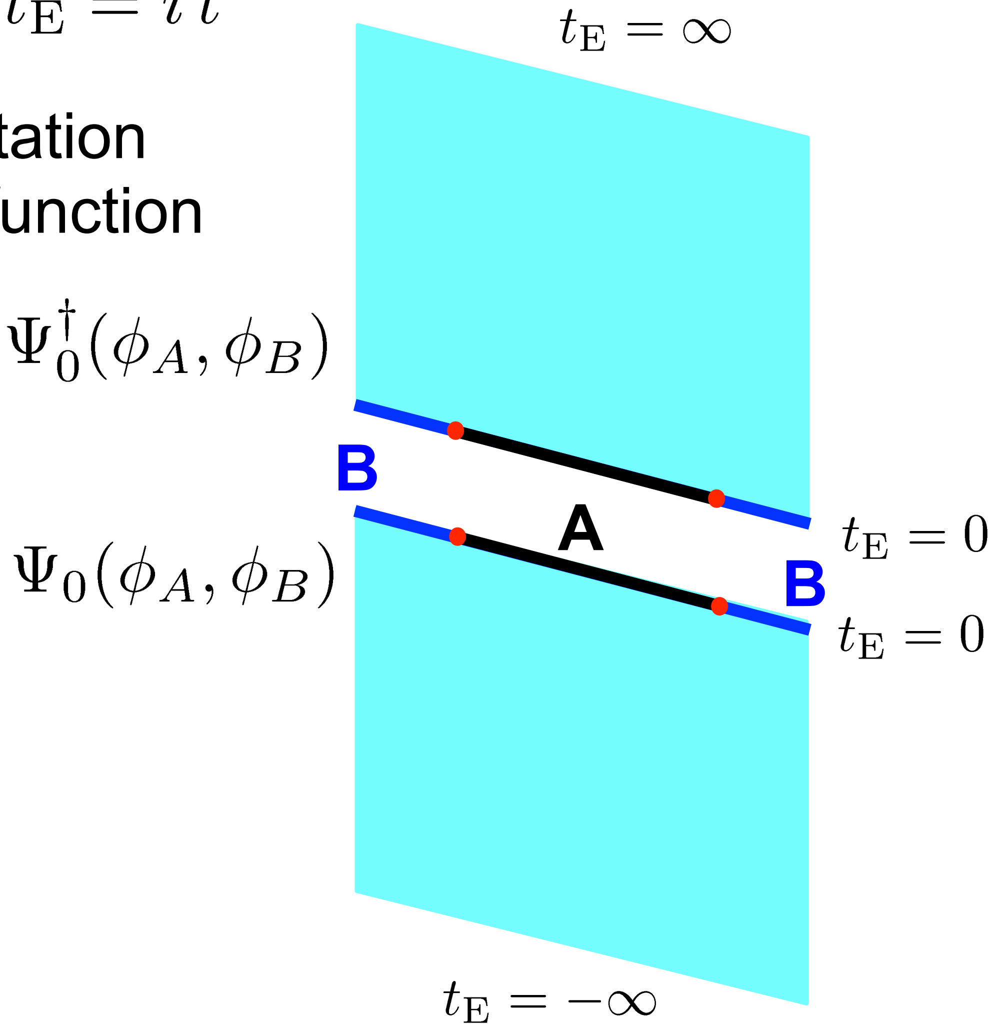
(Metlitski, Fuertes & Sachdev; Hastings, Gonzalez, Kallin & Melko; . . .)

⟶ calculations are demanding;
"standard" approach relies on replica trick

Calculating Renyi Entropy with "Replica Trick":

0. analytically continue: $t_{\mathrm{E}} = i\,t$

1. path integral representation of ground state wave function



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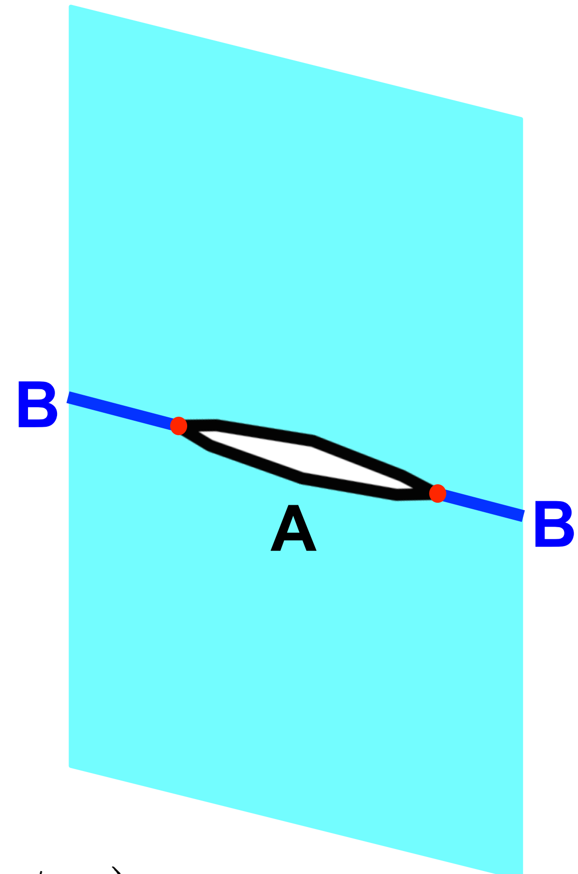
2. trace over ϕ_B to construct density matrix $\rho_A(\phi_A^+, \phi_A^-)$



$$\rho_A(\phi_A^+, \phi_A^-) = \mathrm{Tr}_{\phi_B} \Psi^\dagger(\phi_A^+, \phi_B)\Psi(\phi_A^-, \phi_B)$$

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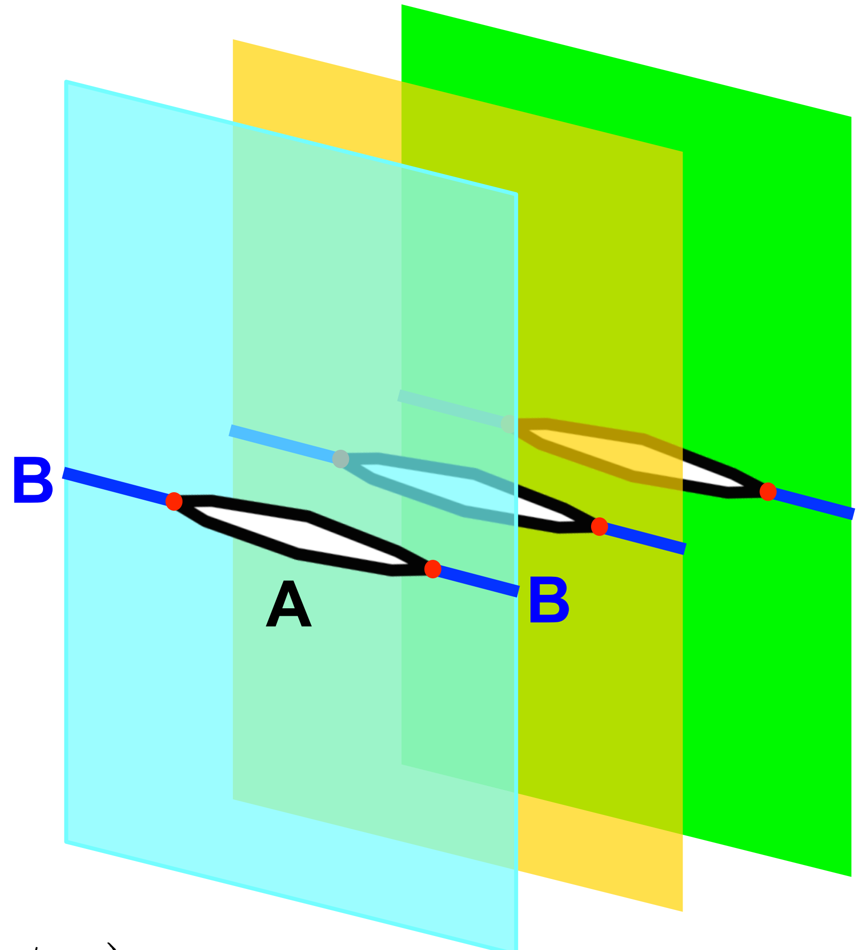

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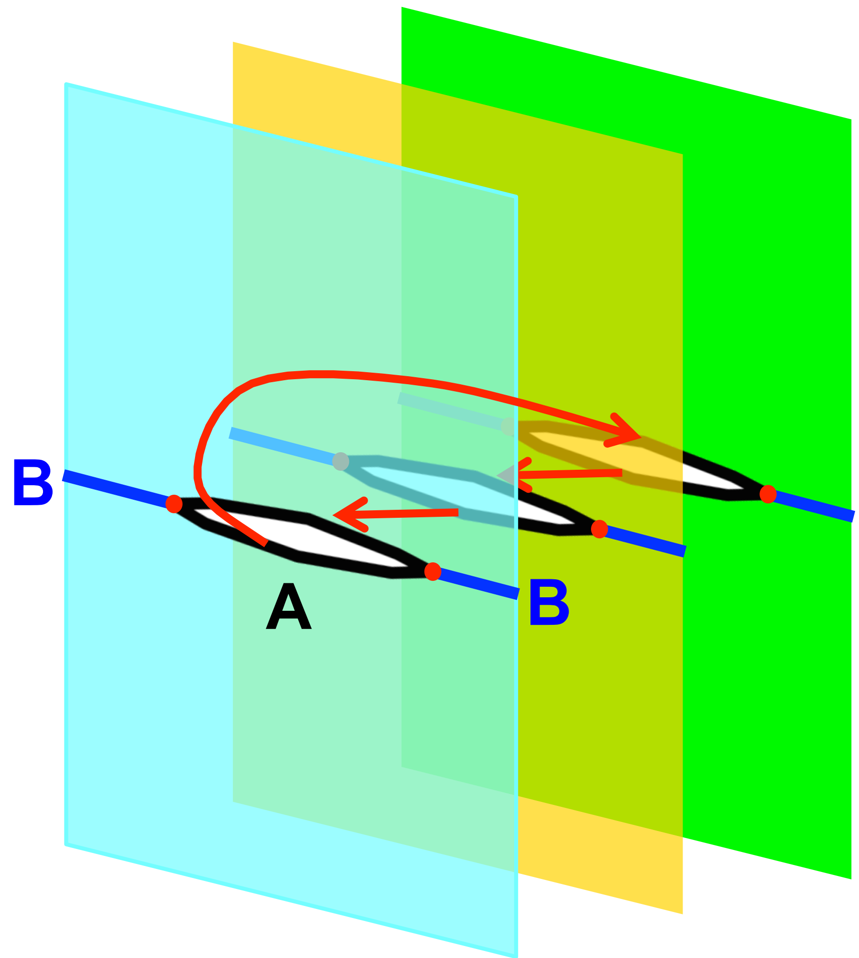
3. evaluate $\mathrm{Tr}(\rho_A^n)$



$$\mathrm{Tr}(\rho_A^n) = \mathrm{Tr}_{\phi_A^i}\left[\rho_A(\phi_A^1, \phi_A^{n-1}) \cdots \rho_A(\phi_A^3, \phi_A^2)\, \rho_A(\phi_A^2, \phi_A^1)\right]$$

Calculating Renyi Entropy with "Replica Trick":

0. analytically continue: $t_{\mathrm{E}} = i\,t$

1. path integral representation
   of ground state wave function

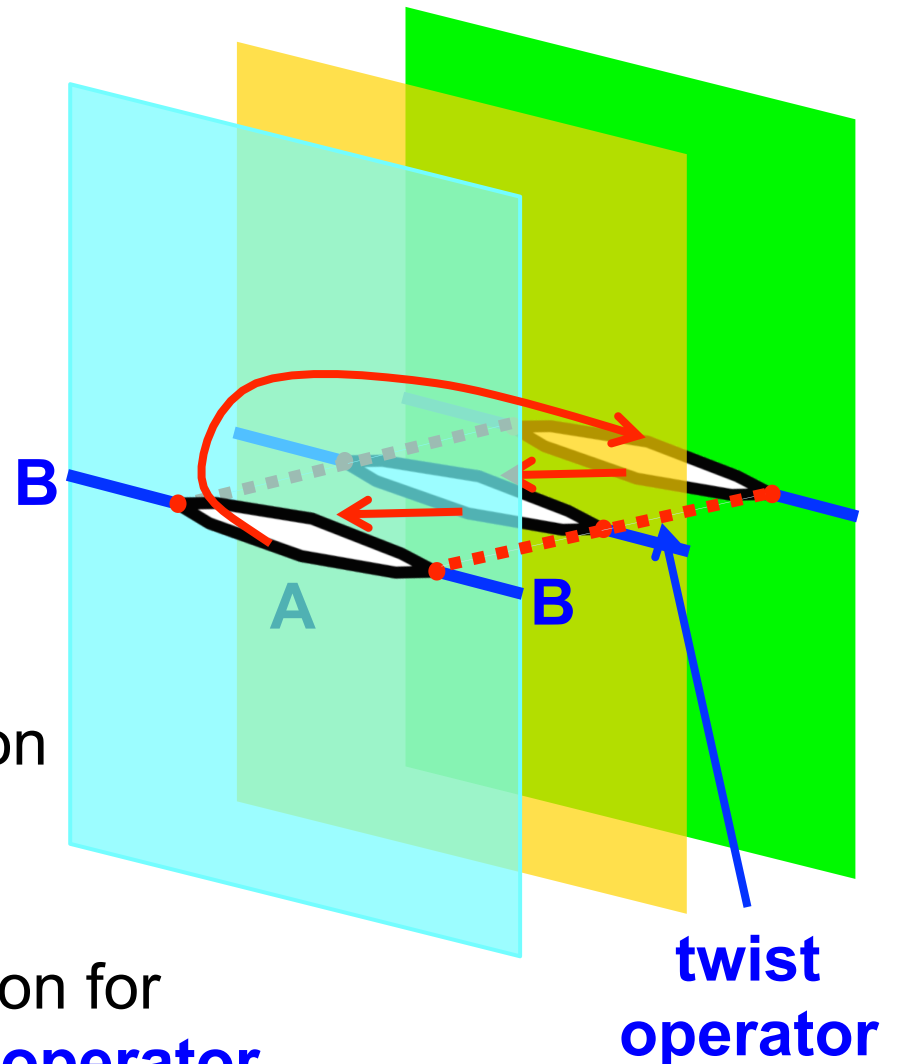
2. trace over ϕ_B to construct
   density matrix $\rho_A(\phi_A^+, \phi_A^-)$

3. evaluate $\mathrm{Tr}(\rho_A^n)$

evaluate euclidean partition function
on n-fold cover of original space

or

evaluate euclidean partition function for
n copies of field theory with **twist operator**
inserted at boundary of region **A**



Calculating Renyi Entropy with Holography:

- "standard" approach to calculate S_n relies on replica trick
- replica trick involves path integral of QFT on singular n-fold cover of background spacetime
- holographic slogan: "its all geometry!"

  $\longrightarrow$ how do we deal with singularity in boundary???

(Fursaev)

- "live with it!" $\longrightarrow$ singularity extends into the bulk and it is effectively "extremized" as part of bulk gravity path integral
- problem: you get the wrong answer

(Headrick)

- "smooth it out!" $\longrightarrow$ use conformal symmetry to "unwrap" singularity; find smooth boundary metric and corresponding smooth bulk solution (particularly "simple" for d=2: all bdy metrics locally conformally flat, all bulk sol's locally AdS_3)
- need another calculation with simpler holographic translation*

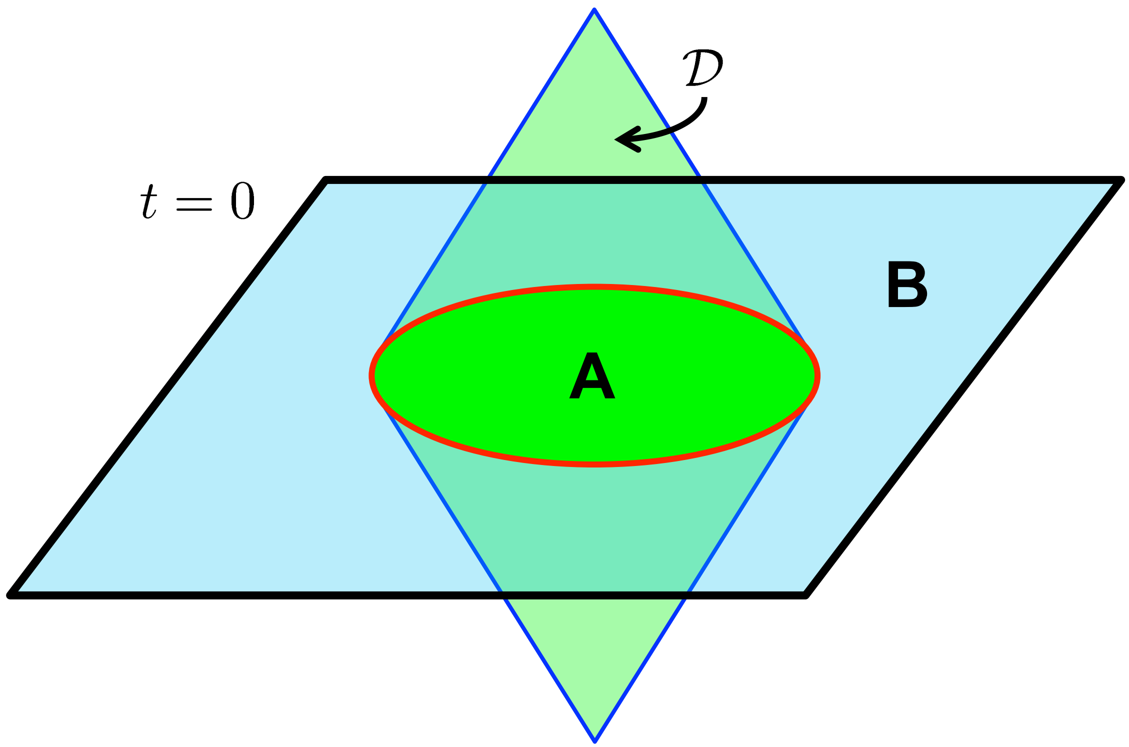
(*realizing "smooth it out!" strategy in disguise)

(Casini, Huerta & RM)

A Simple Calculation of Entanglement Entropy:

- take CFT in d-dim. flat space and choose $\Sigma = S^{d-2}$ with radius R

→ entanglement entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$



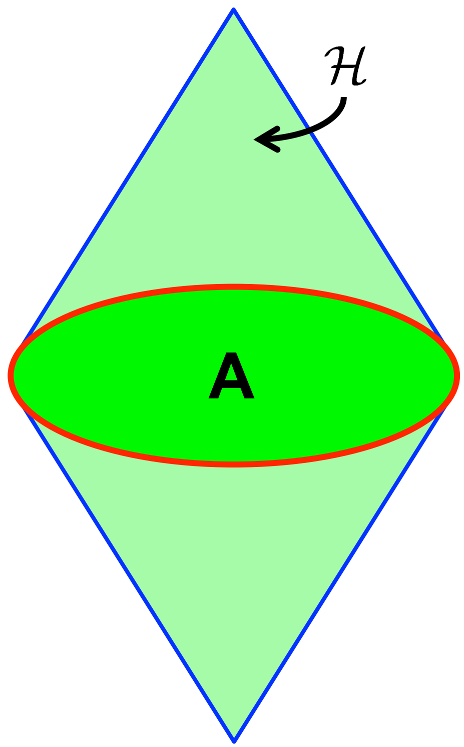
- density matrix ρ_A describes physics in entire causal domain $\mathcal{D}$
- conformal mapping: $\mathcal{D} \to \mathcal{H} = R_t \times H^{d-1}$

A Simple Calculation of Entanglement Entropy:

(Casini, Huerta & RM)

- take CFT in d-dim. flat space and choose $\Sigma = S^{d-2}$ with radius R

$\longrightarrow$ entanglement entropy: $S_{EE} = -Tr\left[\rho_A \, \log \rho_A\right]$



- conformal mapping: $\mathcal{D} \to \mathcal{H} = R_t \times H^{d-1}$

curvature scale: 1/R temperature: T=1/2πR !!

- for CFT: $\rho_{thermal} = U\,\rho_A\,U^{-1}$ $\longrightarrow$ $\boxed{S_{EE} = S_{thermal}}$

(Casini, Huerta & RM)

A Simple Calculation of Entanglement Entropy:

• take CFT in d-dim. flat space and choose S^{d-2} with radius R

→ entanglement entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$

→ by conformal mapping relate to thermal entropy on $\mathcal{H} = R \times H^{d-1}$ with $\mathcal{R} \sim 1/R^2$ and $T=1/2\pi R$

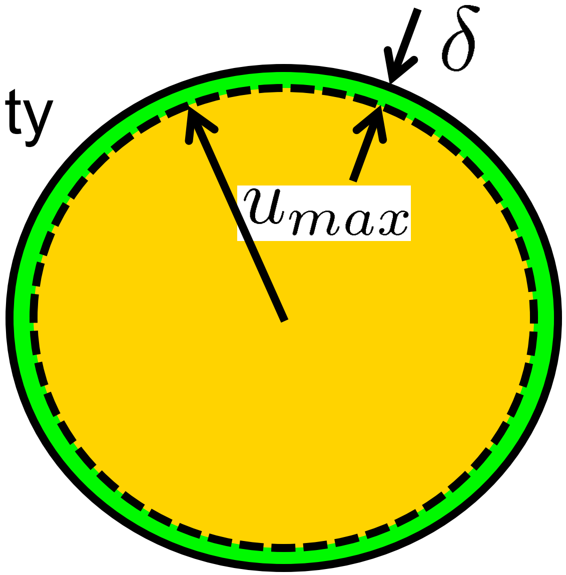
$$\boxed{S_{EE} = S_{thermal}}$$

• note both sides of equality are divergent

→ $S_{thermal}$ sums constant entropy density over infinite volume

• must follow original UV cut-off through conformal mapping to IR cut-off on H^{d-1}

$$u_{max} \simeq R/\delta$$



A Simple Calculation of Entanglement Entropy:

(Casini, Huerta & RM)

- take any CFT in d-dim. flat space and choose S^{d-2} with radius R

→ entanglement entropy: $S_{EE} = -Tr\left[\rho_A \, \log \rho_A\right]$

→ by conformal mapping relate to thermal entropy on $\mathcal{H} = R \times H^{d-1}$ with $\mathcal{R} \sim 1/R^2$ and $T=1/2\pi R$

$$\boxed{S_{EE} = S_{thermal}}$$

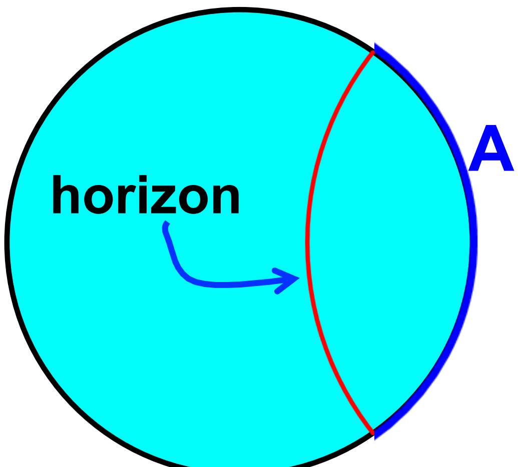
AdS/CFT correspondence:

- thermal bath in CFT = black hole in AdS

$$\boxed{S_{EE} = S_{thermal} = S_{horizon}}$$

- only need to find appropriate black hole

→ topological BH with hyperbolic horizon which intersects ∂A on AdS boundary



(Aminneborg et al; Emparan; Mann; . . .)

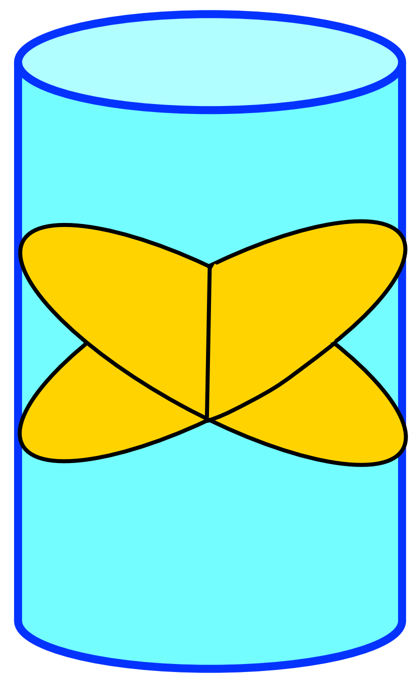
$$\boxed{S_{EE} = S_{thermal} = S_{horizon}}$$

- desired “black hole” is a hyperbolic foliation of empty AdS space

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 - dt^2 + d\vec{x}^2\right) d\tau^2 + \rho^2\, d\Sigma_2^{d-1} \longrightarrow T = \frac{1}{2\pi R}$$

- bulk coordinate transformation implements desired conformal transformation on boundary

- “Rindler coordinates” of AdS space:



$$\boxed{S_{EE} = S_{thermal} = S_{horizon}}$$

- desired "black hole" is a hyperbolic foliation of empty AdS space

$$ds^2 = \frac{L^2\, d\rho^2}{(\rho^2 - L^2)} - \frac{\rho^2 - L^2}{R^2}\, d\tau^2 + \rho^2\, d\Sigma_2^{d-1} \longrightarrow T = \frac{1}{2\pi R}$$

- apply Wald's formula (for any gravity theory) for horizon entropy:

$$S = -2\pi \int d^{d-1}x\sqrt{h}\ \frac{\partial \mathcal{L}}{\partial R^{\mu\nu}{}_{\rho\sigma}}\, \hat{\varepsilon}^{\,\mu\nu}\, \hat{\varepsilon}_{\rho\sigma}$$

$$= \frac{2\pi}{\pi^{d/2}}\Gamma\left(d/2\right) \boxed{a_d^*} V\left(H^{d-1}\right)$$

(RCM & Sinha)

where a_d^* = central charge for "A-type trace anomaly"

contains all of the couplings from the gravity theory

for even d

eg. $a_d^* = \dfrac{\pi^{d/2}}{\Gamma(d/2)} \dfrac{L^{d-1}}{\ell_P^{d-1}}$ for Einstein gravity

= entanglement entropy defines effective central charge

for odd d

$$\boxed{S_{EE} = S_{thermal} = S_{horizon}}$$

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- apply Wald’s formula (for any gravity theory) for horizon entropy:

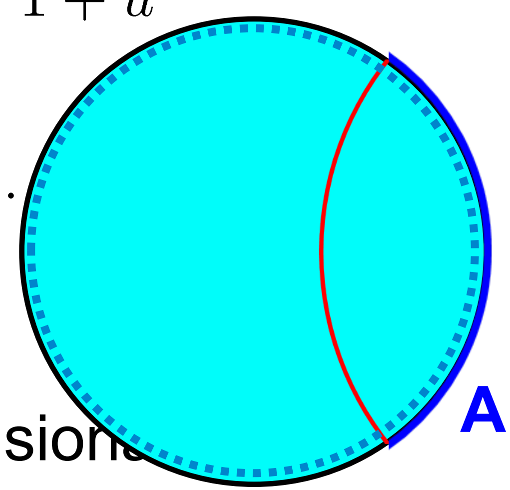
$$S = \frac{2\pi}{\pi^{d/2}}\Gamma\left(d/2\right)\ a_d^* \boxed{V\left(H^{d-1}\right)}$$

$$ds^2 = \frac{du^2}{1+u^2} + u^2\, d\Omega_2^{d-2}$$

intersection with standard regulator surface: $z_{min} = \delta$

$$S = a_d^* \frac{4\pi^{\frac{d-3}{2}}}{(d-2)\Gamma\left(\frac{d-1}{2}\right)} \underbrace{\left(\frac{R}{\delta}\right)^{d-2}}_{} + \cdots$$

“area law” for d-dimensiona



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- apply Wald's formula (for any gravity theory) for horizon entropy:

$$S = \frac{2\pi}{\pi^{d/2}} \Gamma\left(d/2\right)\; a_d^*\; V\left(H^{d-1}\right)$$

$$ds^2 = \frac{du^2}{1+u^2} + u^2\, d\Omega_2^{d-2}$$

universal contributions:

$$S = \;\cdots\; + \;(-)^{\frac{d}{2}-1} 4\, a_d^* \;\log\left(2R/\delta\right)\; + \;\cdots \quad \text{for even d}$$

$$\cdots\; + \quad (-)^{\frac{d-1}{2}} 2\pi\, a_d^* \quad + \;\cdots \quad \text{for odd d}$$

$$\boxed{S_{EE} = S_{thermal} = S_{horizon}}$$

universal contributions:

$$S = \cdots + (-)^{\frac{d}{2}-1} 4\, a_d^* \log(2R/\delta) + \cdots$$ for even d

$$\cdots + (-)^{\frac{d-1}{2}} 2\pi\, a_d^* + \cdots$$ for odd d

- discussion extends to case with background: $R^{1,d-1} \to R \times S^{d-1}$
- for Einstein gravity, coincides with Ryu & Takayanagi result and horizon (bifurcation surface) coincides with R&T surface

$\longrightarrow$ no extremization procedure here?!?

- applies for classical bulk theories beyond Einstein gravity
- can imagine calculating “quantum” corrections (eg, Hawking rad)

(Casini & Huerta)
(Hung, RM, Smolkin & Yale)
(Klebanov, Pufu, Sachdev & Safdi)

Holographic Renyi entropy:

- apply previous approach to calculate Renyi entropy

$$S_n = \frac{1}{1-n} \log Tr\left[\,\rho_A^n\,\right]$$

- there discussion lead to "thermal" density matrix

$$\rho_A = U^{-1} \frac{e^{-H/T_0}}{Tr\left[e^{-H/T_0}\right]} U \qquad \text{with} \qquad T_0 = \frac{1}{2\pi R}$$

$$\Rightarrow \quad Tr\left[\rho_A^n\right] = \frac{Tr\left[e^{-nH/T_0}\right]}{Tr\left[e^{-H/T_0}\right]^n} \quad \longleftarrow$$

partition function at new temperature, $T = T_0/n$

- hence find convenient formulae using $F(T) = -T \log Z(T)$

$$S_n = \frac{n}{1-n} \frac{1}{T_0} \left[F(T_0) - F(T_0/n)\right]$$

Holographic Renyi entropy:

(Casini & Huerta)
(Hung, RM, Smolkin & Yale)
(Klebanov, Pufu, Sachdev & Safdi)

- then use $S=-\partial F/\partial T$ to find:

$$S_n = \frac{n}{n-1}\frac{1}{T_0}\int_{T_0/n}^{T_0} S(T)\,dT \qquad \text{with } T_0 = \frac{1}{2\pi R}$$

Renyi entropy for spherical Σ

thermal entropy on hyperbolic space H^{d-1}

- **turning to AdS/CFT correspondence**, we need topological black hole solutions at arbitrary temperature

$$ds^2 = -\left(\frac{r^2}{L^2}f(r)-1\right)N^2 dt^2 + \frac{dr^2}{\frac{r^2}{L^2}f(r)-1} + r^2 d\Sigma^2_{d-1} \qquad \left[N^2 = \frac{L^2}{f_\infty R^2}\right]$$

- work with gravity theories where we can calculate: Einstein, Gauss-Bonnet, Lovelock, quasi-topological, …..

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

- for example, with Einstein gravity:

$$S_q = \frac{\pi\, q}{q-1} \boxed{\left(\frac{L}{\ell_P}\right)}^{d-1} \left(2 - x_q^{d-2}\left(1+x_q^2\right)\right) \boxed{V\left(H^{d-2}\right)}$$

where $x_q = \frac{1}{qd}\left(1+\sqrt{1-2dq^2+d^2q^2}\right)$

$\longrightarrow$ need to regulate integral over horizon:

$$V\left(H^{d-2}\right) \simeq (-)^{\frac{d}{2}-1} \frac{2\pi^{(d-2)/2}}{\Gamma(d/2)} \log(2R/\delta) \qquad \text{for even d}$$

$\longrightarrow$ translate gravity couplings to CFT parm's:

$$C_T = \frac{\pi^{d/2}}{\Gamma(d/2)} \left(\frac{L}{\ell_P}\right)^{d-1}$$

$$\Rightarrow \quad (S_q)_{univ} = (-)^{\frac{d}{2}-1}\, C_T\, \frac{2\, q}{1-q} \left(2 - x_q^{d-2}\left(1+x_q^2\right)\right) \log(2R/\delta)$$

(for even d)

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

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(for even d)

- compare to d=2 result:

$$S_n \;=\; \frac{c}{6}\left(1+\frac{1}{n}\right)\ \log\left(\ell/\delta\right)$$

⟶ matches universal result of Calabrese & Cardy ✓

- might suggest simple universal form for even d:

$$(S_q)_{univ} = C_T\ \times\ f(d,q)\ \times\ \log(2R/\delta)$$

Holographic Renyi entropy:

- consider Gauss-Bonnet gravity (**with d=4**):

$$I = \frac{1}{2\ell_p^3}\int d^5x\,\sqrt{-g}\left[\frac{12}{L^2} + R + L^2\frac{\lambda}{2}\,\underbrace{\left(R^{abcd}R_{abcd} - 4R_{ab}R^{ab} + R^2\right)}_{\text{4d Euler density}}\right]$$

- higher curvature but eom are still **second order!!** (Lovelock)
- studied in detail for stringy gravity in 1980's

(Zwiebach; Boulware & Deser; Wheeler; Myers & Simon; . . . .)

- interest recently in AdS/CFT studies – a toy model with $c \neq a$

$$c = \pi^2\frac{\tilde{L}^3}{\ell_P^3}\left(1 - 2\lambda f_\infty\right)\,, \qquad a = \pi^2\frac{\tilde{L}^3}{\ell_P^3}\left(1 - 6\lambda f_\infty\right)$$

where $\tilde{L} = L/\sqrt{f_\infty}$ and $f_\infty = (1 - \sqrt{1-4\lambda})/(2\lambda)$

(eg, Brigante, Liu, Myers, Shenker,Yaida, de Boer, Kulaxizi, Parnachev, Camanho, Edelstein, Buchel, Sinha, Paulos, Escobedo, Smolkin, Cremonini, Hofman, ...)

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

- for example, with GB gravity and **d=4**:

$$(S_n)_{univ} = \log(2R/\delta)\,\frac{n}{2}\,\frac{1-x^2}{1-n}\left[(5c-a)x^2 - (13c-5a)\right.$$
$$\left. +16c\frac{2cx^2-c+a}{(3c-a)x^2-c+a}\right]$$

where $\quad 0 = x^3 - \frac{3c-a}{5c-a}\left(\frac{x^2}{n}+x\right) + \frac{1}{n}\frac{c-a}{5c-a}$

- unfortunately indicates no simple universal form:

$$(S_n)_{univ} = a\ \times\ f\left(d,n,\frac{c}{a},t_4,\cdots\right)\ \times\ \log(2R/\delta)$$

- further work (with quasi-topological gravity) shows the universal coefficient depends on more CFTdata than central charges

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

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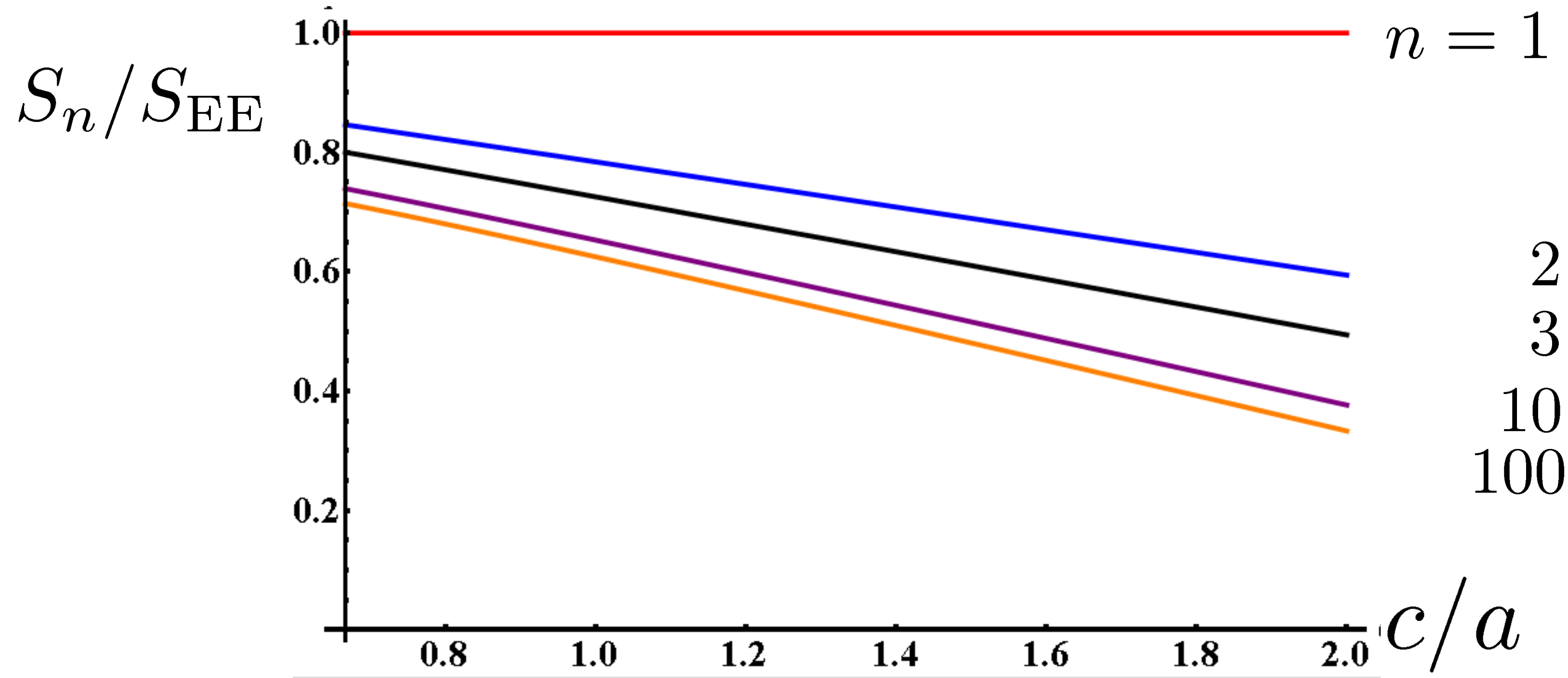
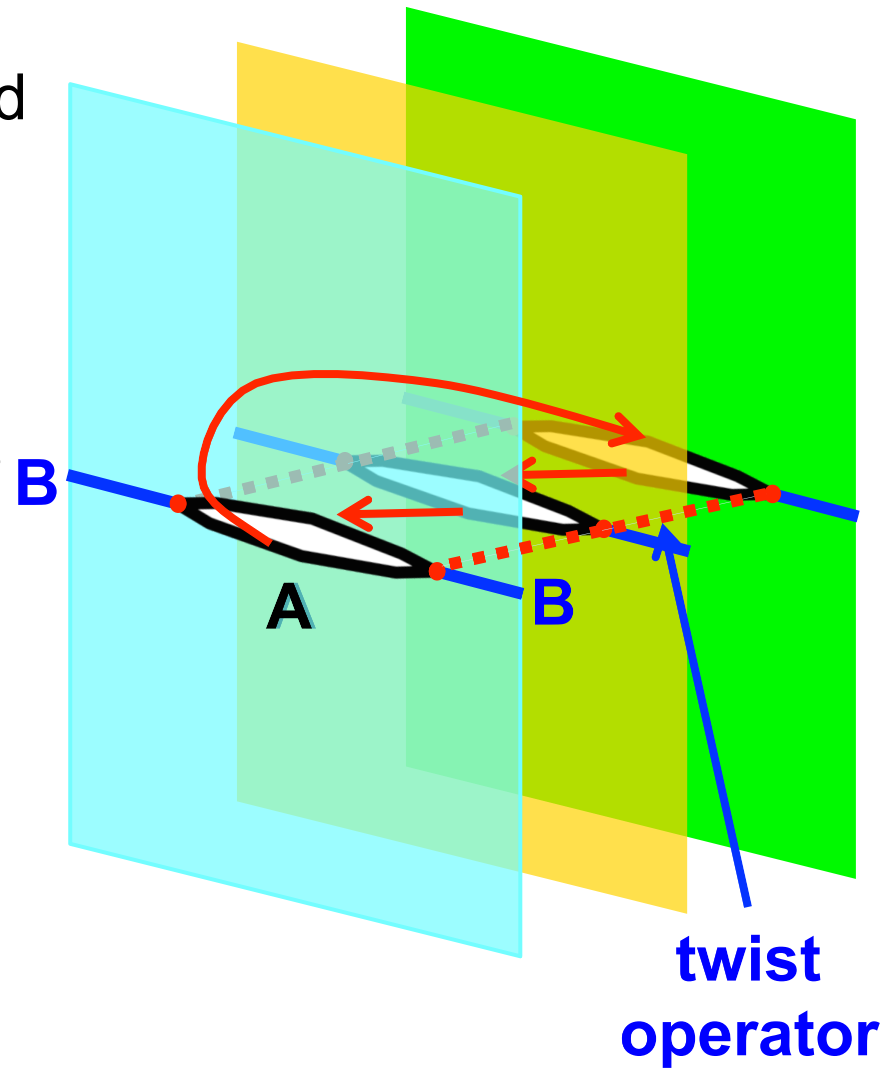
- note despite intimidating expression, results relatively simple:



Twist Operators:

- $\mathrm{Tr}(\rho_A^n)$ evaluated as Euclidean path integral over n copies of field theory inserting **twist operators** at boundary of region **A**
- twist operators introduce n-fold branch cuts where various copies of fields talk to each other



Twist Operators:

- $\mathrm{Tr}(\rho_A^n)$ evaluated as Euclidean path integral over n copies of field theory inserting **twist operators** at boundary of region **A**
- twist operators introduce n-fold branch cuts where various copies talk to each other
- elegant results for d=2, eg, scaling dimension of twist operators

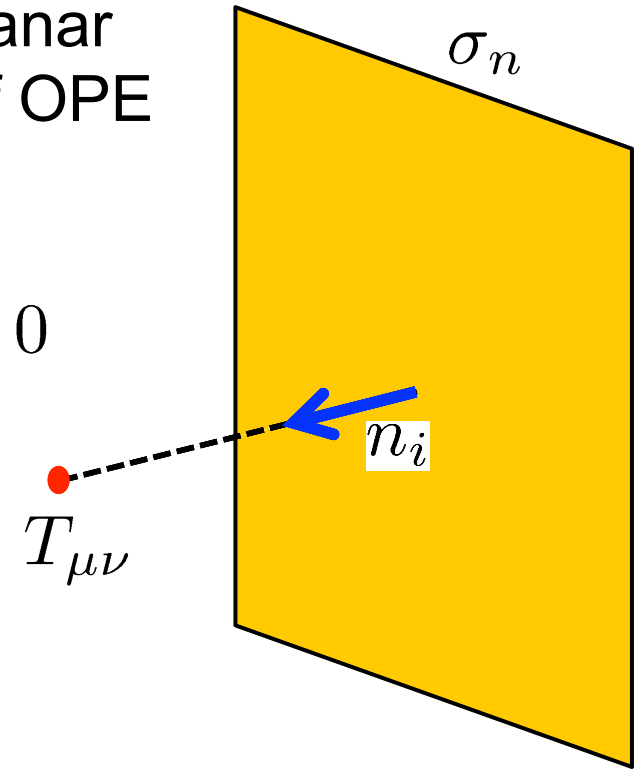
$$h_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

(Calabrese & Cardy)

- in d dimensions, would be (d–2)-dimensional surface operators but little is known about their properties

Twist Operators:

- consider insertion of stress tensor near planar twist operator for CFT in R^d → structure of OPE fixed by symmetry



$$\langle T_{ab}\,\sigma_n\rangle = -\frac{h_n}{2\pi}\,\frac{\delta_{ab}}{r_\perp^d}\,, \qquad \langle T_{ai}\,\sigma_n\rangle = 0$$

$$\langle T_{ij}\,\sigma_n\rangle = \frac{h_n}{2\pi}\,\frac{(d-1)\delta_{ij} - d\,n_i\,n_j}{r_\perp^d}$$

where $a, b \;\parallel\; \sigma_n$ and $i, j \;\perp\; \sigma_n$

- h_n commonly called scaling dimension (precisely matches d=2)

Twist Operators:

- consider previous calculation for spherical entangling surface:
- conformal mapping for spherical entangling surface

→ Euclidean version gives one-to-one map: $S^1 \times H^{d-1} \to R^d$

→ with $\Delta\tau_E = n/T_0 = 2\pi R\, n \;\; (n \in \mathbb{Z})$ get n-fold cover of R^d

$$S^1 \times H^{d-1}: \qquad ds^2 = d\tau_E^2 + R^2 \left(du^2 + \sinh^2\!u\, d\Omega_{d-2}^2\right)$$

coord. transformation: $\exp\left(-u - i\tau_E/R\right) = \dfrac{R - r - it_E}{R + r + it_E}$

$$\left[R^d\right]_n: \qquad ds^2 = \Omega^2 \left[dt_E^2 + dr^2 + r^2\, d\Omega_{d-2}^2\right]$$

$$\Omega^2 = \frac{4R^4}{(R^2 - r^2 + t_E^2)^2 + 4r^2 t_E^2}$$

Holographic aside: (*realizing "smooth it out!" strategy in disguise)

Twist Operators:

- consider previous calculation for spherical entangling surface:
- conformal mapping for spherical entangling surface

→ Euclidean version gives one-to-one map: $S^1 \times H^{d-1} \to R^d$

→ with $\Delta\tau_E = n/T_0 = 2\pi R\, n \quad (n \in \mathbb{Z})$ get n-fold cover of R^d

→ "generates" spherical twist operator σ_n on $S^{d-2}: \; r = R$

Strategy to evaluate h_n

- evaluate $\langle T_{\alpha\beta} \rangle$ in thermal bath; map back to $[R^d]_n$;
  evaluate $\langle T_{\alpha\beta}\, \sigma_n \rangle$ in limit that $T_{\alpha\beta}$ approaches twist operator ;
  read h_n off from singularity in correlator

Twist Operators:

- evaluate $\langle T_{\alpha\beta}\,\sigma_n\rangle$ correlator by mapping from thermal bath

$$\langle T_{\alpha\beta}\,\sigma_n\rangle = \underbrace{\Omega^{d-2}\,\frac{\partial X^\mu}{\partial x^\alpha}\,\frac{\partial X^\nu}{\partial x^\beta}}_{\text{creates singularity near twist operator}}\Big(\underbrace{\langle T_{\mu\nu}(T_0/n)\rangle}_{\text{uniform thermal bath}} - \mathcal{A}_{\mu\nu}(T_0)\rangle\Big)$$

anomalous bit

(compare: Marolf, Rangamani & Van Raamsdonk)

- read off h_n from short distance singularity

$$h_n = 2\pi\frac{n\,R^d}{d-1}\Big(\mathcal{E}(T_0) - \mathcal{E}(T_0/n)\Big)$$

[no holography, yet!!]

Twist Operators:

- evaluate $\langle T_{\alpha\beta}\,\sigma_n\rangle$ correlator by mapping from thermal bath

$$\langle T_{\alpha\beta}\,\sigma_n\rangle = \underbrace{\Omega^{d-2}\,\frac{\partial X^\mu}{\partial x^\alpha}\,\frac{\partial X^\nu}{\partial x^\beta}}_{\text{creates singularity near twist operator}}\Big(\underbrace{\langle T_{\mu\nu}(T_0/n)\rangle}_{\text{uniform thermal bath}} - \langle T_{\mu\nu}(T_0)\rangle\Big)$$

anomalous bit

(compare: Marolf, Rangamani & Van Raamsdonk)

- for example, with GB gravity and **d=4**:

$$h_n = \frac{n}{4\pi}\left(x^2-1\right)\left[c-a-x^2(5c-a)\right]$$

where $\quad 0 = x^3 - \dfrac{3c-a}{5c-a}\left(\dfrac{x^2}{n}+x\right)+\dfrac{1}{n}\,\dfrac{c-a}{5c-a}$

- no simple universal form can be expected
- again, CFT data beyond central charges also appears

Twist Operators:

- holographic results show remarkable simplicity with $n \to 1$

$$\partial_n h_n|_{n=1} = \frac{2}{d-1}\,\pi^{1-\frac{d}{2}}\,\Gamma\left(d/2\right)\, C_T$$

- recall general (non-holographic) formula:

$$h_n = 2\pi\frac{n\,R^d}{d-1}\left(\mathcal{E}(T_0) - \mathcal{E}(T_0/n)\right)$$

$$\longrightarrow \quad \partial_n h_n|_{n=1} = -\frac{2\pi R^d}{(d-1)T_0}\langle\, T_{\tau\tau}\, H\,\rangle$$

- clear that result comes for OPE of two stress tensors!
- verify precise form above holds as general result for any CFT
- generalize:

$$\partial_n^k h_n|_{n=1} = \frac{(-)^k 2\pi R^d}{(d-1)T_0^k}\left(\langle \hat{T}_{\tau\tau}\,\overbrace{H\cdots H}^{k}\rangle - k\,T_0\langle \hat{T}_{\tau\tau}\,\overbrace{H\cdots H}^{k-1}\rangle\right) \quad \text{for } k\geq 2$$

- verified precise form for k=2 as general result for any CFT

Conclusions:

- AdS/CFT correspondence (gauge/gravity duality) has proven an robust tool to study strongly coupled gauge theories
- holographic entanglement/Renyi entropy is part of interesting dialogue has opened between string theorists and physicists in a variety of fields (eg, condensed matter, nuclear physics, . . .)
- potential to learn lessons about issues in boundary theory
  eg, readily calculate Renyi entropies and study twist operators
  for wide class of (holographic) theories in higher dimensions
- potential to learn lessons about issues in bulk gravity theory
  eg, holographic entanglement entropy may give new
  insight into quantum gravity or emergent spacetime

Lots to explore!