Physics 406: Homework 7

1. **Entropy of mixing:**

   (a) Suppose we have two boxes with the same volume, with N molecules of a perfect gas in one and N molecules of a different type of perfect gas in the other, both at the same temperature. If the mass of the particles in each gas is m_a and m_b respectively, what is the total entropy of the two boxes together?

   (b) Now suppose we put our two boxes together and make a hole where the gases can pass from one to the other, so that they mix. When they are fully mixed, what is the partition function of the mixed system? (Remember that atoms of the same type will be indistinguishable, but atoms of different types are obviously distinguishable from one another.)

   (c) Hence what is the entropy of the whole system after mixing? How much did the entropy change by because we mixed the gases?

2. **Phonon specific heat above the Debye temperature:** We have shown that the internal energy U of the phonons in a solid is

$$U = \frac{9N\tau^4}{\theta^3} \int_0^{x_{\max}} \frac{x^3}{e^x - 1}\, dx,$$

   where $x_{\max} = \theta/\tau$ with the Debye temperature θ being given by

$$\theta = \sqrt[3]{6\pi^2\hbar^3 v^3 \rho}, \qquad \rho = \frac{N}{V}.$$

   When $\tau \gg \theta$ show by expanding the integrand above that the value of the heat capacity is $C = 3N$ to leading order *and* to second order—i.e., that including the next order term in the expansion makes no difference to the value of the heat capacity at high temperature.

3. **Gibbs distribution for a simple system:** Suppose we have a simple system that has three possible states. Either it can have no particles in it, in which case it has energy 0, or it can have one particle with either energy 0 or energy ε.

   (a) Write down an expression for the grand partition function $\mathcal{Z}$.

   (b) Find the average number of particles in the system as a function of the temperature τ and the chemical potential μ.

   (c) Find the average internal energy of the system.

4. **Classical perfect gas:** In class we derived the properties of the perfect gas within the grand canonical ensemble by viewing it as the classical limit of a quantum perfect gas. An alternative derivation of the same results is as follows. Recall that the canonical partition function for N particles in a classical perfect gas is

$$Z_N = \frac{1}{N!} Z_1^N, \tag{1}$$

   where the partition function Z_1 for a single particle is

$$Z_1 = \frac{V}{(2\pi\hbar^2/m\tau)^{3/2}}.$$

(a) Write down the general expression for the *grand* partition function $\mathcal{Z}$ of a system with variable particle number and activity λ. Split the sum over states into separate sums over number of particles N and over states s with that number of particles. Hence show that

$$\mathcal{Z} = e^{\lambda Z_1}$$

for the perfect gas. You will need the result that

$$e^x = \sum_{N=0}^{\infty} \frac{x^N}{N!}.$$

(b) Hence find an expression for the grand potential Ω of the perfect gas and thus also for the average number of particles $\langle N \rangle$ and the pressure p from derivatives of Ω.