

## Physics 411: Homework 5

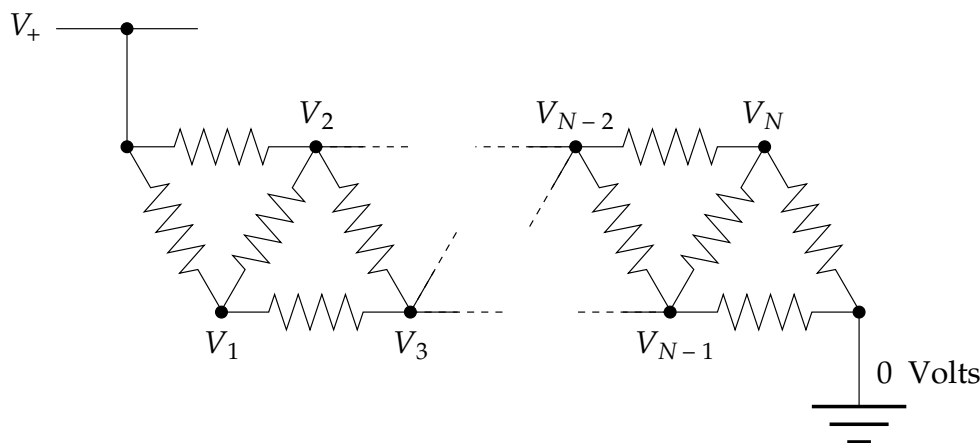
1. **Pivoting:** Download from the web site a copy of the program `gaussback.py` from Example 6.1 on page 184.

(a) Modify the program to incorporate partial pivoting. (You're also welcome to write your own program from scratch if you prefer.) Use your program to solve the problem in Eq. (6.1) and check that you get the same answers as from the original program without pivoting (which are given in Eq. (6.16)).

(b) Now use your program to solve the equations in Eq. (6.17) and show that it can find the solution to these as well, even though Gaussian elimination without pivoting fails.

✓ **For full credit** turn in a printout of your program, plus your results from part (b).

2. **A circuit of resistors:** Consider a long chain of resistors arranged like this:



All the resistors have the same resistance  $R$ . The power rail at the top is at voltage  $V_+ = 5V$ . The problem is to find the voltages  $V_1 \dots V_N$  at the internal points in the circuit.

(a) Using Ohm's law and the Kirchhoff current law, which says that the total net current flow out of (or into) any junction in a circuit must be zero, show that the voltages  $V_1 \dots V_N$  satisfy the equations

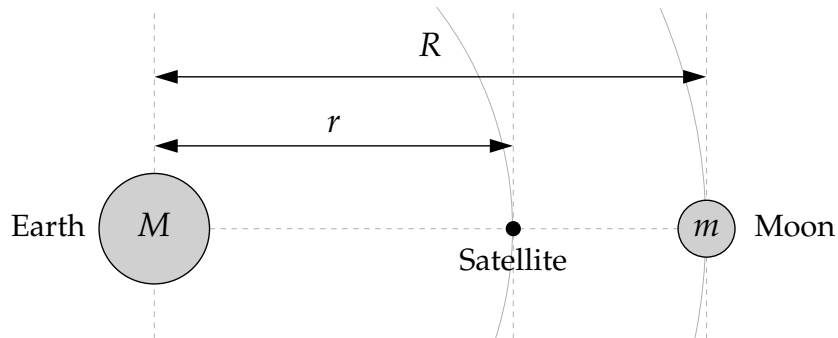
$$\begin{aligned}
 3V_1 - V_2 - V_3 &= V_+, \\
 -V_1 + 4V_2 - V_3 - V_4 &= V_+, \\
 &\vdots \\
 -V_{i-2} - V_{i-1} + 4V_i - V_{i+1} - V_{i+2} &= 0, \\
 &\vdots \\
 -V_{N-3} - V_{N-2} + 4V_{N-1} - V_N &= 0, \\
 -V_{N-2} - V_{N-1} + 3V_N &= 0.
 \end{aligned}$$

Express these equations in vector form  $\mathbf{A}\mathbf{v} = \mathbf{w}$  and find the values of the matrix  $\mathbf{A}$  and the vector  $\mathbf{w}$ .

- (b) Write a program to solve for the values of the  $V_i$  when there are  $N = 6$  internal junctions with unknown voltages. Use any method you like to find the solution, including the `solve` function from `numpy` or your own Gaussian elimination code. Make a graph of the values of  $V_i$  along the chain. (Hint: All the values of  $V_i$  should lie between zero and 5V. If they don't, something is wrong.)
- (c) Now repeat your calculation for the case where there are  $N = 10\,000$  internal junctions. This part is not possible using standard tools like the `solve` function. You need to make use of the fact that the matrix  $\mathbf{A}$  is banded and use the banded function that we discussed in class. A copy of this function can be found on the course web page.

✓ **For full credit** turn in your working from part (a), a printout of your final program, and a printout your plot from part (b) for the  $N = 6$  case.

3. **The Lagrange point:** There is a magical point between the Earth and the Moon, called the  $L_1$  Lagrange point, at which a satellite will orbit the Earth in perfect synchrony with the Moon, staying always in between the two. This works because the inward pull of the Earth and the outward pull of the Moon combine to create exactly the needed centripetal force that keeps the satellite in its orbit. Here's the setup:



- (a) Assuming circular orbits, show that the distance  $r$  from the center of the Earth to the  $L_1$  point satisfies

$$\frac{GM}{r^2} - \frac{Gm}{(R-r)^2} = \omega^2 r,$$

where  $M$  and  $m$  are the Earth and Moon masses,  $G$  is Newton's gravitational constant, and  $\omega$  is the angular velocity of both the Moon and the satellite.

- (b) The equation above is a fifth-order polynomial equation in  $r$  (also called a quintic equation). Such equations cannot be solved exactly in closed form, but it's straightforward to solve them numerically. Write a program to solve for the distance  $r$  from the Earth to the  $L_1$  point. You can use any of the methods we have studied—binary search, the fixed point method, Newton's method, or the secant method all work

well for this problem. Compute a solution accurate to at least four significant figures.

The values of the various parameters are:

$$G = 6.674 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2},$$

$$M = 5.974 \times 10^{24} \text{ kg},$$

$$m = 7.348 \times 10^{22} \text{ kg},$$

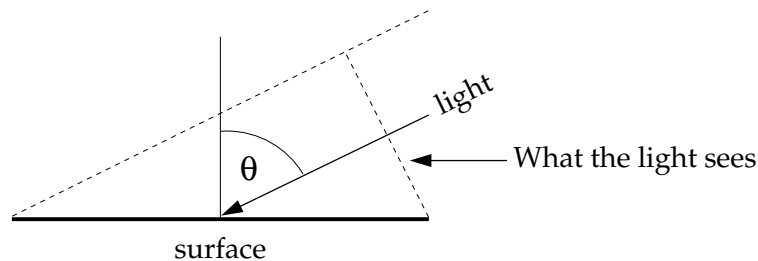
$$R = 3.844 \times 10^8 \text{ m},$$

$$\omega = 2.662 \times 10^{-6} \text{ s}^{-1}.$$

You will also need to choose a suitable starting value for  $r$ , or two starting values if you use the binary search or secant methods.

✓ **For full credit** turn in a copy of your program and your result for the distance  $r$ .

4. **Numerical derivatives and image processing:** When light strikes a surface, the amount falling per unit area depends not only on the intensity of the light, but also on the angle of incidence. If the light makes an angle  $\theta$  to the normal, it only “sees”  $\cos \theta$  of area per unit of actual area on the surface:



So the intensity of illumination is  $a \cos \theta$ , if  $a$  is the raw intensity of the light. This simple physical law is a central element of 3D computer graphics. It allows us to calculate how light falls on three-dimensional objects and hence how they will look when illuminated from various angles.

Suppose, for instance, that we are looking down on the surface of the Earth from above and we see mountains. We know the height of the mountains  $w(x, y)$  as a function of position in the plane, so the equation for the Earth’s surface is simply  $z = w(x, y)$ , or equivalently  $w(x, y) - z = 0$ , and the normal vector  $\mathbf{v}$  to the surface is given by the gradient of  $w(x, y) - z$  thus:

$$\mathbf{v} = \nabla[w(x, y) - z] = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} [w(x, y) - z] = \begin{pmatrix} \partial w/\partial x \\ \partial w/\partial y \\ -1 \end{pmatrix}.$$

Now suppose we have light coming in represented by a vector  $\mathbf{a}$ , with magnitude equal to the intensity of the light. Then the dot product of the vectors  $\mathbf{a}$  and  $\mathbf{v}$  is

$$\mathbf{a} \cdot \mathbf{v} = |\mathbf{a}| |\mathbf{v}| \cos \theta,$$

where  $\theta$  is the angle between the vectors. Thus the intensity of illumination of the surface of the mountains is

$$I = |\mathbf{a}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{a_x(\partial w/\partial x) + a_y(\partial w/\partial y) - a_z}{\sqrt{(\partial w/\partial x)^2 + (\partial w/\partial y)^2 + 1}}.$$

Let's take a simple case where the light is shining horizontally with unit intensity, along a line an angle  $\phi$  counter-clockwise from the east-west axis, so that  $\mathbf{a} = (\cos \phi, \sin \phi, 0)$ . Then our intensity of illumination simplifies to

$$I = \frac{\cos \phi (\partial w/\partial x) + \sin \phi (\partial w/\partial y)}{\sqrt{(\partial w/\partial x)^2 + (\partial w/\partial y)^2 + 1}}.$$

If we can calculate the derivatives of the height  $w(x, y)$  and we know  $\phi$  we can calculate the intensity at any point.

- (a) On the web site you'll find a file called `altitude.dat`, which contains the altitude  $w(x, y)$  in meters above sea level (or depth below sea level) of the surface of the Earth, measured on a grid of points  $(x, y)$ . Write a program that reads this file and stores the data in an array. Then calculate the derivatives  $\partial w/\partial x$  and  $\partial w/\partial y$  at each grid point. Explain what method you used to calculate them and why. (Hint: You'll probably have to use more than one method to get every grid point, because awkward things happen at the edges of the grid.) To calculate the derivatives you'll need to know the value of  $h$ , the distance in meters between grid points, which is about 30 000 m in this case.<sup>1</sup>
- (b) Now using your values for the derivatives calculate the intensity for each grid point from the equation above with  $\phi = 45^\circ$ , and make a density plot of the resulting values using a gray scale where the brightness of each dot depends on the corresponding intensity value. If you get it working right, the plot should look like a relief map of the world—you should be able to see the continents and mountain ranges in 3D. (Common problems include a map that is upside-down or sideways, or a relief map that is "inside-out," meaning the high regions look low and *vice versa*. Work with the details of your program until you get a map that looks right to you.)
- (c) There is another file on the web site called `stm.dat`, which contains a grid of values from scanning tunneling microscope measurements of the (111) surface of silicon. You made a simple visualization of the data in this file for Homework 3. Here you'll make a better one: modify the program you wrote for part (b) above to create a 3D visualization the STM data similar to your visualization of the Earth's surface—you should be able to see what the actual shape of the silicon surface looks like. The value of  $h$  for the derivatives in this case is around  $h = 2.5$  (in arbitrary units).

✓ **For full credit** turn in your final program and printouts of your two density plots.

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<sup>1</sup>It's actually not precisely constant because we are representing the spherical Earth on a flat map, but  $h = 30\,000$  m will give reasonable results.