Complex Systems 535/Physics 508: Homework 7

1. (a) The Fibonacci numbers are 1, 1, 2, 3, 5, 8... They have the definitive property that each is the sum of the previous two. The generating function for the Fibonacci numbers is the power series whose coefficients are the Fibonacci numbers: $f(z) = 1 + z + 2z^2 + 3z^3 + 5z^4 + \ldots$ Show that $f(z) = 1/(1 - z - z^2)$.

   (b) A sequence of numbers a_k with $k = 1, 2, 3, \ldots$ satisfies the recurrence

$$a_k = \begin{cases} 1 & \text{for } k = 1, \\ \sum_{j=1}^{k-1} a_j a_{k-j} & \text{for } k > 1. \end{cases}$$

   Show that the generating function $h(z) = \sum_{k=1}^{\infty} a_k z^k = \frac{1}{2}(1 - \sqrt{1 - 4z})$.

2. Consider a bipartite version of the configuration model, as described in Section 12.11.2, in which there are two types of nodes, and edges run only between nodes of unlike types. There are n_A nodes of type A with mean degree c_A and n_B nodes of type B with mean degree c_B.

   (a) Given that every edge joins a node of type A to a node of type B, show that $n_A c_A = n_B c_B$.

   (b) Depending on the exact form of the degree distributions of the two types of nodes, the network may or may not contain a giant component. Derive a condition in terms of the mean and mean-square degrees of the two types, equivalent to the Molloy-Reed condition for the ordinary configuration model, that tells us when a giant component exists.

   (c) Define u_A to be the probability that the node of type A at the end of an edge is *not* in the giant component, and similarly for u_B and nodes of type B. Show that $u_A = g_1^A(u_B)$ and $u_B = g_1^B(u_A)$ where g_1^A and g_1^B are the generating functions for the excess degrees of nodes of type A and B respectively.

   (d) Give an expression for the fraction S_A of nodes of type A in the giant component.

3. Consider a model of a growing directed network similar to Price's model described in Section 13.1 of the course pack, but without preferential attachment. That is, nodes are added one by one to the growing network and each has c outgoing edges, but those edges now attach to existing nodes uniformly at random, without regard for degrees or any other node property.

   (a) Derive master equations, the equivalent of Eqs. (13.7) and (13.8), that govern the distribution of in-degrees q in the limit of large network size.

   (b) Hence show that in the limit of large size the in-degrees have a geometric distribution $p_q = Cr^q$, where C is a normalization constant and $r = c/(c + 1)$.

4. Consider a model network similar to the model of Barabási and Albert described in Section 13.2, in which undirected edges are added between nodes according to a preferential attachment rule, but suppose that the network does not grow—it starts off with a given number n of nodes and neither gains nor loses any nodes thereafter. In this model, starting with an initial network of n nodes and some specified arrangement of edges, we add

at each step one undirected edge between two nodes, both of which are chosen at random in direct proportion to degree k. Let $p_k(m)$ be the fraction of nodes with degree k when the network has m edges in total.

(a) Show that, when the network has m edges, the probability that the next edge added will attach to node i is k_i/m.

(b) Write down a master equation giving $p_k(m+1)$ in terms of $p_{k-1}(m)$ and $p_k(m)$. Give the equation for the special case of $k = 0$ also.

(c) Eliminate m from the master equation in favor of the mean degree $c = 2m/n$ and take the limit $n \to \infty$ with c held constant to show that $p_k(c)$ satisfies the differential equation

$$c\frac{\mathrm{d}p_k}{\mathrm{d}c} = (k-1)p_{k-1} - kp_k.$$

(d) Define a generating function $g(c,z) = \sum_{k=0}^{\infty} p_k(c)\, z^k$ and show that it satisfies the partial differential equation

$$c\frac{\partial g}{\partial c} + z(1-z)\frac{\partial g}{\partial z} = 0.$$

(e) Show that $g(c,z) = f(c - c/z)$ is a solution of this differential equation, where $f(x)$ is any differentiable function of x.

(f) The particular choice of f depends on the initial conditions on the network. Suppose the network starts off in a state where every node has degree one, which means $c = 1$ and $g(1,z) = z$. Find the function f that corresponds to this initial condition and hence find $g(c,z)$ for all values of c and z.

(g) Show that, for this solution, the degree distribution as a function of c takes the form

$$p_k(c) = \frac{(c-1)^{k-1}}{c^k},$$

except for $k = 0$, for which $p_0(c) = 0$ for all c.

Note that the degree distributions in both this model and the model of question 3 decay exponentially in k, implying that neither preferential attachment nor network growth alone can account for a power-law degree distribution. One must have both growth and preferential attachment to get a power law.