Complex Systems 535/Physics 508: Homework 9

1. Consider a configuration model network that has vertices of degree 1, 2, and 3 only, in fractions p_1, p_2, and p_3, respectively.

   (i) Find the value of the critical vertex occupation probability ϕ_c at which site percolation takes place on the network.

   (ii) Show that there is no giant cluster for any value of the occupation probability ϕ if $p_1 > 3p_3$. In terms of the structure of the network, why is this? And why does the result not depend on p_2?

   (iii) Find the size of the giant cluster as a function of ϕ. (Hint: you may find it useful to remember that $u = 1$ is always a solution of the equation $u = 1 - \phi + \phi g_1(u)$.)

2. Consider the computer algorithm for percolation we discussed in class and which is described in more detail in Chapter 16, but suppose that upon the addition of an edge between two clusters we relabel not the smaller of the two clusters but one or the other chosen at random. Show by an argument analogous to the one in Section 16.4.1 that the worst-case running time of this algorithm is $\mathrm{O}(n^2)$, which is substantially worse than the $\mathrm{O}(n \log n)$ of the algorithm that always relabels the smaller cluster.

3. Consider the spread of an SIR-type disease on a network in which some fraction of the individuals have been vaccinated against the disease. We can model this situation using a joint site/bond percolation model in which a fraction ϕ_s of the vertices are occupied to represent the vertices not vaccinated, and a fraction ϕ_b of the edges are occupied to represent the edges along which contact sufficient for disease transmission takes place.

   (i) Show that the fraction S of individuals infected in the limit of long time is given by the solution of the equations

   $$S = \phi_s[1 - g_0(u)], \qquad u = 1 - \phi_s\phi_b + \phi_s\phi_b g_1(u),$$

   where $g_0(z)$ and $g_1(z)$ are the generating functions for the degree distribution and excess degree distribution, as usual.

   (ii) Show that for a given probability of transmission ϕ_b the fraction of individuals that need to be vaccinated to prevent spread of the disease is $1 - 1/[\phi_b g_1'(1)]$.

4. We have been concerned primarily with *epidemic* disease outbreaks, meaning outbreaks that affect a finite fraction of all individuals in a network. Consider, by contrast, a small SIR outbreak—an outbreak that corresponds to one of the non-giant percolation clusters in the bond percolation approach of Section 17.8—occurring on a configuration model network with degree distribution p_k.

   (i) What is the probability of such an outbreak occurring if the disease starts at a vertex chosen uniformly at random from the whole network (including vertices both within and outside the giant component)?

   (ii) Show that if the probability of transmission along an edge is ϕ then the generating function $h_0(z)$ for the probability π_s that the outbreak has size s is given by the equations

   $$h_0(z) = zg_0(h_1(z)), \qquad h_1(z) = 1 - \phi + \phi z g_1(h_1(z)),$$

   where $g_0(z)$ and $g_1(z)$ are the generating functions for the degree distribution and excess degree distribution respectively.