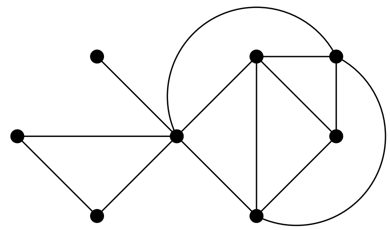
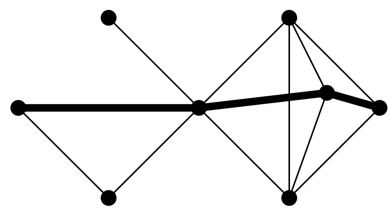
Complex Systems 535/Physics 508: Answers to example problems

1. (a) The graph is planar, as is demonstrated by this redrawing of it:



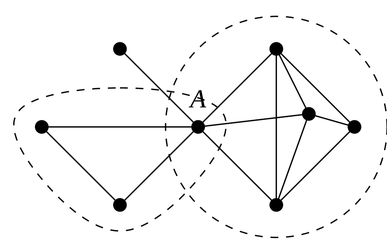
   (b) The diameter is three:



   (c) The distance from A to each of the eight other vertices is $1, 1, 1, 0, 1, 1, 1, 2$. Hence the closeness is 1.

   (d) Vertex A lies on the shortest path from itself to each of the seven other vertices and on the paths from them to itself. It also lies on the shortest path from each of the three vertices in the left half to each of the four on the right, for a total of 12 more paths, and *vice versa* for a further 12. It lies on two of the shortest paths between vertices within the left half and none of those between vertices in the right half, for a further 4 paths (once you count both directions). And finally it lies on the single path from itself to itself. Thus its unnormalized betweenness centrality is $14 + 24 + 4 + 1 = 43$.

   (e)



2. (a) Citation networks, food webs, software call graphs.

   (b) The longest path in an acyclic digraph has $n - 1$ steps, since if it had n or more it would have to visit at least one vertex twice, which would mean there would be a cycle. Hence there are no paths of length n or more, and hence all elements of $\mathbf{A}^n$ are zero.

   (c) The authority and hub weight vectors satisfy

$$\mathbf{x} = \mathbf{A}\mathbf{y} + \alpha\mathbf{1}, \qquad \mathbf{y} = \mathbf{A}^T\mathbf{x}.$$

   Eliminating $\mathbf{y}$, this gives

$$\mathbf{x} = (\mathbf{I} - \mathbf{A}\mathbf{A}^T)^{-1} \cdot \mathbf{1},$$

   to within a multiplicative constant.

3. (a) Each vertex is connected with probability p to each of $n-1$ others, so the mean degree is $c = p(n-1)$.

   (b) A neighboring vertex is connected with probability p to each of the $n-2$ others, for an average of $p(n-2)$ connections, plus it is definitely connected to you (since it is your neighbor), so the mean degree is $1 + p(n-2)$.

   (c) The probability that two of your "friends" are connected is the same as the probability that any two people are connected, which is p. So $C = p$.

   (d) Each of the edges is independent, so the probability of being connected to k others is binomially distributed:

$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}.$$

4. (a) Each vertex contributes $\frac{1}{2} \times 2r(2r-1) = r(2r-1)$ 2-stars, for a total of $nr(2r-1)$ 2-stars in the whole network.

   (b) Each vertex belongs to a triangle with vertices wholely clockwise from it if the two other vertices in the triangle are within distance r. Thus there are $\binom{r}{2} = \frac{1}{2}r(r-1)$ such triangles. Each such triangle is counted only once, so the total number of triangles in the network is $\frac{1}{2}nr(r-1)$.

   (c) The clustering coefficient is 3 times the number of triangles divided by the number of 2-stars:

$$C = \frac{3 \times \frac{1}{2}nr(r-1)}{nr(2r-1)} = \frac{3(r-1)}{2(2r-1)}.$$