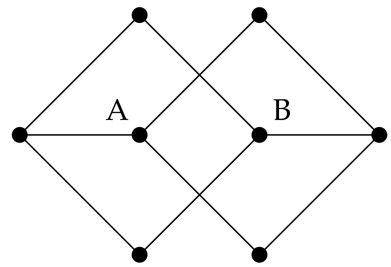
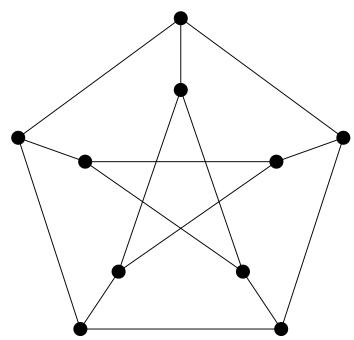
Complex Systems 535/Physics 508: Homework 2

1. Give a proof, visual or otherwise, that the edge connectivity of nodes A and B in this network is 2:



Hint: A correct proof must show both that the connectivity is at least 2 and that it is no more than 2.

2. As described in Section 6.9 of the course pack, Kuratowski's theorem states that every non-planar network contains an expansion of either K_5 or UG or both. Hence prove that this network is not planar:



3. In Section 6.14.5 it is stated that the "algebraic connectivity," i.e., the second-smallest eigenvalue of the graph Laplacian, is non-zero for a network if the network is connected. Construct a proof of this fact as follows.

   (a) Noting the form of the eigenvector corresponding to the zero eigenvalue, argue that every other eigenvector $\mathbf{v}$ must have both positive and negative elements.

   (b) Hence, making use of Eq. (6.53), prove the required result.

4. Consider a k-regular undirected network (i.e., a network in which every vertex has degree k).

   (a) Show that the vector $\mathbf{1} = (1, 1, 1, \ldots)$ is an eigenvector of the adjacency matrix with eigenvalue k.

(b) By making use of the fact that eigenvectors are orthogonal (or otherwise), show that there is no other eigenvector that has all elements positive. The Perron–Frobenius theorem says that the eigenvector with all elements positive has the largest eigenvalue, and hence the eigenvector $\mathbf{1}$ gives, by definition, the eigenvector centrality of our k-regular network and the centralities are the same for every vertex.

(c) Find the Katz centralities of all vertices in a k-regular network.

(d) Name a centrality measure that could give different centralities for different vertices in a regular network.