Physics 406: Homework 6

1. **Number of photons in the universe:**

   (a) Calculate the total number of photons of all frequencies in a cavity of volume V at equilibrium at temperature T. You will need to know that

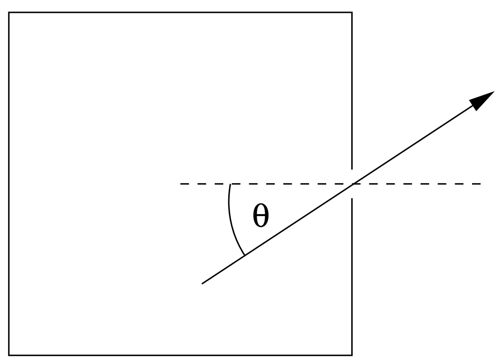
   $$\int_0^\infty \frac{x^2}{\mathrm{e}^x - 1}\,\mathrm{d}x = 2\zeta(3) = 2.404\ldots,$$

   where $\zeta(x)$ is the Riemann ζ-function.

   (b) The diameter of the universe is about 20 billion light years. About how many photons are there in the universe?

   (c) Supposing that no photons enter or leave the universe as it expands, how are the temperature and volume of the universe related? (A different way of working out the answer to this part appeared on the midterm.)

2. **The Stefan-Boltzmann constant:** We have argued that a small hole in the side of a box at temperature T radiates as a black body at that temperature. Suppose the hole has area A and consider the radiation that escapes from the box through the hole:



   We have shown that the energy density in the box is

   $$u = \frac{\pi^2 k^4 T^4}{15\hbar^3 c^3},$$

   where c is the speed of light and k is the Boltzmann constant. Consider the radiation incident on the hole from a direction that makes an angle θ with the normal to the hole, and arrives within a solid angle $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ of that direction.

   (a) What is the energy in a small volume element $r^2\,\mathrm{d}r\,\mathrm{d}\Omega$, where r is measured from the hole? And what fraction of this actually hits the hole, rather than going in some other direction? Thus integrate over r from 0 to c to find the amount of energy transported through the hole per unit time from the given solid angle $\mathrm{d}\Omega$.

   (b) Integrate over θ and ϕ to get the total radiation leaving the hole per unit time. (Hint: be careful about the limits of the integration.)

(c) The amount of radiation J per unit area given off by a black body at temperature T is given by the Stefan-Boltzmann law:

$$J = \sigma_B T^4,$$

where σ_B is the Stefan-Boltzmann constant. Show that the Stefan-Boltzmann constant has the value $5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$.

3. **Radiation pressure:**

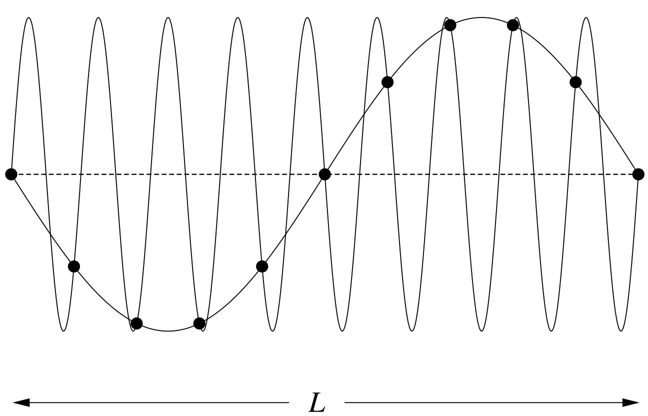
(a) Show that the free energy F of the radiation in a cavity of volume V at temperature T is

$$F = \frac{V\tau^4}{\pi^2\hbar^3c^3}\int_0^\infty x^2\ln(1-\mathrm{e}^{-x})\,\mathrm{d}x = -\frac{V\tau^4\pi^2}{45\hbar^3c^3}.$$

(Hint: look at the midterm if you don't know how to do this.)

(b) Hence find the pressure exerted by the radiation on the walls of the box. This pressure is called **radiation pressure**.

4. **Aliasing:** Here's a picture representing the aliasing phenomenon that's much better than the one I drew in class:



It depicts a one-dimensional version of the aliasing problem. The dots are atoms. The solid lines represent two possible waves, both of which correspond to the same positions of the atoms.

Suppose the length of the system in atom spacings is L, as shown. In the picture $L = 10$. Let the atoms be numbered $k = 0\ldots L$. Then the displacement y_k of the kth atom is

$$y_k = \sin(2\pi k/\lambda_h),$$

where λ_h is the wavelength of the high-frequency wave, which is related to the frequency by $\lambda_h\omega_h = 2\pi v$, with v being the speed of sound.

(a) Show that if $\lambda_h = 2L/n$, where n is any integer, then the wave is zero at both ends of the system. How many such modes are there between frequencies ω and $\omega + \mathrm{d}\omega$?

(b) Show that another similar wave with the longer wavelength $\lambda_l = 2L/(2L-n)$ gives the same displacements of the atoms (except for a minus sign).

(c) Hence what is the Debye frequency $\omega_{\max}$ for this one dimensional solid (i.e., the maximum frequency of a unique wave in the system)?