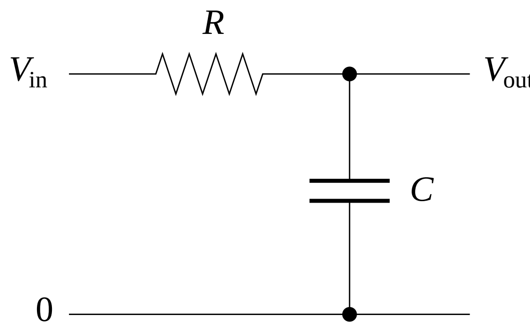
Physics 411: Homework 7

1. **Low-pass filter:** Here is a simple electronic circuit with one resistor and one capacitor:



This circuit acts as a low-pass filter: you send a signal in on the left and it comes out filtered on the right.

Using Ohm's law and the capacitor law and assuming that the output load has very high impedance, so that a negligible amount of current flows through it, we can write down the equations that govern this circuit as follows. Let I be the current that flows through R and into the capacitor, and let Q be the charge on the capacitor. Then:

$$IR = V_{\text{in}} - V_{\text{out}}, \qquad Q = CV_{\text{out}}, \qquad I = \frac{\mathrm{d}Q}{\mathrm{d}t}.$$

Substituting the second equation into the third, then substituting the resulting relation into the first, we find that $V_{\text{in}} - V_{\text{out}} = RC\,(\mathrm{d}V_{\text{out}}/\mathrm{d}t)$, or equivalently

$$\frac{\mathrm{d}V_{\text{out}}}{\mathrm{d}t} = \frac{1}{RC}\left(V_{\text{in}} - V_{\text{out}}\right).$$

(a) Write a program (or modify one from the book) to solve this equation for $V_{\text{out}}(t)$ using the fourth-order Runge–Kutta method when in the input signal is a square-wave with frequency 1 and amplitude 1:

$$V_{\text{in}}(t) = \begin{cases} 1 & \text{if } \lfloor 2t \rfloor \text{ is even,} \\ -1 & \text{if } \lfloor 2t \rfloor \text{ is odd,} \end{cases} \tag{1}$$

where $\lfloor x \rfloor$ means x rounded down to the next lowest integer. Use the program to make plots of the output of the filter circuit from $t = 0$ to $t = 10$ when $RC = 0.01$, 0.1, and 1, with initial condition $V_{\text{out}}(0) = 0$. You will have to make a decision about what value of h to use in your calculation. Small values give more accurate results, but the program will take longer to run. Try a variety of different values and choose one for your final calculations that seems sensible to you.

(b) Based on the graphs produced by your program, describe what you see and explain what the circuit is doing.

A program similar to the one you wrote is running inside your stereo or music player, to create the effect of the "bass" control. In the old days, the bass control on a stereo would have been connected to a real electronic low-pass filter in the amplifier circuitry, but these days there is just a computer processor that simulates the behavior of the filter in a manner similar to your program.

☑ **For full credit** turn in a printout of your program and the plots it produces, along with your answer to part (b).

2. **The Lotka–Volterra equations:** The Lotka–Volterra equations are a mathematical model of predator–prey interactions between biological species. Let two variables x and y be proportional to the size of the populations of two species, traditionally called "rabbits" (the prey) and "foxes" (the predators). You could think of x and y as being the population in thousands, say, so that $x = 2$ means there are 2000 rabbits. Strictly the only allowed values of x and y would then be multiples of 0.001, since you can only have whole numbers of rabbits or foxes. But 0.001 is a pretty close spacing of values, so it's a decent approximation to treat x and y as continuous real numbers so long as neither gets very close to zero.

   In the Lotka–Volterra model the rabbits reproduce at a rate proportional to their population, but are eaten by the foxes at a rate proportional to both their own population and the population of foxes:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \beta xy,$$

   where α and β are constants. At the same time the foxes reproduce at a rate proportional the rate at which they eat rabbits—because they need food to grow and reproduce—but also die of old age at a rate proportional to their own population:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \gamma xy - \delta y,$$

   where γ and δ are also constants.

   (a) Write a program to solve these equations using the fourth-order Runge–Kutta method for the case $\alpha = 1$, $\beta = \gamma = 0.5$, and $\delta = 2$, starting from the initial condition $x = y = 2$. Have the program make a graph showing both x and y as a function of time on the same axes from $t = 0$ to $t = 30$. (Hint: Notice that the differential equations in this case do not depend explicitly on time t—in vector notation, the right-hand side of each equation is a function $f(\mathbf{r})$ with no t dependence. You may nonetheless find it convenient to define a Python function `f(r,t)` including the time variable, so that your program takes the same form as programs given in the book.)

   (b) Describe in words what is going on in the system, in terms of rabbits and foxes.

☑ **For full credit** turn in a printout of your program and the graph it produces, along with your answer to part (b).

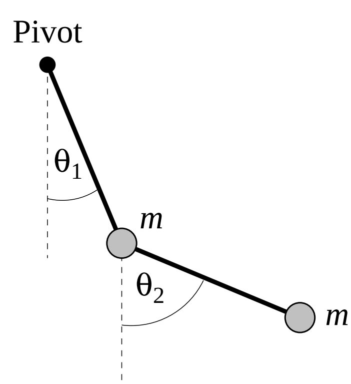
3. **Nonlinear pendulum:** Building on the results from Example 8.6 on page 327 of the book, calculate the motion of a nonlinear pendulum as follows.

   (a) Write a program to solve the two first-order equations, Eqs. (8.45) and (8.46), using the fourth-order Runge–Kutta method for a pendulum with a 10 cm arm. Use your program to calculate the angle θ of displacement for several periods of the pendulum when it is released from a standstill at $\theta = 179°$ from the vertical (i.e., pointing almost exactly upward). Make a graph of θ as a function of time.

   (b) Extend your program to create an animation of the motion of the pendulum. Your animation should, at a minimum, include a representation of the moving pendulum bob and the pendulum arm. (Hint: You will probably find the function `rate` discussed in Section 3.5 useful for making your animation run at a sensible speed. Also, you may want to make the step-size for your Runge–Kutta calculation smaller than the framerate of your animation, i.e., do several Runge–Kutta steps per frame on screen. This is certainly allowed and may help to make your calculation more accurate.)

   ☑ **For full credit** turn in a printout of your plot from part (a), your final program from part (b), and a snapshot of your animation in action.

4. **The double pendulum:** The pendulum in the last question is nonlinear, but its movement is nonetheless perfectly regular and periodic—there are no surprises. A *double pendulum* on the other hand is completely the opposite—chaotic and unpredictable. A double pendulum consists of a normal pendulum with another pendulum hanging from its end:



   For simplicity let us ignore friction, and assume that both pendulums have bobs of the same mass m and massless arms of the same length ℓ. The position of the arms at any moment in time is uniquely specified by the two angles θ_1 and θ_2. The equations of motion for the angles are most easily derived using the Lagrangian formalism, as follows.

   The heights of the two bobs, measured from the level of the pivot are

$$h_1 = -\ell \cos\theta_1, \qquad h_2 = -\ell(\cos\theta_1 + \cos\theta_2),$$

so the potential energy of the system is

$$V = mgh_1 + mgh_2 = -mg\ell(2\cos\theta_1 + \cos\theta_2),$$

where g is the acceleration due to gravity. The (linear) velocities of the two bobs are given by

$$v_1 = \ell\dot{\theta}_1, \qquad v_2^2 = \ell^2\big[\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\big],$$

where $\dot{\theta}$ means the derivative of θ with respect to time t. (If you don't see where the second velocity equation comes from, it's a good exercise to derive it for yourself from the geometry of the pendulum.) Now the total kinetic energy is

$$T = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}mv_2^2 = m\ell^2\big[\dot{\theta}_1^2 + \tfrac{1}{2}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\big],$$

and the Lagrangian of the system is

$$\mathscr{L} = T - V = m\ell^2\big[\dot{\theta}_1^2 + \tfrac{1}{2}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\big] + mg\ell(2\cos\theta_1 + \cos\theta_2).$$

Then the equations of motion are given by the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{\theta}_1}\right) = \frac{\partial\mathscr{L}}{\partial\theta_1}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{\theta}_2}\right) = \frac{\partial\mathscr{L}}{\partial\theta_2},$$

which in this case give

$$2\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1 - \theta_2) + \dot{\theta}_2^2\sin(\theta_1 - \theta_2) + 2\frac{g}{\ell}\sin\theta_1 = 0,$$
$$\ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1 - \theta_2) - \dot{\theta}_1^2\sin(\theta_1 - \theta_2) + \frac{g}{\ell}\sin\theta_2 = 0,$$

where the mass m has cancelled out.

These are second-order equations, but we can convert them into first-order ones by the usual method, defining new variables ω_1 and ω_2 thus:

$$\dot{\theta}_1 = \omega_1, \qquad \dot{\theta}_2 = \omega_2.$$

In terms of these variables our equations of motion become

$$2\dot{\omega}_1 + \dot{\omega}_2\cos(\theta_1 - \theta_2) + \omega_2^2\sin(\theta_1 - \theta_2) + 2\frac{g}{\ell}\sin\theta_1 = 0,$$
$$\dot{\omega}_2 + \dot{\omega}_1\cos(\theta_1 - \theta_2) - \omega_1^2\sin(\theta_1 - \theta_2) + \frac{g}{\ell}\sin\theta_2 = 0.$$

Finally we have to rearrange these into the standard form of Eq. (8.28) with a single derivative on the left-hand side of each one, which gives

$$\dot{\omega}_1 = -\frac{\omega_1^2\sin(2\theta_1 - 2\theta_2) + 2\omega_2^2\sin(\theta_1 - \theta_2) + (g/\ell)\big[\sin(\theta_1 - 2\theta_2) + 3\sin\theta_1\big]}{3 - \cos(2\theta_1 - 2\theta_2)},$$
$$\dot{\omega}_2 = \frac{4\omega_1^2\sin(\theta_1 - \theta_2) + \omega_2^2\sin(2\theta_1 - 2\theta_2) + 2(g/\ell)\big[\sin(2\theta_1 - \theta_2) - \sin\theta_2\big]}{3 - \cos(2\theta_1 - 2\theta_2)}.$$

(This last step is quite tricky and involves some trigonometric identities. You may find it useful to go through the derivation for yourself.)

These two equations, along with the equations $\dot{\theta}_1 = \omega_1$ and $\dot{\theta}_2 = \omega_2$, give us four first-order equations which between them define the motion of the double pendulum.

(a) Derive an expression for the total energy $E = T + V$ of the system in terms of the variables θ_1, θ_2, ω_1, and ω_2.

(b) Write a program using the fourth-order Runge–Kutta method to solve the equations of motion for the case where $\ell = 40\,\text{cm}$, with the initial conditions $\theta_1 = \theta_2 = 90°$ and $\omega_1 = \omega_2 = 0$. Use your program to calculate the total energy of the system assuming that the mass of the bobs is 1 kg each, and make a graph of energy as a function of time from $t = 0$ to $t = 100$ seconds.

Because of energy conservation, the total energy should be constant over time (actually it should be zero for this particular set of initial conditions), but you will find that it is not perfectly constant because of the approximate nature of the solution of the differential equation. Choose a suitable value of the step size h to ensure that the variation in energy is less than 10^{-5} Joules over the course of the calculation.

(c) Make a copy of your program and modify the copy to create a second program that does not produce a graph, but instead makes an animation of the motion of the double pendulum over time. At a minimum, the animation should show the two arms and the two bobs.

☑ **For full credit** turn your plot of the energy from part (b), your final program from part (c), and a snapshot of your animation in action.