Complex Systems 535/Physics 508: Homework 6

1. Consider a configuration model in which every node has the same degree k.

   (a) What is the degree distribution p_k? What are the generating functions g_0 and g_1 for the degree distribution and the excess degree distribution?

   (b) Show that the probability of a node belonging to the giant component is 1 for all $k \geq 3$, i.e., that the giant component is the essentially size of the entire network, at least in the limit of large network size.

   (c) What happens when $k = 1$?

   (d) **Extra credit:** For $k = 2$ show that in the limit of large n the probability π_s that a node belongs to a component of size s is given by $\pi_s = 1/\left[2\sqrt{n(n-s)}\right]$.

2. Let us model the internet as a configuration model with a perfect power-law degree distribution as in Eq. (12.57) in the course pack, with $\alpha \simeq 2.5$ and $k \geq 1$. There is no closed-form expression for the generating functions g_0 and g_1 in this case (though they can be written in terms of special functions). Nonetheless, using any means you like, estimate the fraction of the nodes on the internet you expect to be functional at any one time, where functional means they belong to the largest component. Hint: Reasonable ways to answer this question include writing a computer program to approximate the generating functions numerically, or creative use of mathematical software such as Mathematica.

3. Consider the configuration model with (properly normalized) degree distribution $p_k = (1-a)a^k$ with $a < 1$.

   (a) Show that the probability u of Eq. (12.30) satisfies the cubic equation
   $$a^2u^3 - 2au^2 + u - (1-a)^2 = 0.$$

   (b) Noting that $u = 1$ is always a trivial solution of Eq. (12.30), show that in this case the non-trivial solution corresponding to the existence of a giant component satisfies the quadratic equation $a^2u^2 - a(2-a)u + (1-a)^2 = 0$.

   (c) Hence show that as a fraction of the size of the network, the size of the giant component, if there is one, is
   $$S = \tfrac{3}{2} - \sqrt{a^{-1} - \tfrac{3}{4}}.$$

   (d) Show that the giant component exists only if $a > \frac{1}{3}$.

4. Consider the binomial probability distribution $p_k = \binom{n}{k} p^k (1-p)^{n-k}$.

   (a) Show that the probability generating function for this distribution is $g(z) = (pz + 1 - p)^n$.

   (b) Find the first and second moments of the distribution from Eqs. (12.88) and (12.90) in the course pack and hence show that the variance of the distribution is $\sigma^2 = np(1-p)$.

   (c) Show that the sum k of two numbers drawn independently from the same binomial distribution is distributed according to $\binom{2n}{k} p^k (1-p)^{2n-k}$.

5. **Extra credit:** Write a program in the computer language of your choice (or modify your program from last week) to generate a configuration model network in which there are nodes of degree 1 and 3 only and then calculate the size of the largest component.

   (a) Use your program to calculate the fraction S of the network filled by the largest component in a network with $n = 100\,000$ nodes and $p_1 = 0.6$, $p_3 = 0.4$ (and $p_k = 0$ for all other values of k). Compare your result to the exact (analytic) answer for the same values of p_1 and p_3 when $n \to \infty$.

   (b) Modify your program to make a graph of the size of the largest component for values of p_1 from 0 to 1 in steps of 0.01. Hence estimate the value of p_1 at the phase transition at which the giant component disappears.

   For full extra credit, turn in a printout of your program from part (a), along with the output it produces showing the result for the size of the largest component and your comparison to the exact analytic result, plus the graph you produced for part (b) and your estimate of the position of the phase transition.