Complex Systems 535/Physics 508: Homework 2

1. Consider a k-regular undirected network (i.e., a network in which every vertex has degree k).

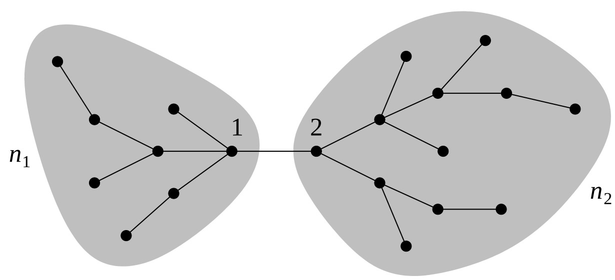
   (i) Show that the vector $\mathbf{1} = (1, 1, 1, \ldots)$ is an eigenvector of the adjacency matrix with eigenvalue k.

   (ii) By making use of the fact that eigenvectors are orthogonal (or otherwise), show that there is no other eigenvector that has all elements positive. The Perron–Frobenius theorem says that the eigenvector with all elements positive has the largest eigenvalue, and hence the eigenvector $\mathbf{1}$ gives, by definition, the eigenvector centrality of our k-regular network and the centralities are the same for every vertex.

   (iii) Find the Katz centralities of all vertices in a k-regular network.

   (iv) Name a centrality measure that could give different centralities for different vertices in a regular network.

2. Consider an undirected tree of n vertices. A particular edge in the tree joins vertices 1 and 2 and divides the tree into two disjoint regions of n_1 and n_2 vertices as sketched here:



   Show that the closeness centralities C_1 and C_2 of the two vertices, defined according to Eq. (7.29), are related by

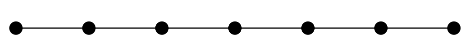
$$\frac{1}{C_1} + \frac{n_1}{n} = \frac{1}{C_2} + \frac{n_2}{n}.$$

3. Consider an undirected connected tree of n vertices. Suppose that a particular vertex in the tree has degree k, so that its removal would divide the tree into k disjoint regions, and suppose that the sizes of those regions are $n_1 \ldots n_k$.

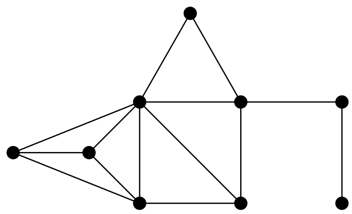
   (i) Show that the unnormalized betweenness centrality x of the vertex, defined by $x_i = \sum_{st} n^i_{st}/g_{st}$, where n^i_{st} is the number of paths from s to t through i and g_{st} is the total number of paths from s to t, is

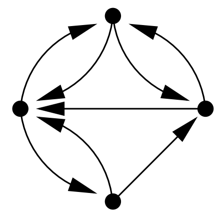
$$x = n^2 - \sum_{m=1}^{k} n_m^2.$$

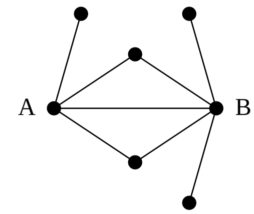
   (ii) Hence or otherwise calculate the betweenness of the ith vertex from the end of a "line graph" of n vertices, i.e., n vertices in a row like this:



4. Consider these three networks:







   (i) Find a 3-core in the first network.

   (ii) What is the reciprocity of the second network?

   (iii) What is the cosine similarity of vertices A and B in the third network?

5. Consider the modularity Q defined in Eq. (7.69) in the course-pack, and define the quantities

$$e_{rs} = \frac{1}{2m} \sum_{ij} A_{ij}\, \delta(c_i, r)\, \delta(c_j, s), \tag{1}$$

which is the fraction of edges that join vertices of type r to vertices of type s, and

$$a_r = \frac{1}{2m} \sum_i k_i\, \delta(c_i, r), \tag{2}$$

which is the fraction of ends of edges attached to vertices of type r.

   (i) Show that

$$Q = \sum_r \left(e_{rr} - a_r^2\right).$$

   (ii) In a survey of couples in the city of San Francisco in 1992, Catania *et al.* recorded, among other things, the ethnicity of interviewees and calculated the fraction of couples whose members were from each possible pairing of ethnic groups. The fractions were as follows:

		Women				
		Black	Hispanic	White	Other	Total
Men	Black	0.258	0.016	0.035	0.013	0.323
	Hispanic	0.012	0.157	0.058	0.019	0.247
	White	0.013	0.023	0.306	0.035	0.377
	Other	0.005	0.007	0.024	0.016	0.053
	Total	0.289	0.204	0.423	0.084	

   Assuming the couples interviewed to be a representative sample of the edges in the undirected network of relationships for the community studied, and treating the vertices as being of four types—black, hispanic, white, and other—what are the numbers e_{rr} and a_r for each type? Using these numbers calculate the modularity of the network with respect to ethnicity.