Chapter 9

Acoustic musical instruments

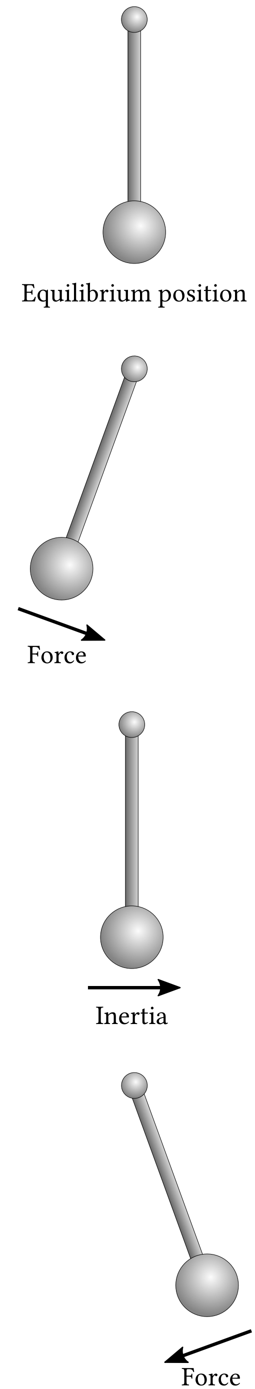
In previous chapters we have studied the nature of sound and how we hear it. We now turn to the question of how musical sound is made, using the devices we call musical instruments. We will examine the workings of a wide range of musical instruments, including both traditional acoustic instruments, such as the instruments of the orchestra, and electric and electronic instruments such as the electric guitar and synthesizers. We begin with acoustic instruments.

9.1 Vibration and sound production

The fundamental working principle of all acoustic musical instruments is the same: some part of the instrument is made to vibrate in some way, which in turn makes the air vibrate to produce sound. Drums are a good example. A drum consists of a drum head—a flexible skin or membrane—stretched over a rigid frame. The drum head is made to vibrate by striking it, for instance with a drumstick, and the vibrating head pushes on the adjacent air, producing pressure variations that we hear as sound, in a manner somewhat similar to the operation of a loudspeaker with its vibrating cone (see Section 7.5). The frequency of vibration translates directly into the frequency of the sound wave and hence to the pitch of the sound one hears.

The vibrating element of an instrument varies from one instrument to another. On a drum it is the drum head. On a string instrument it is the strings. On a wind or brass instrument it is a vibrating column of air. However, the basic scientific principles that govern the vibration are the same in all these cases. In order for a mechanical system to vibrate it must have three crucial properties: an equilibrium, a restoring force, and inertia. Let us take a look at how these properties conspire to produce the effects we want.

For a moment let us forget about musical instruments and consider a very simple example of vibration, a pendulum, consisting of a weight or bob attached to the end of a rod which can pivot around its other end, as shown in the figures on the right.



A pendulum typically only vibrates slowly, perhaps once a second, which is below the frequency of audible sound, so it is not a useful musical instrument, but it is a good tool to help us understand the mechanics of vibration before we move onto the more complicated mechanics of real instruments.

If a pendulum is hanging straight down vertically, as in the top figure on the right, it will be motionless. As long as we don't disturb it, it will not go anywhere on its own. This is the *equilibrium* we spoke of. Every vibrating system has an equilibrium position where it will remain motionless unless disturbed.

Now suppose we pull the pendulum to one side, as in the second figure. The action of gravity now pulls the bob back toward the equilibrium position as indicated by the arrow. If we were holding the bob we would feel this pull as a weight against our hand. This is the *restoring force*. The restoring force acts to pull us back toward the equilibrium position. In the figure the bob has been pulled to the left and the restoring force is to the right. If we were to pull the bob to the right then the force would be to the left.

If we now let go of the bob it will fall back toward the center and after a little while it will have traveled back to the equilibrium position, as shown in the third figure. At this point one might imagine it would stop moving: as we have said the equilibrium position is a stationary point where the bob will sit motionless indefinitely. Now, however, our third property of vibration comes into play: *inertia*. By the time the bob has fallen back to the equilibrium position it is moving quite fast and moving objects have inertia. Once you set them moving they go on moving, at least for a while. This inertia means that the bob does not stop when it reaches the equilibrium position, but keeps on moving and swings over the other way, to the right as shown in the bottom figure.

Now, however, the restoring force on the bob is acting in the opposite direction, to the left, which slows it down, turns it around, and pushes it back toward the center again. And so the process repeats. The bob swings one way, passes through the equilibrium position, slows down, and swings the other way, over and over in the repeated motion familiar to us from pendulum clocks and swing sets everywhere.

9.1.1 The mathematics of vibration

We have described the mechanics of vibration in qualitative terms. We can be more precise about it by invoking Newton's laws of motion, specifically the first and second laws. In colloquial terms, Newton's first law says that an object in motion will continue moving unless something or someone acts to stop it. This is inertia. One might argue that objects do not continue in motion indefinitely—in the end a rolling ball stops moving. This however is only because of friction with the ground. In other words something is acting to stop the ball, as Newton says. In the absence of friction an object really will continue moving forever. There are rocks in space (where there is no friction) that have been moving for millions of years and will continue moving for millions more.

Newton's second law tells us how the motion of an object will change if we do take action. Specifically, if we apply a force F to an object, the second law says that the object will accelerate in the direction of the force with an acceleration a given by

$$a = \frac{F}{m}, \tag{9.1}$$

where F is measured in newtons, m is the mass of the object in kilograms, and the resulting acceleration is measured in meters per second per second. In the case of our pendulum, for instance, the force F is provided by gravity acting on the bob and m represents the mass of the bob. Commonly, Newton's second law is written in slightly different form as $F = ma$, but for our purposes here Eq. (9.1) is actually more convenient.

We can use Newton's laws to deduce some crucial properties of vibrating motion. Take a look at the pictures of the pendulum on page 321 again and consider the motion of the bob starting in its leftmost position where, momentarily at least, it is standing still, before accelerating back to the center. Let us suppose the amount of time it takes to reach the center point is t and it has velocity v when it gets there. We can calculate the average acceleration of the bob by dividing the change in velocity (from zero to v) by the time t, giving

$$a = \frac{v}{t}. \tag{9.2}$$

Thus, for instance, if the pendulum is initially stationary and ends up with velocity $v = 2\,\mathrm{m/s}$ half a second later, the acceleration is $2/0.5 = 4$ meters per second per second, or $4\,\mathrm{m/s^2}$.

Now suppose that we somehow contrive to make our pendulum swing twice as fast. In other words we double the frequency of the vibration. We could do this for instance by increasing the restoring force so that the bob is pushed faster toward the center. Now the amount of time the bob takes to reach the center will be reduced to $\frac{1}{2}t$ and its velocity at the center point will be twice as large at $2v$, so its acceleration is now

$$a = \frac{2v}{\frac{1}{2}t} = 4\frac{v}{t}, \tag{9.3}$$

which is four times what it was previously in Eq. (9.2).

In other words, if you multiply the frequency of vibration by two you multiply the acceleration by four. Following the same argument one can easily show that in fact if you multiply the frequency by any number the acceleration is multiplied by that number squared. In mathematical terms acceleration is proportional to frequency squared: $a \propto f^2$.

Now we apply Newton's second law $a = F/m$ to write this as

$$\frac{F}{m} \propto f^2, \tag{9.4}$$

or, taking the square root and rearranging,

$$f \propto \sqrt{\frac{F}{m}}. \tag{9.5}$$

This result says that the frequency of vibration is proportional to the square root of the restoring force F divided by the amount of inertia as measured by the mass m. This fundamental rule, or some version of it, governs all vibrations.

Example 9.1: The frequency of a pendulum

For the pendulum the restoring force F is provided by gravity. If we wanted to increase the restoring force on a pendulum, one way to do it would be to increase gravity, which we could do by taking the pendulum to a different planet with stronger gravity, or by flying it into space where gravity is weaker. This would be a rather extravagant approach, but it would work. For instance, if we took the pendulum high above the Earth to a place where gravity was a quarter as strong, how would the frequency of the pendulum change, all other things being equal?

Equation (9.5) tells us the answer: frequency is proportional to the square root of the restoring force F, so the frequency would be a half of what it is on the surface of the planet, since $\sqrt{1/4} = \frac{1}{2}$.

A more practical way to change the restoring force would be to change the weight of the bob of the pendulum. Doubling the weight, for instance, would double the restoring force. How will this change the frequency of pendulum? This is a more tricky question. Doubling the weight of the bob does double the restoring force, but it also doubles the mass m, so the value of F/m does not change at all—the increase just cancels out. Equation (9.5) then tells us that the frequency will not change when we change the weight of the bob. This can be a useful feature, making a pendulum a reliable timekeeper that has historically been used in clocks as well as scientific instruments such as seismometers.

Advanced material

9.1.2 Simple harmonic motion

We have motivated our results so far by appealing to qualitative arguments, but we can make things more precise with a little mathematics. Consider a vibrating system and let x represent the displacement of the system from its equilibrium position. For instance, x might be how far the bob of a pendulum is displaced from the vertical rest position. The restoring force on the bob of a pendulum is zero when the pendulum is at equilibrium but grows larger the further we push it to one side or the other, meaning it increases with x, and this is true of most vibrating systems. Let us suppose the force F is simply proportional to x with some proportionality constant k, so that $F = -kx$. The minus sign indicates that the restoring force is always in the opposite direction to the displacement—if we push a pendulum to left the restoring force is to the right and *vice versa*.

At the same time we can write the acceleration of the motion as the second derivative $\mathrm{d}^2x/\mathrm{d}t^2$, and hence we can write Newton's second law $F = ma$ as

$$-kx = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \tag{9.6}$$

or equivalently

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{k}{m}x = 0, \tag{9.7}$$

where m is the mass of the vibrating system—the mass of the bob in the case of the pendulum, for example.

Equation (9.7) is the fundamental equation of *simple harmonic motion*, the most straightforward type of vibration and the main one that applies to musical instruments. The equation has a solution of the form

$$x(t) = A\sin(2\pi ft), \tag{9.8}$$

where A is the width or amplitude of the motion and f is its frequency. We can check that this is indeed a solution by substituting it into Eq. (9.7) and performing the derivative, which gives us

$$-(2\pi f)^2 A\sin(2\pi ft) + \frac{k}{m}A\sin(2\pi ft) = 0. \tag{9.9}$$

Cancelling some factors and rearranging, we find that the equation is satisfied provided $(2\pi f)^2 = k/m$ or

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{9.10}$$

This equation tells us the frequency of vibration in terms of the quantities k and m. This is a version of the result we derived previously in Eq. (9.5), with the constant k measuring the size of the restoring force. Note that the frequency does not depend on the amplitude A of the vibration. One might expect that larger amplitude vibrations would be slower because there is more distance to cover, but larger amplitude also means larger restoring force (since restoring force is proportional to displacement), which in turn makes the motion faster. The faster motion exactly compensates for the greater distance traveled so that the frequency stays the same. This turns out to be crucial for musical purposes, since it means that the frequencies of the notes produced by a musical instrument that employs simple harmonic motion do not depend on whether the instrument is played loudly or softly.

9.2 Vibrating strings

The strings of a string instrument are an archetypal example of the phenomena we have been talking about, a simple mechanical system providing a steady vibration that forms the basis for producing musical notes. String instruments differ in the details of their form and construction but there are basic principles that are common to all of them. One or more strings are attached at both ends to a rigid frame that pulls them taught, allowing them to vibrate when plucked, struck, or bowed. Other parts of the instrument, such as neck and soundboard, can also play important roles, but let us ignore these for the moment and concentrate our attention on the string itself, as depicted in Fig. 9.1.

Figure 9.1a shows a string in its equilibrium position, straight and at rest. If left undisturbed the string will remain in this position indefinitely. In order to vibrate the string must be under tension, stretched between the two end supports. The tension T is defined to be the force, measured in newtons, with which these end supports are pulling on the string. Imagine holding it taught with your hands: the tension is the force you would pull with, which is necessarily the same at both ends.

Be careful not to confuse the tension with the period of a sound wave, which is also represented by the letter T.

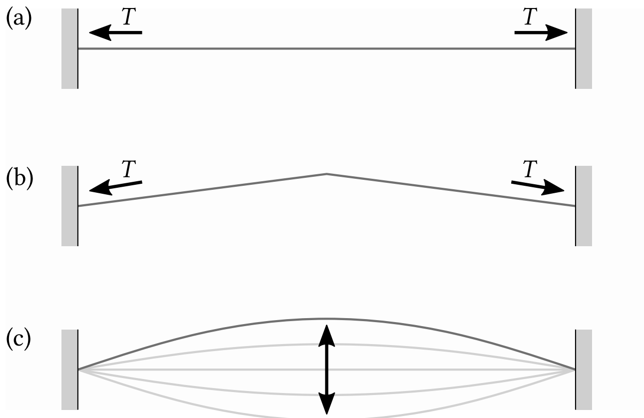


Figure 9.1: Vibration of a stretched string. A string is stretched between two fixed supports. (a) The equilibrium position of the string occurs when the string lies in a straight line between the supports. The tension T is the force exerted by the supports on the string to keep it stretched. This force is necessarily the same at both ends. (b) If the string is pulled upward in the middle, the tension force now acts at an angle, outward but also partly downward. It is this downward component of the tension that pulls the string back toward the equilibrium position and provides the restoring force. (c) If we let go of the string, the combination of restoring force and inertia makes it vibrate.

The tension provides the restoring force needed to make the string vibrate. If you grab the string in the middle and pull it upward as shown in Fig. 9.1b, it will resist you and pull back. The reason is that by pulling the string up in this way, you change the direction of the tension force as shown in the figure. Now the tension is acting not just sideways but also partly downward. It is this downward component of the tension that pulls the string back towards being straight. The restoring force is not equal to the tension, but it is caused by the tension and proportional to it: if you double the tension then you double the restoring force. The string also has mass—it weighs something, which gives is inertia—so it has all the features necessary for vibration: equilibrium, restoring force, and inertia. When we let it go, it will vibrate as shown in Fig. 9.1c. Making use of our earlier result that the frequency f of vibration is proportional to $\sqrt{F/m}$ where m is the mass, we have in this case

$$f \propto \sqrt{\frac{T}{m}}. \tag{9.11}$$

Example 9.2: A vibrating string

How will the frequency of vibration of a string change if we multiply either the tension or the mass by four?

Since f is proportional to $\sqrt{T}$, multiplying the tension by four will double the frequency, because $\sqrt{4} = 2$. On the other hand, f is inversely proportional $\sqrt{m}$, so multiplying the mass by four will half the frequency—heavier strings have lower frequency. Both of these adjustments are useful in musical situations. Variations in tension are used to tune musical instruments, the tension being adjusted by turning a tuning peg or machine head. Variations in mass are useful on instruments that have more than one string, tuned to different notes: strings intended for lower notes are commonly made heavier than those for higher notes, allowing the instrument to play a wide range of different pitches, potentially spanning many octaves.

9.2.1 Mersenne's law

Equation (9.11) tells us how the frequency of vibration of a string depends on tension and mass, but it does not tell the complete story because there is another important variable we have yet to discuss: the length of the string, which we will denote by L.

In the 17th century, French scientist and philosopher Marin Mersenne performed a series of experiments and determined that the frequency of vibration of a string in hertz is given in terms of its tension, mass, and length by the formula

$$f = \frac{1}{2}\sqrt{\frac{T}{Lm}}. \tag{9.12}$$

Note that this is not now a "proportional to" type of equation, but a full equality. Given the tension, mass, and length, you can use this formula to calculate the actual frequency of a string.

Sometimes you may see references to "Mersenne's laws" (plural) because the effects of length, tension, and mass are considered by some to be three separate laws, even though they are all captured by a single equation.

Equation 9.12 is commonly referred to as *Mersenne's law* or *Mersenne's formula*, although others before Mersenne, including Galileo, had worked out at least parts of it. Mersenne arrived at the formula by performing experiments, but it can also be derived theoretically from Newton's laws of motion. The derivation, which involves some calculus, is given in Sections 9.2.3 and 9.2.4.

For musical purposes, Eq. (9.12) is not in the most useful form of Mersenne's law because measuring the mass m of a string can be quite difficult. We can make the equation more useful by the following trick. In almost all cases, the strings of a string instrument take the form of cylinders, circular in cross-section. The mass of such a string is by definition equal to the density of the material it is made of (the weight in kilograms per cubic meter) times the volume of material (the number of cubic meters). Suppose our string has length L as before and radius r, as sketched in Fig. 9.2. The formula for the volume V of a cylinder says that $V = \pi r^2 L$ and, multiplying by the density ρ, we find the mass to be

$$m = \pi r^2 L\rho. \tag{9.13}$$

In fact it is usually more convenient to talk about the diameter d of a string than its

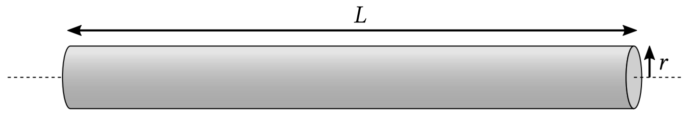


Figure 9.2: Volume of a string. A string in the form of a cylinder of length L and radius r has volume $V = \pi r^2 L$. For clarity the width of the cylinder is exaggerated in this picture. A real-life instrument string would be much narrower compared to its length.

radius. The radius is half the diameter $r = \frac{1}{2}d$, so

$$m = \pi\left(\tfrac{1}{2}d\right)^2 L\rho = \tfrac{1}{4}\pi d^2 L\rho. \tag{9.14}$$

Substituting this form into Eq. (9.12) we get

$$f = \frac{1}{2}\sqrt{\frac{4T}{\pi d^2 L^2 \rho}} \tag{9.15}$$

which simplifies to

$$f = \frac{1}{Ld}\sqrt{\frac{T}{\pi\rho}}. \tag{9.16}$$

This is the form of Mersenne's law we will use in this book. Given a string's tension, length, and diameter (which can be easily measured), and the density of the material from which it is made, this equation allows us to calculate the frequency at which it will vibrate.

Example 9.3: An acoustic guitar string

The vibrating part of a string on a standard acoustic guitar is $25\frac{1}{2}$ inches long, which is 0.648 meters. The second string (the one that plays the second-highest note) normally has a diameter of 16 thousandths of an inch or 0.406 mm. Suppose it is made of solid steel, which has a density of $7900\,\mathrm{kg/m^3}$, and its tension is 10.7 kg. What musical note does it play?

To answer this question we must first convert the tension into newtons, which we do by multiplying by 9.81:

$$T = 9.81 \times 10.7 = 105\,\text{newtons}. \tag{9.17}$$

Feeding this figure into Mersenne's law, Eq. (9.16), along with the given numbers for the length, diameter, and density, we find that

$$f = \frac{1}{0.648 \times 0.406 \times 10^{-3}}\sqrt{\frac{105}{\pi \times 7900}} = 247\,\text{Hz}. \tag{9.18}$$

Consulting Fig. 2.11 on page 34, we see that this frequency is very close to that of the note B3, which is indeed the note played by the second string of a guitar.

Example 9.4: An electric guitar string

An electric guitar has strings the same length as those of an acoustic guitar and they play the same notes, but they are thinner. The second string, for instance, still plays the note B3, but typically has a diameter of only 13 thousandths of an inch, or 0.330 mm. What will be the tension of such a string if it is again made of steel?

To answer this question, we rearrange Mersenne's law, Eq. (9.16), to give us the tension thus:

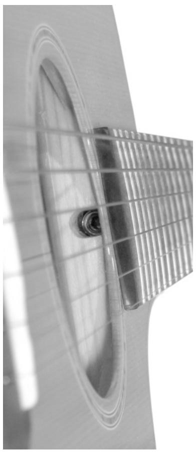
$$T = \pi\rho(fLd)^2. \tag{9.19}$$

The density, frequency, and length are as before; only the diameter has changed, giving us a tension of

$$T = \pi \times 7900 \times (247 \times 0.648 \times 0.330 \times 10^{-3})^2 = 69.2\,\text{newtons}. \tag{9.20}$$

If we like we can convert this into kilograms by dividing by 9.81, which gives a tension of 7.1 kg, significantly less than then 10.7 kg of the acoustic guitar. In practice, this means that the strings of the electric guitar feel less rigid to the player and are easier to move and pluck. This difference plays an important role in the contrasting styles of play for acoustic and electric guitars. In particular, the ability to "bend" the strings of an electric guitar, which causes their pitch to go slightly sharp, is a central technique in rock guitar styles, and is much easier to do when the strings have lower tension (see Section 10.3.3).

In passing, we note that 7 to 10 kg of tension in a guitar string is a significant amount. Most guitars have six strings and, assuming the tension is similar on all the strings (which it is), the total tension force from all six is in the range of 40 to 60 kg—roughly the weight of a adult human being. All of this force has to be borne by the neck of the guitar without warping or breaking. To help with this a guitar normally has a steel reinforcing bar called a truss rod embedded inside the neck.



The tip of the reinforcing truss rod can be seen inside the soundhole of this acoustic guitar.

9.2.2 Turning vibration into sound

The strings of a string instrument do not on their own produce much sound. They do push on the air around them and cause it to vibrate, but only to a small extent. Think of the sound made by a stretched rubber band or a plucked archery bow: you can hear a note, but it is not loud enough to be a practical musical instrument.

The problem is two-fold. First, a typical string is rather thin, so it doesn't move very much air. It can produce quite a large sound pressure p at the string surface where it compresses the air, and hence a large sound intensity $p^2/\rho c$ (see Eq. (3.7) on page 57). But intensity is sound energy *per square meter* and the surface of the string does not cover very many square meters. Hence the total sound energy produced is small. We encountered a similar issue when we considered the operation of loudspeakers in Section 7.5. There we saw that the sound power produced by a speaker is proportional to the area of the speaker cone, so smaller speakers produce less sound—see Eq. (7.81) on page 260. The same is true of the vibrating string and its small surface area means that it will never produce much sound.

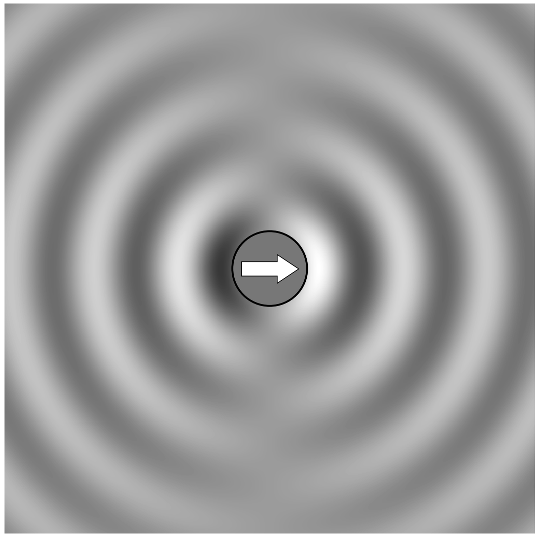


Figure 9.3: Sound pressure produced by a moving string. This computer calculation shows the sound pressure produced by a string as it vibrates and pushes on the air around it. The string, seen in cross-section, is represented by the circle. The air is compressed in front of the string, creating a region of high pressure (light shading), but it is also rarefied behind the string, creating a region of low pressure (dark). The two partly cancel, so the net amount of sound produced by the string is relatively small.

But this is not the only problem. Take a look at Fig. 9.3, which shows a visualization of the sound pressure created by a vibrating string. As the string moves through the air it does indeed compress the air in front of it and raise the pressure, but at the same time it also leaves a rarefied region of low pressure behind it, and the two partially cancel out, so that the net change in air pressure is smaller than we might expect. In the language of acoustics, we say that the string is a "dipole radiator" of sound, meaning a combination of high and low sound pressure at the same time.

To turn vibration into sound, therefore, a string instrument makes use of a *soundboard*, a thin board, typically made of wood. In the most common arrangement, the strings run over a short bar, called a *bridge*, which rests on the soundboard. When the string vibrates the vibration is transmitted through the bridge to the soundboard, causing it to vibrate too. On an acoustic guitar, for instance, the whole of the front of the guitar body acts as the soundboard—see Fig. 9.4. On a grand piano, the soundboard is a wooden board several feet long mounted on the bottom of the instrument. You can see it if you look underneath the piano.

Because it is much larger than the size of the string, the soundboard makes a larger sound than the string does. We discussed sound production from a soundboard

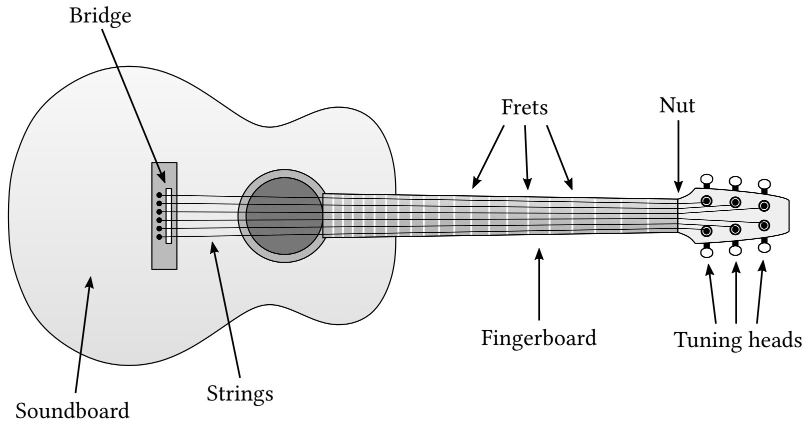


Figure 9.4: An acoustic guitar. The vibrating parts of the strings on an acoustic guitar run from the bridge to the nut at the end of the neck. The vibration is transmitted through the bridge to the front of the guitar body, which acts as the soundboard.

previously in Section 1.3, and the mechanics is also similar to sound production by a loudspeaker cone, as discussed in Section 7.5. A soundboard does still suffer from the dipole radiator problem described above: every time it compresses the air in front of it, it also creates a region of lower pressure behind. This issue is minimized in an instrument like a guitar, however, because the back of the soundboard is mostly hidden inside the body of the instrument where the sound cannot escape. (We discussed a similar strategy for loudspeakers in Section 7.5.5, in which the sound from the back of a speaker cone is minimized by enclosing the speaker in a sealed cabinet.)

The basic principles described here are common to all string instruments, but there is a great deal of craft that goes into the design and manufacture of individual instruments. In Chapter 10 we will look in detail at the workings of a range of string instruments, including the guitar and other plucked instruments, bowed orchestral strings, and the piano.

Advanced material

9.2.3 Equation of motion for a stretched string

Marin Mersenne derived his law for the frequency of vibration of a string experimentally, by careful measurement of the behavior of real strings. It can, however, also be derived from first principles using a knowledge of mechanics and Newton's second law of motion.

Consider a stretched string fixed at both ends and under tension T. Figure 9.5 shows a portion of such a string, in the process of vibrating. Let us measure position along the string by x, which for a string of length L will run

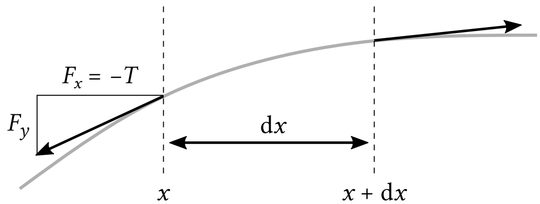


Figure 9.5: Forces acting on a segment of a string. The curve in this diagram represents a portion of a vibrating string. Because the string is curved the tension forces acting on the two ends of a small segment of width dx are at slightly different angles.

from $x = 0$ to $x = L$, and consider a short segment of the string of length dx, between positions x and $x+\mathrm{d}x$, as indicated by the dashed lines in the figure. We will denote the vertical position of the string at point x, measured from the equilibrium position, by $y(x)$. Note that as the vibration pushes the string up and down it causes it to curve as shown, which means the slope $\partial y/\partial x$ of the string at the two ends of our segment is not exactly the same.

The tension force acts along the line of the string and has a horizontal component F_x that is, to a good approximation, just equal to T, the resting tension on the string. At the left-hand side of our segment it is $F_x = -T$ since it acts to the left, and at the right-hand side it is $+T$. The horizontal component does increase a little when the string vibrates up and down because the string gets stretched slightly, which increases the tension, but this effect is small and we will ignore it.[1] The ratio of the vertical and horizontal components of the force, F_x and F_y, is equal to the slope of the string, and hence the vertical force at the left-hand end of our segment is

$$F_y = F_x\left(\frac{\partial y}{\partial x}\right)_x = -T\left(\frac{\partial y}{\partial x}\right)_x, \tag{9.21}$$

where the subscript x indicates that we are calculating the slope at position x. Similarly at the right-hand end the vertical force is

$$F_y = T\left(\frac{\partial y}{\partial x}\right)_{x+\mathrm{d}x}, \tag{9.22}$$

and the total vertical force on the segment is the sum of Eqs. (9.21) and (9.22):

$$T\left(\frac{\partial y}{\partial x}\right)_{x+\mathrm{d}x} - T\left(\frac{\partial y}{\partial x}\right)_x = T\frac{\partial^2 y}{\partial x^2}\,\mathrm{d}x. \tag{9.23}$$

Now we apply Newton's second law. Let the total mass of the string, from one end to the other, be m, so that the mass per unit length is m/L and the mass of our small segment is $(m/L)\,\mathrm{d}x$. Applying Newton's law in the form $F = ma$ in the vertical direction, with this mass and the force from Eq. (9.23), we have

$$T\frac{\partial^2 y}{\partial x^2}\,\mathrm{d}x = \frac{m}{L}\mathrm{d}x\,\frac{\partial^2 y}{\partial t^2}, \tag{9.24}$$

where $\partial^2 y/\partial t^2$ is the vertical acceleration. Cancelling a factor of dx and rearranging, we then find that

$$\frac{\partial^2 y}{\partial x^2} - \left(\frac{m}{LT}\right)\frac{\partial^2 y}{\partial t^2} = 0. \tag{9.25}$$

This is the equation of motion for a vibrating string. Generically it is a form of the the wave equation. We encountered the same equation, in a slightly different form, in Eq. (1.15) on page 12, as the equation that governs the motion of sound through air.

9.2.4 Solution of the equation

There is more than one solution to Eq. (9.25), but the primary one for our purposes is:

$$y(x,t) = A\sin\frac{\pi x}{L}\sin(2\pi f t), \tag{9.26}$$

[1] The small increase in the tension can have a musical effect: it makes the note go slightly sharp when the displacement of the string from side to side is large, for instance when a guitar string is plucked particularly vigorously. This slight change in pitch causes a twanging sound that is a characteristic element of some musical styles. See Section 10.3.3 for more details.

which is a vibration with frequency f. Note that this expression has $y = 0$ when $x = 0$ or $x = L$, i.e., at the ends of the string, as must be the case since the string is fixed and motionless at these two points.

To verify that (9.26) is a solution, we can substitute it into Eq. (9.25) and perform the derivatives, which gives

$$\begin{aligned} &-A\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi x}{L}\sin(2\pi f t) \\ &\qquad + A\frac{m}{LT}(2\pi f)^2 \sin\frac{\pi x}{L}\sin(2\pi f t) = 0. \end{aligned} \tag{9.27}$$

Cancelling several factors and rearranging, we find that we do indeed have a solution provided that

$$f = \frac{1}{2}\sqrt{\frac{T}{Lm}}. \tag{9.28}$$

Comparing with Eq. (9.12), we see that this is precisely Mersenne's law. In other words, a string fixed at both ends will vibrate at a frequency given by Mersenne's law. As we will see later, there are other solutions to Eq. (9.25) too, but all of them have higher frequencies of vibration and hence correspond to overtones of the sound. Equation (9.28) gives the fundamental frequency of the string.

A solution of the form (9.26) is a *standing wave*: it is an oscillatory wave-shaped motion, but it is not a traveling wave like a sound wave moving through air. It remains stationary, confined to the length of the string, with its peaks and valleys always in the same place.

Equation (9.26) also tells us that the waveform of the vibration is a sine wave and it tells us that the *spatial* shape of the vibrating string is also a sine wave. These observations will become important later when we look at the timbre of the notes produced by string instruments.

9.3 Other sources of vibration

Vibrating strings are not the only means by which musical instruments generate vibration. Instruments come in an enormous range of types and styles that generate vibration using many different kinds of vibrating elements.

Membranes and skins: Drums use flexible vibrating membranes—drum heads—to produce sound. A nice feature of a drum head is that it acts as its own soundboard. A drum head is large enough to produce a significant amount of sound on its own by pushing directly against the air next to it. On the other hand, the vibrations of drum heads are more complicated than those of strings, and in particular they do not normally have simple periodic vibrations, and hence they do not produce a clear musical note. This makes drums useful primarily as rhythmic instruments and not as melodic ones. We discuss the working of drums in Section 13.6.

Rigid metal sheets: Instruments like cymbals and gongs make use of vibrating sheets of metal to produce sound. These are similar in some respects to drum heads, but they have an intrinsic rigidity of their own and hence do not need to be stretched across a frame as a drum head is. Like a drum head they act as their own soundboard, having an area large enough to make a significant amount of sound without further help. Bells are related to cymbals, consisting again of a vibrating sheet, but one that has now been formed into a cup-like shape, often with a hammer or clapper inside to produce sound. We discuss bells and cymbals in Sections 13.4 and 13.7.

Bars of wood or metal: Solid bars of wood or metal can produce a sound when struck, and this principle is used in instruments like the xylophone and glockenspiel, which employ free-standing bars, typically resting on a frame or support of some kind, and in handheld percussion instruments like woodblocks and claves. Chimes, such as triangle and wind chimes, use free-hanging solid metal cylinders rather than bars, but are otherwise similar in their mechanics, while orchestral chimes (also called tubular bells) use hollow cylinders. A slightly different approach employs a bar or cylinder fastened at one end and free to vibrate at the other, as found for instance in the tuning fork, the kalimba or mbira, and the Rhodes piano. We discuss vibrating bars and cylinders in Sections 13.1, 13.3, and 14.5.2.

Air columns: After string instruments, the most common class of pitched (melodic) instruments are the wind and brass instruments. These are somewhat different from the others we have mentioned in that they produce their sound not through the vibration of a solid object, but through the vibration of a column of air. The interesting physics of this approach we discuss in detail in Chapter 11. Like drums and cymbals, wind instruments have no need of a soundboard, though for a different reason: since wind instruments work by making air vibrate, they directly produce sound themselves, without the need for any mechanism to turn vibration into sound.

Having seen the basic principles of musical instruments, we now turn to an examination of the specific workings of some of the most common and popular instruments, starting in the next chapter with the string instruments.

Chapter summary:

- Acoustic musical instruments—those that work without the benefit of electrical amplification—are all based on the same fundamental principle. Some part of the instrument is made to **vibrate**, and that vibration is translated into vibration of the air and hence sound. Examples including the vibrating strings of a string instrument and the vibrating drum head on a drum.
- In order for something to vibrate it must have three properties: an **equilibrium position**, a **restoring force**, and **inertia**. A simple example is the pendulum, although a pendulum is not usually useful as a musical instrument. A more complicated but also more useful example is a vibrating string. The frequency of vibration f of a vibrating object obeys

  $$f \propto \sqrt{\frac{F}{m}},$$

  where F measures the size of the restoring force and m measures the inertia in terms of the mass or density of the vibrating object.

- As an example, the frequency of vibration of a string is given in terms of its length L, diameter d, tension T, and density ρ by **Mersenne's law**:

  $$f = \frac{1}{Ld}\sqrt{\frac{T}{\pi\rho}}.$$

  The frequency of vibration of the string translates directly into the frequency of the sound produced by a string instrument, and hence we can use this formula to calculate what musical note such an instrument will produce.
- The motion of a string alone is not enough to produce a significant sound. The string is too small to move much air. To get around this issue, string instruments have a **soundboard**, a thin board, usually made of wood, that is attached to or in contact with the strings in some way, so that the vibration of the strings is transferred to the soundboard. It is the vibration of the soundboard that actually produces most of the musical sound.
- Other instruments produce vibration in a variety of ways. **Percussion instruments** use vibrating membranes (drums), metal sheets (cymbals and bells), and wooden or metal bars (xylophone and glockenspiel). Typically these instruments do not require a soundboard. The vibrating elements produce enough sound on their own.
- **Wind and brass instruments** take a different approach, with the vibrating element being a column of air. The vibration of the air itself constitutes sound, so wind instruments also do not need a soundboard—they generate sound directly.

Exercises

9.1 A pendulum oscillates back and forth once per second. If you took the same pendulum to a planet where the gravity was twice as strong as Earth's how fast would it then oscillate?

9.2 The vibrating part of a guitar string is 64.8 cm long and the string is made of solid steel with density 7900 kg/m^3 and diameter 0.33 mm. If its tension is 123 newtons what note does the string play?

9.3 The highest string on a violin is tuned to the note E5 and the vibrating part of the string is 32.5 cm long. The string is made of solid steel with a density of 7900 kg/m^3 and diameter 0.25 mm. What tension does it have?

9.4 A piano string that plays the note C4 is 65 cm long, 1 mm in diameter, and made of steel with density 7900 kg/m^3.

a) What is the tension on the string?
b) A piano has about 230 strings in total. If they all have about the same tension, what is the total tension force on the piano frame from all of the strings together?

9.5 Suppose that instead of being circular a steel instrument string is made square in cross-section, with width d along each side of the square. What would be the equivalent of Mersenne's law, Eq. (9.16), for the frequency of vibration of such a string?

9.6 The guzheng is a traditional Chinese string instrument with plucked steel strings about 0.5 mm in diameter, strung over a wooden frame. There is no fingerboard—each string plays its own note, as on a harp.

a) Given that the density of steel is $7900\,\mathrm{kg/m^3}$, what is the tension of a 60 cm long guzheng string that plays the note C4?
b) The guzheng plays a pentatonic scale and has a range spanning about four octaves. About how many strings does it have?
c) Hence what is the total tension on the frame of a guzheng?

9.7 A piano string is made of solid steel with density $7900\,\mathrm{kg/m^3}$, and has length 50 cm and diameter 1 mm.

a) What is its mass m in grams?
b) The string plays the note C5. How many times is it vibrating back and forth per second?
c) Suppose it moves back and forth by 2 mm. What is the total distance it moves per second? This is its average velocity v.
d) Kinetic energy is equal to $\frac{1}{2}mv^2$. About how much energy does the string have in joules?
e) Suppose the note lasts 10 seconds before it dies away. Approximately what is the rate at which the string loses energy, in joules per second?

9.8 Suppose a guitar plays the note C3.

a) What is the frequency of vibration of the string?
b) If the soundboard moves back and forth by 0.001 mm as it vibrates, what is the total distance it travels per second? This is its average velocity u.
c) Using Eq. (1.2) on page 5 estimate the sound pressure produced right next to the soundboard.

9.9 As we will see in later chapters of this book, there are equivalents of Mersenne's law, Eq. (9.16), for vibrating objects other than strings.

a) For a thin circular membrane like a drum head the equivalent equation is

$$f = \frac{0.7655}{L}\sqrt{\frac{T}{h\rho}},$$

where L is the diameter of the drum, h is the thickness of the membrane, ρ is its density, and T is the tension. If we double the diameter of the drum head, without changing anything else, how will the musical pitch of the note change? How about if we double the thickness?

b) For a solid bar like the bar of a xylophone, with length L and thickness h, the frequency is

$$f = \frac{3\sqrt{3}\pi h}{16L^2}\sqrt{\frac{E}{\rho}},$$

where ρ is again the density and E is a quantity called the Young's modulus, which measures the hardness of the material. If we double the length of such a bar, how will the musical pitch of the note change? How about if we double the thickness?