Complex Systems 535/Physics 508: Homework 7

1. Consider a configuration model in which every vertex has the same degree k.

   (i) What is the degree distribution p_k? What are the generating functions g_0 and g_1 for the degree distribution and the excess degree distribution?

   (ii) Show that the giant component fills the whole network for all $k \geq 3$.

   (iii) What happens when $k = 1$?

   (iv) **Extra credit:** What happens when $k = 2$?

2. Recall the rate equation for Price's model of a citation network in the limit of large n:

$$p_k = \frac{c}{c+a}\left[(k-1+a)p_{k-1} - (k+a)p_k\right] \qquad \text{for } k > 0,$$

$$p_0 = 1 - \frac{c}{c+a}p_0.$$

   (i) Write down the special case of these equations for $c = a = 1$.

   (ii) Show that the degree distribution generating function $g_0(x) = \sum_{k=0}^{\infty} p_k x^k$ for this case satisfies the differential equation

$$g_0(x) = 1 + \tfrac{1}{2}(x-1)\left[xg_0'(x) + g_0(x)\right].$$

   (iii) Show that the function

$$h(x) = \frac{x^3 g_0(x)}{(1-x)^2}$$

   satisfies

$$\frac{dh}{dx} = \frac{2x^2}{(1-x)^3}.$$

   (iv) Hence find a closed-form solution for the generating function $g_0(x)$. Confirm that your solution has the correct limiting values $g_0(0) = p_0$ and $g_0(1) = 1$.

   (v) Thus find a value for the mean in-degree of a vertex in Price's model. Is this what you expected?

3. Consider the following simple model of a growing network. Vertices are added to a network at a rate of one per unit time. Edges are added at a mean rate of β per unit time, where β can be anywhere between zero and ∞. (That is, in any small interval Δt of time, the probability of an edge being added is $\beta \Delta t$.) Edges are placed uniformly at random between any pair of vertices that exist at that time. They are never moved after they are first placed.

   We are interested in the component structure of this model, which we will tackle using a rate equation method. Let $a_k(n)$ be the fraction of vertices that belong to components of size k when there are n vertices in the graph. That is, if we choose a vertex at random from the n vertices currently in the graph, $a_k(n)$ is the probability the vertex will fall in a component of size k.

   (i) What is the probability that a newly appearing edge will fall between a component of size r and another of size s? (You can assume that n is large and the probability of both ends of an edge falling in the same component is small.) Hence what is the probability that a newly appearing edge will join together two pre-existing components to form a new one of size k?

(ii) What is the probability that a newly appearing edge joins a component of size k to a component of *any* other size, thereby creating a new component of size larger than k?

(iii) Thus write down a rate equation that gives the fraction of vertices $a_k(n+1)$ in components of size k for $n+1$ vertices, in terms of the values for n vertices.

(iv) The only exception to the previous result is that components of size 1 appear at a rate of one per unit time. Write a separate rate equation for $a_1(n+1)$.

(v) If a steady state solution exists for the component size distribution, show that it must satisfy the equations

$$(1+2\beta)a_1 = 1, \qquad (1+2\beta k)a_k = \beta k \sum_{j=1}^{k-1} a_j a_{k-j}.$$

(vi) Multiply by z^k and sum over k from 1 to ∞ and hence show that the generating function $g(z) = \sum_k a_k z^k$ satisfies the ordinary differential equation

$$2\beta \frac{dg}{dz} = \frac{1 - g/z}{1 - g}.$$

(vii) **Lots of extra credit:** A humongous number of extra points go to anyone who can find a nontrivial solution to this last equation in closed form for the appropriate boundary conditions ($g(0) = 0$). Series expansion solutions count if the series coefficients are in closed form, or solutions making use of special functions. (I should point out that I don't know of any solution to this equation, but I don't claim to be great at solving nonlinear equations.)