Physics 411: Homework 8

1. **The driven pendulum:** A pendulum can be driven by, for example, exerting a small oscillating force horizontally on the mass. Then the equation of motion for the pendulum becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta + C\cos\theta\sin\Omega t,$$

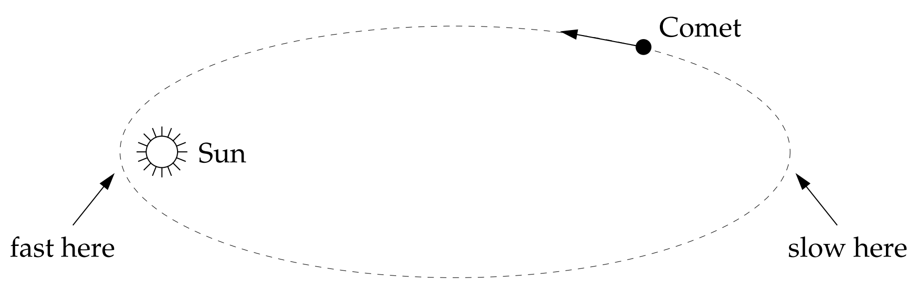
where C and Ω are constants.

   (a) Write a program to solve this equation for θ as a function of time with $\ell = 10\,\text{cm}$, $C = 2\,\text{s}^{-2}$ and $\Omega = 5\,\text{s}^{-1}$ and make a plot of θ as a function of time from $t = 0$ to $t = 100\,\text{s}$. Start the pendulum at rest, $\theta = 0$ and $d\theta/dt = 0$.

   (b) Now change the value of Ω, while keeping C the same, to find a value for which the pendulum resonates with the driving force and swings widely from side to side. Make a plot for this case also.

☑ **For full credit** turn in a printout of your program, your plots of $\theta(t)$ from parts (a) and (b) and a note of the approximate value of Ω at which you found the resonance.

2. **Cometary orbits:** Many comets travel in highly elongated orbits around the Sun. For much of their lives they are far out in the solar system, moving very slowly, but on rare occasions their orbit brings them close to the Sun for a fly-by and for a brief period of time they move very fast indeed:



This is a classic example of a system for which an adaptive step size method is useful, because for the large periods of time when the comet is moving slowly we can use long time-steps, so that the program runs quickly, but short time-steps are crucial in the brief but fast-moving period close to the Sun.

The differential equation obeyed by a comet is straightforward to derive. The force between the Sun, with mass M at the origin, and a comet of mass m with position vector $\mathbf{r}$ is GMm/r^2 in direction $-\mathbf{r}/r$ (i.e., the direction towards the Sun), and hence Newton's second law tells us that

$$m\frac{d^2\mathbf{r}}{dt^2} = -\left(\frac{GMm}{r^2}\right)\frac{\mathbf{r}}{r}.$$

Canceling the m and taking the x component we have

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -GM\frac{x}{r^3},$$

and similarly for the other two coordinates. We can, however, throw out one of the coordinates because the comet stays in a single plane as it orbits. If we orient our axes so that this plane is perpendicular to the z-axis, we can forget about the z coordinate and we are left with just two second-order equations to solve:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -GM\frac{x}{r^3}, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -GM\frac{y}{r^3},$$

where $r = \sqrt{x^2 + y^2}$.

(a) Turn these two second-order equations into four first-order equations, using the methods you have learned.

(b) Write a program to solve your equations using the fourth-order Runge–Kutta method with a *fixed* step size. You will need to look up the mass of the Sun and Newton's gravitational constant G. As an initial condition, take a comet at coordinates $x = 4$ billion kilometers and $y = 0$ (which is somewhere out around the orbit of Neptune) with initial velocity $v_x = 0$ and $v_y = 500\,\mathrm{m\,s^{-1}}$. Make a graph showing the trajectory of the comet (i.e., a plot of y against x).

    Choose a fixed step size h that allows you to accurately calculate at least two full orbits of the comet. Since orbits are periodic, a good indicator of an accurate calculation is that successive orbits of the comet lie on top of one another on your plot. If they do not then you need a smaller value of h. Give a short description of your findings. What value of h did you use? What did you observe in your simulation? How long did the calculation take?

(c) Make a copy of your program and modify the copy to do the calculation using an adaptive step size. Set a target accuracy of $\delta = 1$ kilometer per year in the position of the comet and again plot the trajectory. What do you see? How do the speed, accuracy, and step size of the calculation compare with those in part (b)?

(d) Modify your program to place dots on your graph showing the position of the comet at each Runge–Kutta step around a single orbit. You should see the steps getting closer together when the comet is close to the Sun and further apart when it is far out in the solar system.

Calculations like this can be extended to cases where we have more than one orbiting body. We can include planets, moons, asteroids, and others. Analytic calculations are impossible for such complex systems, but with careful numerical solution of differential equations we can calculate the motions of objects throughout the entire solar system.

☑ **For full credit** turn in a printout of your final program, your plots from parts (a) and (c), and your answers to the questions in parts (b) and (c).

3. **Quantum oscillators:** Consider the one-dimensional, time-independent Schrödinger equation in a harmonic (i.e., quadratic) potential $V(x) = V_0 x^2/a^2$, where V_0 and a are constants.

   (a) Write down the Schrödinger equation for this problem and convert it from a second-order equation to two first-order ones. Write a program, or modify the one from Example 8.8, to find the energies of the ground state and the first two excited states for these equations to an accuracy of 0.001 eV when m is the electron mass, $V_0 = 50\,\text{eV}$, and $a = 10^{-11}\,\text{m}$. The wavefunction goes to $x = \pm\infty$, but you can get good answers by using a large but finite interval. Try using $x = -10a$ to $+10a$, with the wavefunction $\psi = 0$ at both boundaries. (In effect, you are putting the harmonic oscillator in a box with impenetrable walls.) The wavefunction is real everywhere, so you don't need to use complex variables, and you can use evenly spaced points for the solution—no need to use an adaptive method for this problem.

   The quantum harmonic oscillator is known to have energy states that are equally spaced. Check that this is true, to the precision of your calculation, for your answers. (Hint: The ground state has energy in the range 100 to 200 eV.)

   (b) Now modify your program to calculate the energies of the same three states for the anharmonic oscillator with $V(x) = V_0 x^4/a^4$, with the same parameter values.

   (c) Modify your program further to calculate the properly normalized wavefunctions of the anharmonic oscillator for the three states and make a plot of them, all on the same axes, as a function of x over a modest range near the origin—say $x = -5a$ to $x = 5a$.

   To normalize the wavefunctions you will have to calculate the integral $\int_{-\infty}^{\infty} |\psi(x)|^2\,\mathrm{d}x$ and then rescale ψ appropriately to ensure that the wavefunctions integrate to 1. Either the trapezoidal rule or Simpson's rule will give you a reasonable value for the integral. Note, however, that you may find a few very large values at the end of the array holding the wavefunction. Where do these large values come from? Are they real, or spurious?

   One simple way to deal with the large values is to make use of the fact that the system is symmetric about its midpoint and calculate the integral of the wavefunction over only the left-hand half of the system, then double the result. This neatly misses out the large values.

☑ **For full credit** turn in a printout of your final program, your results for the energies in parts (a) and (b), your answer to the questions in part (c) and your plot of the wavefunctions.