Complex Systems: A Survey

M. E. J. Newman

Department of Physics, University of Michigan, Ann Arbor, MI 48109 and
Center for the Study of Complex Systems, University of Michigan, Ann Arbor, MI 48109

A complex system is a system composed of many interacting parts, often called agents, which displays collective behavior that does not follow trivially from the behaviors of the individual parts. Examples include condensed matter systems, ecosystems, stock markets and economies, biological evolution, and indeed the whole of human society. Substantial progress has been made in the quantitative understanding of complex systems, particularly since the 1980s, using a combination of basic theory, much of it derived from physics, and computer simulation. The subject is a broad one, drawing on techniques and ideas from a wide range of areas. Here I give a short survey of the main themes and methods of complex systems science and an annotated bibliography of resources, ranging from classic papers to recent books and reviews.

I. INTRODUCTION

Complex systems is a relatively new and broadly interdisciplinary field that deals with systems composed of many interacting units, often called "agents." The foundational elements of the field predate the current surge of interest in it, which started in the 1980s, but substantial recent advances in the area coupled with increasing interest both in academia and industry have created new momentum for the study and teaching of the science of complex systems.

There is no precise technical definition of a "complex system," but most researchers in the field would probably agree that it is a system composed of many interacting parts, such that the collective behavior of those parts together is more than the sum of their individual behaviors. The collective behaviors are sometimes also called "emergent" behaviors, and a complex system can thus be said to be a system of interacting parts that displays emergent behavior.

Classic examples of complex systems include condensed matter systems, ecosystems, the economy and financial markets, the brain, the immune system, granular materials, road traffic, insect colonies, flocking or schooling behavior in birds or fish, the Internet, and even entire human societies.

Unfortunately, complex systems are, as their name makes clear, complex, which makes them hard to study and understand. Experimental observations are of course possible, though these fall largely within the realm of the traditional scientific disciplines and are usually not considered a part of the field of complex systems itself, which is primarily devoted to theoretical developments.

Complex systems theory is divided between two basic approaches. The first involves the creation and study of simplified mathematical models that, while they may not mimic the behavior of real systems exactly, try to abstract the most important qualitative elements into a solvable framework from which we can gain scientific insight. The tools used in such studies include dynamical systems theory, information theory, cellular automata, networks, computational complexity theory, and numerical methods. The second approach is to create more comprehensive and realistic models, usually in the form of computer simulations, which represent the interacting parts of a complex system, often down to minute details, and then to watch and measure the emergent behaviors that appear. The tools of this approach include techniques such as Monte Carlo simulation and, particularly, agent-based simulation, around which a community of computer scientists and software developers has grown up to create software tools for sophisticated computational research in complex systems.

This review focuses on the methods and theoretical tools of complex systems, including both the modeling and simulation approaches above, though I also include a short section of references to individual specific complex systems, such as economies or ecosystems, which can serve as a concrete foundation motivating the theoretical studies.

II. GENERAL REFERENCES

Complex systems is a relatively young subject area and one that is evolving rapidly, but there are nonetheless a number of general references, including books and reviews, that bring together relevant topics in a useful way.[1]

A. Books

The first two books listed are elementary and require little mathematics for their comprehension. The first, by Mitchell, is recent and aimed at the popular audience. The second is older but wider ranging and contains more technical content.

1. **Complexity: A Guided Tour**, M. Mitchell (Oxford University Press, Oxford, 2009). (E)

2. **The Computational Beauty of Nature**, G. W. Flake (MIT Press, Cambridge, MA, 1998). (E)

The following three books are more advanced. Each covers important topics in complex systems, but none covers the field comprehensively. The authors of the second book are economists rather than physicists and their book has, as a result, more of a social science flavor. The book by Mandelbrot

[1] Each reference in this paper is labeled "(E)", "(I)", or "(A)," to denoted elementary, intermediate, or advanced material.

is, by now, quite old, predating "complex systems" as a recognized field, but is considered a classic and very readable, although not all of the ideas it contains have become accepted.

3. **Modeling Complex Systems**, N. Boccara (Springer, New York, NY, 2004). (I)

4. **Complex Adaptive Systems**, J. H. Miller and S. E. Page (Princeton University Press, Princeton, 2007). (I)

5. **The Fractal Geometry of Nature**, B. B. Mandelbrot (W. H. Freeman, New York, 1983). (I)

B. Journals

A number of journals focus specifically on complex systems, of which the best known are

Advances in Complex Systems
Complexity
Complex Systems

However, the vast majority of research on complex systems is not published in these journals, but appears either in subject-specific journals, such as physics journals, or in general science journals. Some of the most prominent physics journals publishing on complex systems are

Chaos, Solitons, and Fractals
Europhysics Letters
European Physical Journal B
Nature Physics
Physical Review E
Physical Review Letters
Physica A
Physica D

Among general science journals, *Science*, *Nature*, and *Proceedings of the National Academy of Sciences* all publish regularly on complex systems.

III. EXAMPLES OF COMPLEX SYSTEMS

Many individual complex systems are studied intensively within their own academic fields—ecosystems in ecology, stock markets in finance and business, and so forth. It is not the purpose of this paper to review this subject-specific literature, but this section outlines some of the literature on specifically complex-systems approaches to individual systems.

Physical systems: Although they are not always thought of in that way, many physical systems, and particularly those studied in condensed matter and statistical physics, are true examples of complex systems. Physical systems that fall within the realm of complex systems science include classical condensed matter systems such as crystals, magnets, glasses, and superconductors; hydrodynamical systems including classical (Newtonian) fluids, nonlinear fluids, and granular flows; spatiotemporal pattern formation in systems like chemical oscillators and excitable media; molecular self-assembly, including tiling models, biomolecules, and nanotechnological examples; biophysical problems such as protein folding and the physical properties of macromolecules; and physical systems that perform computation, including analog and quantum computers. It is perhaps in condensed matter physics that the fundamental insight motivating the study of complex systems was first clearly articulated, in the classic 1972 article by Anderson:

6. "More is different," P. W. Anderson, Science **177**, 393–396 (1972). In this paper Anderson points out the misconception of basic physical theories, such as quantum mechanics, as "theories of everything." Although such theories do, in principle, explain the action of the entire universe, the collective behaviors of particles or elements in a complex system often obey emergent physical laws—like the equation of state of a gas, for instance—that cannot be derived easily (or in some cases at all) from the underlying microscopic theory. In other words, there are physical laws at many "levels" in the phenomenology of the universe, and only one of those levels is described by fundamental theories like quantum mechanics. To understand the others, new theories are needed. (E)

Many of the physicists who have made careers working on complex systems got their start in condensed matter physics, and an understanding of that field will certainly help the reader in understanding the ideas and language of complex systems theory. Two recent books written by physicists directly involved in research on complex systems are:

7. **Statistical Mechanics: Entropy, Order Parameters and Complexity**, J. P. Sethna (Oxford University Press, Oxford, 2006). This book is accompanied by a set of online programs and simulations that are useful for explaining and understanding some of the concepts. (A)

8. **Advanced Condensed Matter Physics**, L. M. Sander (Cambridge University Press, Cambridge, 2009). (A)

Both are sophisticated treatments, but for the mathematically inclined reader these books provide a good starting point for understanding physical theories of complex systems.

Ecosystems and biological evolution: The biosphere, both in its present state and over evolutionary history, presents an endlessly fascinating picture of a complex system at work.

9. **Signs of Life: How Complexity Pervades Biology**, R. Solé and B. Goodwin (Basic Books, New York, 2002). A good introduction, which includes some significant mathematical elements, but confines the most challenging of them to sidebars. The authors are a physicist and a biologist, and the combination makes for a book that is accessible and relevant to those interested in how physics thinking can contribute outside of the traditional boundaries of physics. (I)

10. **Evolutionary Dynamics: Exploring the Equations of Life**, M. A. Nowak (Belknap Press, Cambridge, MA, 2006). A more technical work that also includes an introduction, in the biological arena, to several of the areas of complex systems theory discussed later in this paper. (I)

The following two papers provide useful discussions from the ecology viewpoint:

11. "Ecosystems and the biosphere as complex adaptive systems," S. A. Levin, Ecosystems **1**, 431–436 (1998). (I)

12. "Understanding the complexity of economic, ecological, and social systems," C. S. Holling, Ecosystems **4**, 390–405 (2001). As its title suggests, this article provides a comparative review of ecosystems along side economies and human societies, from the viewpoint of an ecologist. (I)

Some classic works in complex systems also fall into the areas of ecology and evolutionary biology:

13. "Will a large complex system be stable?" R. M. May, Nature **238**, 413–414 (1972). This important early paper applies complex systems ideas to the stability of ecosystems, and is a significant precursor to more recent work in network theory (see Section IV.A). (A)

14. "Towards a general theory of adaptive walks on rugged landscapes," S. A. Kauffman and S. Levin, J. Theor. Bio. **128**, 11–45 (1987). In this paper, Kauffman and Levin described for the first time their NK model, which is now one of the standard models of macroevolutionary theory. (A)

15. **At Home in the Universe**, S. A. Kauffman (Oxford University Press, Oxford, 1995). This later book by Kauffman gives an accessible introduction to the NK model. (E)

Human societies: Human societies of course have many aspects to them, not all of which are amenable to study by quantitative methods. Three aspects of human societies, however, have proved of particular interest to scientists working on complex systems: (1) urban planning and the physical structure of society, (2) the social structure of society and social networks, and (3) differences between societies as revealed by sociological experiments. I address the first two of these in this section. Experimental approaches are addressed in Section IV.E.

One of the most influential works on urban planning is the 1961 book by Jacobs which, while predating modern ideas about complex systems, has nonetheless inspired many of those ideas. It is still widely read today:

16. **The Death and Life of Great American Cities**, J. Jacobs (Random House, New York, 1961). (E)

The following papers provide a sample of recent work on urban societies viewed as complex systems. The articles by Bettencourt *et al.*, which address the application of scaling theory to urban environments, have been particularly influential, although their results are not universally accepted. The first is at a relatively high technical level while second is a non-technical overview. I discuss scaling theory in more detail in Section IV.D.

17. "The size, scale, and shape of cities," M. Batty, Science **319**, 769–771 (2008). Batty is an architect who has in recent years championed the application of complex systems theory in urban planning. In this nontechnical article he gives an overview of current ideas, drawing on spatial models, scaling, and network theory. (E)

18. **Cities and complexity**, M. Batty (MIT Press, Cambridge, MA, 2007). In this book Batty expands widely on the topic of his article above. Although technical, the book is approachable and the author makes good use of models and examples to support his ideas. (I)

19. "Growth, innovation, scaling, and the pace of life in cities," L. M. A. Bettencourt, J. Lobo, D. Helbing, C. Kühnert, and G. B. West, Proc. Natl. Acad. Sci. USA **104**, 7301–7306 (2007). The work of Bettencourt and collaborators on the application of scaling theory to the study of urban environments has been particularly influential. They find that a wide variety of parameters describing the physical structure of US cities show "power-law" behavior. Power laws are discussed further in Section IV.D. (A)

20. "A unified theory of urban living," L. M. A. Bettencourt and G. B. West, Nature **467**, 912–913 (2010). This nontechnical paper discusses the motivations and potential rewards of applying complex systems approaches to urban planning. (E)

Turning to social networks, there has been a substantial volume of work on networks in general by complex systems researchers, which we review in Section IV.A, but there is also an extensive literature on human social networks in sociology, which, while not specifically aimed at readers in complex systems, nonetheless contains much of interest. The two books below are good general references. The article by Watts provides an interesting perspective on what complex systems theory has to add to a field of study that is now almost a hundred years old.

21. **Social Network Analysis: A Handbook**, J. Scott (Sage, London, 2000), 2nd edition. (I)

22. **Social Network Analysis**, S. Wasserman and K. Faust (Cambridge University Press, Cambridge, 1994). (A)

23. "The 'new' science of networks," D. J. Watts, Annual Review of Sociology **30**, 243–270 (2004). (I)

Economics and markets: Markets are classic examples of complex systems, with manufacturers, traders, and consumers interacting to produce the emergent phenomenon we call the economy. Physicists and physics-style approaches have made substantial contributions to economics and have given rise to the new subfield of "econophysics," an area of lively current research activity.

24. **An Introduction to Econophysics: Correlations and Complexity in Finance**, R. N. Mantegna and H. E. Stanley (Cambridge University Press, Cambridge, 1999). This book is a standard reference in the area. (I)

25. **Why Stock Markets Crash: Critical Events in Complex Financial Systems**, D. Sornette (Princeton University Press, Princeton, 2004). Though it addresses primarily financial markets, and not economics in general, this highly-regarded book is a good example of the physics approach to these problems. (I)

26. "Is economics the next physical science?" J. D. Farmer, M. Shubik, and E. Smith, Physics Today **58** (9), 37–42 (2005). An approachable introductory paper that asks what physics can contribute to our understanding of economic and financial problems. (E)

A fundamental debate that has characterized the influence of complex systems ideas on economics is the debate over the value of the traditional "equilibrium" models of mathematical economics, as opposed to newer approaches based on ideas such as "bounded rationality" or on computer simulation methods. A balanced overview of the two viewpoints is given by Farmer and Geanakoplos.

27. "The virtues and vices of equilibrium and the future of financial economics," J. D. Farmer and J. Geanakoplos, Complexity **14** (3), 11–38 (2009). (E)

A number of books have also appeared that make connections between economic theory and other areas of interest in complex systems. A good recent example is the book by Easley and Kleinberg, which draws together ideas from a range of fields to help illuminate economic behaviors and many other things in a lucid though quantitative way.

28. **Networks, Crowds, and Markets**, D. Easley and J. Kleinberg (Cambridge University Press, Cambridge, 2010). (E)

Pattern formation and collective motion: In two- or three-dimensional space the interactions of agents in a complex system can produce spatial patterns of many kinds and systems that do this are seen in many branches of science, including physics (e.g., Rayleigh–Bénard convection, diffusion limited aggregation), chemistry (the Belousov–Zhabotinsky reaction), and biology (embryogenesis, bacterial colonies, flocking and collective motion of animals and humans). The paper by Turing below is one of the first and best-known efforts to develop a theory of pattern formation in the context of biological morphogenesis, and a classic in the complex systems literature. The book by Winfree is an unusual and thought-provoking point of entry into the literature that makes relatively modest mathematical demands of its reader (and addresses many other topics in addition to pattern formation).

29. "The chemical basis of morphogenesis," A. M. Turing, Phil. Trans. R. Soc. London B **237** (37-72) (1952). (A)

30. **The Geometry of Biological Time**, A. T. Winfree (Springer, New York, 2000), 2nd edition. (I)

Collective motions of self-propelled agents, such as road and pedestrian traffic and animal flocking, have been actively studied using methods from physics. Vehicular traffic shows a number of interesting behaviors that emerge from the collective actions of many drivers, like the propagation of traffic disturbances such as tailbacks in the opposite direction to traffic flow, and the so-called jamming transition, where cars' speeds drop suddenly as traffic density passes a critical point. Some similar phenomena are visible in pedestrian traffic as well, although pedestrians are not always confined to a one-dimensional road the way cars are, and the added freedom can give rise to additional phenomena.

31. "A cellular automaton model for freeway traffic," K. Nagel and M. Schreckenberg, J. Phys. I France **2**, 2221–2229 (1992). The classic Nagel–Schreckenberg model of road traffic is a beautiful example of the application of now-standard ideas from complex systems theory to a real-world problem. The model is a "cellular automaton" model. Cellular automata are discussed further in Section IV.C. (I)

32. "Traffic and related self-driven many-particle systems," D. Helbing, Rev. Mod. Phys. **73**, 1067–1141 (1997). The Nagel–Schreckenberg model and many other models and theories of traffic flow are examined in detail in this extensive review by Helbing. (I)

Flocking or schooling in birds or fish is a cooperative phenomenon in which the animals in a flock or school collectively fly or swim in roughly the same direction, possibly turning as a unit. It's believed that animals achieve this by simple self-enforced rules that involve copying the actions of their nearby neighbors while at the same time keeping a safe distance.

33. "Novel type of phase transition in a system of self-driven particles," T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, Phys. Rev. Lett. **75**, 1226–1229 (1995). This paper introduces what is now the best studied model of flocking behavior, and a good example of a drastic but useful simplification of a complex problem. (A)

34. "Collective motion," T. Vicsek and A. Zafiris, Rev. Mod. Phys. (in press). This recent review summarizes progress on theories of flocking. (I)

35. "Effective leadership and decision-making in animal groups on the move," I. D. Couzin, J. Krause, N. R. Franks, and S. A. Levin, Nature **433**, 513–516 (2005). Another good example of the use of a simplified model to shed light on a complex phenomenon, this paper shows how the coordinated movement of a large group of individuals can self-organize to effectively achieve collective goals even when only a small fraction of individuals know where they are going. (I)

36. "Empirical investigation of starling flocks: A benchmark study in collective animal behaviour," M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, A. Orlandi, G. Parisi, A. Procaccini, M. Viale, and V. Zdravkovic, Animal Behaviour **76**, 201–215 (2008). An interesting recent development in the study of flocking is the appearance of quantitative studies of large flocks of real birds using video techniques. This paper describes a collaborative project that brought together field studies with theories based on ideas from statistical and condensed matter physics. (I)

IV. COMPLEX SYSTEMS THEORY

The remainder of this review deals with the general theory of complex systems. Perhaps "general theories" would be a better term, since complex systems theory is not a monolithic body of knowledge. Borrowing an analogy from Doyne Farmer of the Santa Fe Institute, complex systems theory is not a novel, but a series of short stories. Whether it will one day become integrated to form a single coherent theory is a matter of current debate, although my belief is that it will not.

A. Lattices and networks

The current theories of complex systems typically envisage a large collection of agents interacting in some specified way. To quantify the details of the system one must specify first its topology—who interacts with whom—and then its dynamics—how the individual agents behave and how they interact.

Topology is usually specified in terms of lattices or networks, and this is one of the best developed areas of complex systems theory. In most cases, regular lattices need little introduction—almost everyone knows what a chess board looks like. Some models built on regular lattices are considered in Section IV.C. Most complex systems, however, have more complicated non-regular topologies that require a more general network framework for their representation.

Several books on the subject of networks have appeared in recent years. The book by Watts below is at a popular level, although it contains a small amount of mathematics. The book by Newman is lengthy and covers many aspects in technical detail; the book by Cohen and Havlin is shorter and more selective. I also list two reviews, one brief and one encyclopedic, of research in the field, for advanced readers.

37. **Six Degrees: The Science of a Connected Age**, D. J. Watts (Norton, New York, 2003). (E)

38. **Networks: An Introduction**, M. E. J. Newman (Oxford University Press, Oxford, 2010). (I)

39. **Complex Networks: Structure, Stability and Function**, R. Cohen and S. Havlin (Cambridge University Press, Cambridge, 2010). (I)

40. "Exploring complex networks," S. H. Strogatz, Nature **410**, 268–276 (2001). (A)

41. "Complex networks: Structure and dynamics," S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Physics Reports **424**, 175–308 (2006). (A)

The book by Easley and Kleinberg cited above, Refs. 28, also includes material on networks.

B. Dynamical systems

Turning to the behavior of the agents in a complex system, many different theories have been developed. One of the most mature is dynamical systems theory, in which the behaviors of agents over time are represented individually or collectively by simple mathematical models, coupled together to represent interactions. Dynamical systems theory is divided into continuous dynamics, addressed in this section, and discrete dynamics, addressed in the following one.

Continuous dynamical systems are typically modeled using differential equations and show a number of emergent behaviors that are characteristic of complex systems, such as chaos and bifurcations (colorfully referred to as "catastrophes" in the 1970s, although this nomenclature has fallen out of favor). Three elementary references are the following:

42. **Sync: The Emerging Science of Spontaneous Order**, S. Strogatz (Hyperion, New York, 2003). A popular book introducing some of the basic ideas of dynamical systems theory by one of the pioneers of the field. The book focuses particularly on the phenomenon of synchronization, but also includes useful material on other topics in the field. (E)

43. **Chaos and Fractals**, H.-O. Peitgen, H. Jürgens, and D. Saupe (Springer, Berlin, 2004). A lavishly illustrated introduction suitable for undergraduates or even advanced high-school students. (E)

44. **Dynamics: The Geometry of Behavior**, R. Abraham and C. D. Shaw (Addison-Wesley, Reading, MA, 1992), 2nd edition. This unusual book is, sadly, out of print now, though one can still find it in libraries. It is essentially a picture book or comic illustrating the principles of dynamical systems. The field being one that lends itself well to visual representation, this turns out to be an excellent way to grasp many of the basic ideas. (E)

There are also many more advanced sources for material on dynamical systems, including the following.

45. **Nonlinear Dynamics and Chaos**, S. H. Strogatz (Addison-Wesley, Reading, MA, 1994). A substantial college-level text on the standard methods of dynamical systems theory. (I)

46. "Deterministic nonperiodic flow," E. N. Lorenz, J. Atmos. Sci. **20**, 130–141 (1963). This is a classic in the field, the first paper to really spell out the origin of chaotic behavior in a simple system, and is clear and well written, although it requires a strong mathematical background. (A)

47. "Controlling chaos," E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196–1199 (1990). Another seminal paper in the field, which studies the technically important subject of controlling chaotic systems. (A)

C. Discrete dynamics and cellular automata

Discrete dynamical systems, those whose evolution in time progresses via a succession of discrete "time steps," were a subject of considerable research interest in the 1970s and 1980s. A classic example is the logistic map, which displays a transition (actually several transitions) from an ordered regime to a chaotic one that inspired a substantial literature on the "edge of chaos" in complex systems.

48. "Simple mathematical models with very complicated dynamics," R. M. May, Nature **261**, 459–467 (1976). A classic pedagogical review of the logistic map and similar discrete dynamical systems from one of the fathers of complex systems theory. The mathematics is elementary in principle, involving only algebra and no calculus, but some of the concepts are nonetheless quite tricky to visualize. (I)

49. "Universal behavior in nonlinear systems," M. J. Feigenbaum, Physica D **7**, 16–39 (1983). In 1978 Mitchell Feigenbaum proved one of the most important results in dynamical systems theory, the existence of universal behavior at the transition to chaos, deriving in the process a value for the quantity now known as Feigenbaum's constant. His original research papers on the topic are technically challenging, but this later paper is relatively approachable and provides a good outline of the theory. (I)

A pedagogical discussion of Feigenbaum's theory can also be found in the book by Strogatz, Ref. 45 above.

Dynamical systems that are discrete in both time and space are called *cellular automata*, or CAs for short, and these fall squarely into the realm of complex systems, being precisely systems of many interacting agents. The simplest and best studied cases are on lattices, although cellular automata with other geometries are also studied. Well known examples of cellular automata include J. H. Conway's "Game of Life," the "Rule 110" automaton, which is capable of universal computation, and the Nagel–Schreckenberg traffic model mentioned in Section III.

50. "Mathematical Games: The fantastic combinations of John Conway's new solitaire game "life"," M. Gardner, Scientific American **223**, 120–123 (1970). One of Martin Gardner's excellent

"Mathematical Games" columns for *Scientific American*, in which the most famous CA of them all, Conway's Game of Life, made its first appearance. Decades later the article is still an excellent introduction. (E)

51. **Winning Ways for Your Mathematical Plays**, J. H. Conway, R. K. Guy, and E. R. Berlekamp, volume 2 (A. K. Peters, Natick, MA, 2003), 2nd edition. This is the second of four excellent volumes about games—such as board games and card games—and their mathematical analysis, originally published in the 1980s but recently republished. It contains a thorough discussion of the Game of Life, which was invented by one of the book's authors. (I)

52. **Brainchildren: Essays on Designing Minds**, D. C. Dennett (MIT Press, Cambridge, MA, 1998). This book is not, principally, a book about CAs and its author is not principally a CA researcher, but the chapter entitled "Real Patterns" is an excellent introduction not only to CAs but also to why those who study complex systems are interested in them as models of processes in the wider world. (E)

53. **A New Kind of Science**, S. Wolfram (Wolfram Media, Champaign, IL, 2002). Most of this large volume is devoted to a discussion of Wolfram's research, but the first part of the book, particularly the first hundred pages or so, provides a very readable introduction to CAs, laying out the basics of the field clearly while making only modest mathematical demands of the reader. (I)

54. "Studying artificial life with cellular automata," C. G. Langton, Physica D **22**, 120–149 (1986). An influential early paper on the theory of cellular automata, which made connections with other areas of complex systems research, including chaos theory and "artificial life" (see Section IV.H). Among other things, the paper contains some (in retrospect) rather charming figures of simulation results, created by directly photographing the screen of a computer terminal. (I)

55. **Cellular Automata: A Discrete Universe**, A. Ilachinski (World Scientific, Singapore, 2001). For the advanced reader this book provides most of what one might want to know about cellular automata. (A)

Chapter 11 of the book by Mitchell, Ref. 1, also provides a good overview of the study of cellular automata. For those interested in pursuing the topic further, an excellent and entertaining resource is the free computer program *Golly*, by Andrew Trevorrow and Tomas Rokicki, which simulates a wide range of cellular automata and illustrates their dynamics with instructive and elegant computer graphics.

D. Scaling and criticality

Among the fundamental tools in the theory of complex systems, some of the most important have been the physical ideas of scaling, phase transitions, and critical phenomena. One example of their application is mentioned above, the study by Feigenbaum of critical behavior in discrete dynamical systems at the "edge of chaos," Ref. 49, but there are many others.

A startling phenomenon observed in a number of complex systems is the appearance of "power-law" distributions of measured quantities. Power-law distributions are said to "scale" or "show scaling" because they retain their shape even when the measured quantity is "rescaled," meaning it is multiplied by a constant. The observation and origin of power laws and scaling in complex systems has been a subject of discussion and research for many decades. The following two papers provide general overviews of the area:

56. "A brief history of generative models for power law and lognormal distributions," M. Mitzenmacher, Internet Mathematics **1**, 226–251 (2004). (I)

57. "Power laws, Pareto distributions and Zipf's law," M. E. J. Newman, Contemporary Physics **46**, 323–351 (2005). (I)

Power laws have been the topic of some of the most influential publications in complex systems theory, going back as far as the work of Pareto in the 1890s. The mechanisms for power-law behavior have been a particular focus of interest and the claim has been made that there may be a single mathematical mechanism responsible for all power laws and hence a unified theory of complex systems that can be built around that mechanism. One candidate for such a universal mechanism is "self-organized criticality." Current thinking, however, is that there are a number of different mechanisms for power-law behavior, and that a unified theory probably does not exist.

58. "On a class of skew distribution functions," H. A. Simon, Biometrika **42**, 425–440 (1955). One of the first, and still most important, mechanisms suggested for power laws, the "rich get richer" or "preferential attachment" mechanism. Simon was the first to write down the theory in its modern form, although many of the ideas were present in significantly earlier work: see for instance "A mathematical theory of evolution based on the conclusions of Dr. J. C. Willis," G. U. Yule, Philos. Trans. R. Soc. London B **213**, 21–87 (1925). (A)

59. "Self-organized criticality: An explanation of the $1/f$ noise," P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381–384 (1987). Physicists have long been aware that physical systems tuned precisely to a special "critical point" will display power-law behavior, but on its own this appears to be a poor explanation for power laws in naturally occurring complex systems, since such systems will not normally be tuned to the critical point. Bak *et al.* in this paper proposed an ingenious way around this problem, pointing out that certain classes of system tune themselves to the critical point automatically, simply by the nature of their dynamics. This process, dubbed "self-organized criticality" is illustrated in this paper with a cellular automaton model, the "self-organizing sandpile." (A)

60. "Robust space–time intermittency and $1/f$ noise," J. D. Keeler and J. D. Farmer, Physica D **23**, 413–435 (1986). Sometimes overlooked in the literature on self-organized criticality, this paper actually preceded the paper by Bak *et al.* by more than a year and described many of the important concepts that formed the basis for the approach of Bak *et al.* (A)

61. "Self-organized critical forest-fire model," B. Drossel and F. Schwabl, Phys. Rev. Lett. **69**, 1629–1632 (1992). Perhaps the simplest of self-organized critical models is the forest fire model of Drossel and Schwabl. Although it came after the sandpile model of Bak *et al.* it is easier to understand and may make a better starting point for understanding the theory. (A)

62. **How Nature Works: The Science of Self-Organized Criticality**, P. Bak (Copernicus, New York, 1996). A self-contained and readable, if somewhat partisan, introduction to the science of self-organized criticality, written by the theory's greatest champion. (E)

63. "Highly optimized tolerance: A mechanism for power laws in designed systems," J. M. Carlson and J. Doyle, Phys. Rev. E **60**, 1412–1427 (1999). An alternative general theory for the appearance of power laws is the "highly optimized tolerance" (HOT) the-

ory of Carlson and Doyle. While its inventors would not claim it as an explanation of all power laws, it may well be a better fit to observations than self-organized criticality in some cases. This paper introduces the best-known model in the HOT class, the "highly optimized forest fire" model, which is analogous to the self-organized forest fire model above. (I)

64. "A general model for the origin of allometric scaling laws in biology," G. B. West, J. H. Brown, and B. J. Enquist, Science **276**, 122–126 (1997). Perhaps the biggest stir in this area in recent years has been created by the theory of biological allometry, i.e., power-law scaling in biological organisms, put forward by West *et al.* This is the original paper on the theory, although West *et al.* have published many others since. (A)

65. "Life's universal scaling laws," G. B. West and J. H. Brown, Physics Today **57** (9), 36–42 (2004). A general introduction to the theory of West *et al.* for physicists. (E)

The book by Mandelbrot, Ref. 5, is also an important historical reference on this topic, making a connection between power laws and the study of fractals—curves and shapes having non-integer dimension.

E. Adaptation and game theory

A common property of many though not all complex systems is adaptation, meaning that the collective behavior of the agents in the system results in the optimization of some feature or quantity. Biological evolution by means of natural selection is the classic example: evolution takes place as a result of the competition among the members of a breeding population for resources and is thus exclusively a result of agent interactions—precisely an emergent phenomenon in the complex systems sense.

Complex systems displaying adaptation are sometimes called "complex adaptive systems." In constructing theories and models of complex adaptive systems the fundamental concept is that of "fitness," a measure or value that conveys how well an individual, group, species, or strategy is doing in comparison to the competition, and hence how likely it is to thrive. In the simplest models, one posits a fitness function that maps descriptive parameters, such as body size or foraging strategy, to fitness values and then looks for parameter values that maximize the fitness.

The following three books are not specifically about complex systems, but nonetheless all provide an excellent background for the reader interested in theories of adaptation.

66. **The Theory of Evolution**, J. Maynard Smith (Cambridge University Press, Cambridge, 1993), 3rd edition. This updated version of Maynard Smith's widely read introduction to evolutionary theory is still a good starting point for those who want to know the basics. (I)

67. **Climbing Mount Improbable**, R. Dawkins (Norton, New York, 1997). Dawkins is one of the best known science writers of the last century and his many books on evolutionary biology have been particularly influential. His earlier book *The Selfish Gene* is, after Darwin's *Origin of Species*, perhaps the most influential book written about evolution. *Climbing Mount Improbable* is more elementary and, for the beginner, an excellent introduction to our current understanding of the subject. (E)

68. **The Structure of Evolutionary Theory**, S. J. Gould (Belknap Press, Cambridge, MA, 2002). (I)

Biologically derived ideas concerning adaptation have also inspired applications in computer science, wherein practitioners arrange for programs or formulas to compete against one another to solve a problem, the winners being rewarded with "offspring" in the next generation that then compete again. Over a series of generations one can use this process to evolve good solutions to difficult problems. The resulting method, under the names *genetic algorithms* or *genetic programming*, has become a widely used optimization scheme and a frequent tool of complex systems researchers.

69. "Genetic algorithms," J. H. Holland, Scientific American **267** (1), 66–72 (1992). A nontechnical introduction to genetic algorithms by their originator and greatest proponent, John Holland. (E)

70. "Evolving inventions," J. R. Koza, M. A. Keane, and M. J. Streeter, Scientific American **288** (2), 52–59 (1992). A discussion of genetic programming, which is the application of genetic-algorithm-type methods directly to the evolution of computer software. (E)

71. **Introduction to Genetic Algorithms**, M. Mitchell (MIT Press, Cambridge, MA, 1996). Although relatively old, Mitchell's book on genetic algorithms is probably still the foremost general text on the subject and a good resource for those looking for more depth. (A)

While fitness can depend on simple physical parameters like body size, significant contributions to fitness at the organismal level often come from the behaviors of agents—the way they interact with each other and their environment. The mapping between the parameters of behavior and the fitness is typically a complex one and a body of theory has grown up to shed light on it. This body of theory goes under the name of *game theory*.

A "game," in this sense, is any scenario in which "players" choose from a set of possible moves and then receive scores or "payoffs" based on the particular choice of moves they and the other players made. Game theory is used in the context of biological evolution to model mating strategies, in economics as a model of the behavior of traders in markets, in sociology to model individuals' personal, financial, and career decisions, and in a host of other areas ranging from ecology and political science to computer science and engineering.

Although almost a quarter of a century old, Morton Davis's "nontechnical introduction" to game theory remains a good starting point for those interested in understanding the ideas of game theory without getting into a lot of mathematics. The book has been recently reprinted in an inexpensive paperback edition that makes it a good buy for students and researchers alike. For a more mathematical introduction, the book by Myerson is a classic, written by one of the leading researchers in the field, while the book by Watson gives a lucid modern presentation of the material.

72. **Game Theory: A Nontechnical Introduction**, M. D. Davis (Dover, New York, 1997). (E)

73. **Game Theory: Analysis of Conflict**, R. B. Myerson (Harvard University Press, Cambridge, MA, 1997). (A)

74. **Strategy: An Introduction to Game Theory**, J. Watson (Norton, New York, 2007), 2nd edition. (I)

The book by Nowak, Ref. 10, also provides an introduction to game theoretical methods specifically in the area of biological evolution, while the book by Easley and Kleinberg, Ref. 28, includes a discussion of games played on networks.

Some specific topics within game theory are so important and widely discussed that a knowledge of them is a must for anyone interested in the area.

75. **The Evolution of Cooperation**, R. Axelrod (Basic Books, New York, 2006). The "prisoner's dilemma" is probably the best known (and also one of the simplest) of game theoretical examples. A famous event in the history of game theory is the contest organized by Robert Axelrod in which contestants devised and submitted strategies for playing the (iterated) prisoner's dilemma game against one another. Among a field of inventive entries, the contest was won by mathematical biologist Anatol Rapoport using an incredibly simple strategy called "tit-for-tat," in which on each round of the game the player always plays the same move their opponent played on the previous round. Axelrod uses this result as a starting point to explain why people and animals will sometimes cooperate with one another even when it is, at first sight, not in their own best interests. (E)

76. "Emergence of cooperation and organization in an evolutionary game," D. Challet and Y.-C. Zhang, Physica A **246**, 407–418 (1997). The minority game, proposed by physicists Challet and Zhang, is a remarkably simple game that nonetheless shows complex and intriguing behavior. In this game a population of n players, where n is odd, repeatedly choose one of two alternative moves, move 1 or move 2. On any one round of the game you win if your choice is in the minority, i.e., if fewer players choose the same move as you than choose the alternative. It's clear that there is no universal best strategy for playing this game since if there were everyone would play it, and then they'd all be in the majority and would lose. The minority game is a simplified version of an earlier game proposed by Brian Arthur, usually called the *El Farol problem*, in honor of a famous bar of that name in Santa Fe, New Mexico. (A)

77. **The Bounds of Reason: Game Theory and the Unification of the Behavioral Sciences**, H. Gintis (Princeton University Press, Princeton, NJ, 2009). An intriguing line of work in the last couple of decades has been the development of experimental game theory (also called behavioral game theory or experimental economics), in which instead of analyzing games theoretically, experimenters get real people to play them and record the results. The remarkable finding is that, although for many of these games it is simple to determine the best move—even without any mathematics—people often don't play the best move. Even if the experimenters offer real money in return for winning plays, people routinely fail to comprehend the best strategy. Results of this kind form the basis for the economic theory of "bounded rationality," which holds that it is not always correct to assume that people act in their own best interests with full knowledge of the consequences of their actions. (This may seem like an obvious statement, but it is a surprisingly controversial point in economics.) (A)

F. Information theory

Information theory is not usually regarded as a part of complex systems theory itself, but it is one of the tools most frequently used to analyze and understand complex systems. As its name suggests, information theory describes and quantifies information and was originally developed within engineering as a way to understand the capabilities and limitations of electronic communications. It has found much wider application in recent years, however, including applications to the analysis of patterns of many kinds. A pattern is precisely recognizable as a pattern because its information content is *low*. For instance, there is little information in a periodically repeating sequence of symbols, numbers, colors, etc. If we can accurately predict the next symbol in a sequence then that symbol contains little information since we knew what it was going to be before we saw it. This idea and its extensions has been applied to the detection of patterns in DNA, in networks, in dynamical systems, on the Internet, and in many kinds of experimental data.

78. **An Introduction to Information Theory**, J. R. Pierce (Dover, New York, 1980), 2nd edition. Although relatively old, this book is still the best introduction to information theory for the beginner. The subject requires some mathematics for its comprehension, but the level of mathematical development in Pierce's book is quite modest. (I)

79. **Elements of Information Theory**, T. M. Cover and J. A. Thomas (John Wiley, New York, 1991). A thorough introduction to modern information theory, this book demands some mathematical sophistication of the reader. (A)

80. "A mathematical theory of communication I," C. E. Shannon, Bell System Technical Journal **27**, 379–423 (1948). The original paper by the father of information theory, Claude Shannon, in which he lays out the theory, in remarkably complete form, for the first time. As well as being the first paper on the topic, this is also a well-written and palatable introduction for those willing to work through the mathematics. (A)

An active area of current research in complex systems is the application of information theory to measure the complexity of a system. This work aims to answer quantitatively the question, "What is a complex system?" by creating a measure that will, for instance, take a large value when a system is complex and a small one when it is not. One of the best-known examples of such a measure is the Kolmogorov complexity, which is defined as the length of the shortest computer program (in some agreed-upon language) that will generate the system of interest or a complete description of it. If a system is simple to describe then a short program will suffice and the Kolmogorov complexity is low. If a larger program is required then the complexity is higher. Unfortunately the Kolmogorov complexity is usually extremely hard—and in some cases provably impossible—to calculate, and hence researchers have spent considerable effort to find measures that are more tractable.

81. "How to define complexity in physics, and why," C. H. Bennett, in W. H. Zurek (editor), "Complexity, Entropy, and the Physics of Information," pp. 443–454 (Addison-Wesley, Reading, MA, 1990). A nontechnical description of the problem and why it is interesting by one of the leading researchers in the field. (E)

82. **Complexity: Hierarchical Structure and Scaling in Physics**, R. Badii and A. Politi (Cambridge University Press, Cambridge, 1997). Chapters 8 and 9 of this book provide a useful introduction to measures of complexity, and provide a connection to the topic of

the next section of this review, computational complexity theory. (A)

G. Computational complexity

Somewhat peripheral to the main thrusts of current complex systems research, but nonetheless of significant practical value, is the study of *computational complexity*. Computational complexity theory deals with the difficulty of performing certain tasks, such as calculating a particular number or solving a quantitative problem. Although typically discussed in the language of algorithms and computer science, computational complexity in fact has much wider applications, in evolutionary biology, molecular biology, statistical physics, game theory, engineering, and other areas. For instance, one might ask how difficult is it, in terms of time taken or number of arithmetic computations performed, to the find the ground state of a physical system, meaning the state with the lowest energy. For some systems this is an easy task but for others it is difficult because there are many possible states and no general principle for determining which energies are lowest. Indeed it is possible to prove, subject to basic assumptions, that in some cases there exists no general technique that will find the ground state quickly, and the only reliable approach is to search exhaustively through every state in turn, of which there may be a huge number. But if this is true for computations performed by hand or on a computer, it is no less true of nature itself. When nature finds the lowest energy state of a system it is, in effect, performing a computation, and if you can prove that no method exists for doing that computation quickly then this tells you that the physical system will not reach its ground state quickly, or in some cases at all, if the number of states that need to be searched through is so vast that the search would take years or centuries. Thus results about the theory of computation turn out to give us very real insight into how physical (or social or biological) systems must behave.

The best known issue in computational complexity theory, one that has made it to the pages of the newspapers on occasion, is the question of whether two fundamental classes of problems known as P and NP are in fact identical. The class P is the class of problems that can be solved rapidly, according to a certain definition of "rapidly." An example is the problem of multiplying two matrices, for which there is a simple well-known procedure that will give you the answer in short order. The class NP, on the other hand, is the class of problems such that if I hand you the solution you can *check that it's correct* rapidly, which is not the same thing at all. Obviously NP includes all problems in P—if you tell me a purported solution for the product of two matrices I can check it rapidly just by calculating the product myself from scratch and making sure I agree with your answer. But NP can also include problems whose answer is easy to check but difficult to compute. A classic example is the "traveling salesman problem," which asks whether there exists a route that will take a salesman to each of n cities while traveling no more than a set number of miles. (It is assumed, for simplicity, that the salesman can fly in a straight line from each city on his route to the next—he is not obliged to follow the path of the established roads.) If you hand me a purported solution to such a problem I can check it quickly. Does the route visit every city? Is it below the given number of miles? If the answer is yes to both questions then the solution is good. But if you give me only the list of cities and I have to find a solution for myself then the problem is much harder and indeed it widely is believed (though not known for certain) that no method exists that will find the solution rapidly in all cases. Unless this belief is wrong and there exists a (currently unknown) way to solve such problems easily so that problems in the NP class also belong to P, then NP is a bigger class than P and hence the two classes are not identical. Most researchers in computational complexity theory believe this to be the case, but no one has yet been able to prove it, nor indeed has any clue about how one should even begin.

83. "NP-complete problems in physical reality," S. Aaronson, ACM SIGACT News **36** (1), 30–52 (2005). In this article Aaronson discusses the application of computational complexity theory, and particularly the central idea of "NP-completeness," to a wide range of scientific problems including protein folding, quantum computing, and relativity, introducing in the process many of the main ideas of computational complexity. (I)

84. **The Nature of Computation**, C. Moore and S. Mertens (Oxford University Press, Oxford, 2011). A readable and informative introduction to the theory of computational complexity and its applications from two leading complex systems researchers. This book emphasizes the important idea that it is not only computers that perform computation: all sorts of systems in the natural and man-made world are effectively performing computations as part of their normal functioning, and so can be viewed through the lens of computational theories. (I)

85. **Introduction to the Theory of Computation**, M. Sipser (Thomson, Boston, MA, 2006), 2nd edition. A general and widely used text on computational complexity within computer science. (A)

H. Agent-based modeling

Many types of computer modeling are used to study complex systems. Most of the standard methods of numerical analysis—finite-element methods, linear algebra and spectral methods, Monte Carlo methods, and so forth—have been applied in one branch of the field or another. However, there is one method that is particular to the study of complex systems and has largely been developed by complex systems scientists, and that is *agent-based modeling*. The goal of agent-based computer models, sometimes also called "individual-based," is to separately and individually simulate the agents in a complex system and their interactions, allowing the emergent behaviors of the system to appear naturally, rather than putting them in by hand. The first two papers listed here both give pedagogical introductions to agent-based methods, but from quite different viewpoints. The third reference is an entire journal volume devoted to discussions of agent-based modeling, including a number of accessible overview articles.

86. “Agent based models,” S. E. Page, in L. Blume and S. Durlauf (editors), **The New Palgrave Encyclopedia of Economics,** (Palgrave Macmillan, Basingstoke, 2008), 2nd edition. (E)

87. “From factors to actors: Computational sociology and agent-based modeling,” M. W. Macy and R. Willer, Annual Review of Sociology **28**, 143–166 (2002). (E)

88. **Adaptive agents, intelligence, and emergent human organization: Capturing complexity through agent-based modeling**, B. J. L. Berry, L. D. Kiel, and E. Elliott (editors), volume 99, Suppl. 3, *Proc. Natl. Acad. Sci. USA* (2002). (E)

The book by Miller and Page, Ref. 4, also contains a useful introduction to agent-based methods. There also exist a number of books that tackle the subject in the context of specific fields of scientific study, such as:

89. **Individual-based Modeling and Ecology**, V. Grimm and S. F. Railsback (Princeton University Press, Princeton, NJ, 2005). An introduction to agent-based modeling in ecology. (I)

90. **Agent-Based Models**, N. Gilbert (Sage Publications, London, 2007). A very short introduction to social science applications of agent-based models. (I)

A few classic examples of agent-based models are also worthy of mention:

91. “Dynamic models of segregation,” T. Schelling, J. Math. Soc. **1**, 143–186 (1971). One of the first true agent-based models is the model of racial segregation proposed by Thomas Schelling in 1971. Schelling did not have access to a computer at the time he proposed his model (or perhaps was not interested in using one), and so simulated it by hand, using coins on a grid of squares. However, many computer simulations of the model have subsequently been performed. Schelling was awarded the Nobel Prize in Economics for 2005, in part for this work, and to date this is the only Nobel Prize awarded for work on traditional complex systems (although one could argue that, for instance, condensed matter systems are complex systems, and several prizes in physics have been awarded for condensed matter research). (E)

92. **Growing Artificial Societies: Social Science from the Bottom Up**, J. M. Epstein and R. L. Axtell (MIT Press, Cambridge, MA, 1996). The “Sugarscape” models of Epstein and Axtell provide a beautiful example of the emergence of complex behaviors from the interactions of simple agents. This set of models would also be a good starting point for experimenting with agent-based simulations: the rules are simple and easy to implement, and the results lend themselves nicely to computer graphics and visualization, making the models relatively straightforward to interpret. Versions of some of the models are available already programmed in standard agent-based simulation software packages (see below). (I)

93. “Artificial economic life: a simple model of a stockmarket,” R. G. Palmer, W. B. Arthur, J. H. Holland, B. LeBaron, and P. Tayler, Physica D **75**, 264–274 (1994). A good example of an agent-based model is the “artificial stock market” created by Palmer *et al.* at the Santa Fe Institute in the early 1990s. In this study, the researchers simulated individually the behavior of many traders in a stock market, giving them a deliberately heterogeneous selection of trading strategies and limited knowledge of market conditions. They observed regimes of the model in which it displayed the equilibrium behavior of neoclassical economics, but others in which it displayed chaotic behavior more akin to that of real stock markets. (I)

94. “An approach to the synthesis of life,” T. S. Ray, in C. Langton, C. Taylor, J. D. Farmer, and S. Rasmussen (editors), “Artificial Life II,” volume XI, **Santa Fe Institute Studies in the Sciences of Complexity**, pp. 371–408 (Addison-Wesley, Redwood City, CA, 1991). An inventive and influential example of an agent-based simulation is the Tierra evolution model created by Ray. In this simulation, computer programs reproduce by explicitly copying themselves into new memory locations, competing and mutating to make best use of computer resources, meaning CPU time and memory. Although similar in some respects to the genetic programming studies discussed in Section IV.E, Tierra is different in that no fitness function is imposed externally upon its programs. Instead, fitness emerges naturally in the same way it does in biological evolution: those programs that manage to reproduce themselves survive and spread, while those that do not die out. Tierra was the first such simulation to be constructed, but others, such as the Avida system, have appeared in recent years. Systems such as these are referred to generally as “artificial life” simulations. Artificial life was a major thrust in complex systems research in the 1990s. (I)

Finally, there are a variety of software packages available for performing agent-based simulations. Some of them are highly advanced programming libraries suitable for cutting-edge research, while others are designed as easy-to-use educational tools requiring little prior knowledge. Among the former, *Repast* and *Mason* are currently the most widely used and mature systems, while among the latter *NetLogo* is a good starting point.

V. CONCLUSION

Complex systems is a broad field, encompassing a wide range of methods, many of them drawn from physics, and having an equally wide range of applications, within physics and in many other areas. The resources reviewed here cover only a fraction of this rich and active field of scientific endeavor. For the interested reader there is an abundance of further resources to be explored when those in this article are exhausted, and for the scientist intrigued by the questions raised there are ample opportunities to contribute. Science has only just begun to tackle the questions raised by the study of complex systems and the areas of our ignorance far outnumber the areas of our expertise. For the scientist looking for profound and important questions to work on, complex systems offers a wealth of possibilities.

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