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Econ 609

Christian Gollier's "The Economics of Risk and Time"
Exercise 17

Meyer (1977) introduces the concept of stochastic dominance with respect to a function. Consider any increasing function u_1 and define the associated function set $\gamma(u_1) = \left\{ u \mid -\frac{u''(z)}{u'(z)} \geq -\frac{u_1''(z)}{u_1'(z)} \right\}$. $\gamma(u_1)$ is the set of functions that are more risk averse than u_1.

(a) Show that $\gamma(u_1)$ is a convex set.

Let, u_A be more risk averse than $u_1 \Rightarrow -\frac{u_A''(z)}{u_A'(z)} \geq -\frac{u_1''(z)}{u_1'(z)}$ ....(1)

and, u_B be more risk averse than $u_1 \Rightarrow -\frac{u_B''(z)}{u_B'(z)} \geq -\frac{u_1''(z)}{u_1'(z)}$ ....(2)

and, $u_C = \lambda u_A + (1-\lambda) u_B$ ....(3)

We have to show that, $-\frac{u_C''(z)}{u_C'(z)} \geq -\frac{u_1''(z)}{u_1'(z)}$

$$-\frac{u_C''(z)}{u_C''(z)} = \frac{-\lambda u_A''(z) - (1-\lambda)u_B''(z)}{\lambda u_A'(z) + (1-\lambda)u_B'(z)} = -\lambda \frac{u_A''(z)}{u_c'(z)} - (1-\lambda)\frac{u_B''(z)}{u_C'(z)}$$

$$\geq -\lambda \frac{u_A'(z)u_1''(z)}{u_C'(z)u_1'(z)} - (1-\lambda)\frac{u_B'(z)u_1''(z)}{u_C'(z)u_1'(z)}$$

$$\geq -\lambda \frac{u_1''(z)}{u_1'(z)} - (1-\lambda)\frac{u_1''(z)}{u_1'(z)} \qquad [\therefore \lambda u_A \leq u_C]$$

$$\geq -\frac{u_1''(z)}{u_1'(z)}$$

$\therefore \gamma(u_1)$ is a convex set.

(b) Find a simple basis for $\gamma(u_1)$, and derive the integral condition for the corresponding stochastic dominance order.

A basis for the set of utility functions that are more concave than u_1 is,

$$b_\theta(z) = \min(u_1(z), u_1(\theta)) \ \ \forall z, \theta \text{ and } \theta \in [\mathrm{a}, \mathrm{b}]$$

Let, F_1 and F_2 denote the cumulative distribution function for $\tilde{z}_1$ and $\tilde{z}_2$. The integral condition for the corresponding stochastic dominance order,

$$\int_a^b \min(u_1(z), u_1(\theta)) dF_1(z) \leq \int_a^b \min(u_1(z), u_1(\theta)) dF_2(z))$$

$$\Rightarrow \int_a^\theta u_1(z) dF_1(z) + \int_\theta^b u_1(\theta) dF_1(z) \leq \int_a^\theta u_1(z) dF_2(z) + \int_\theta^b u_1(\theta) dF_2(z)$$

$$\Rightarrow \int_a^\theta u_1(z) dF_1(z) + u_1(\theta) \int_\theta^b dF_1(z) \leq \int_a^\theta u_1(z) dF_2(z) + u_1(\theta) \int_\theta^b \ dF_2(z)$$

$$\Rightarrow \int_a^\theta u_1(z) dF_1(z) + u_1(\theta) F_1(z) |_\theta^b \leq \int_a^\theta u_1(z) dF_2(z) + u_1(\theta) F_2(z) |_\theta^b$$

$$\Rightarrow \int_a^\theta u_1(z) dF_1(z) + u_1(\theta)[F_1(b) - F_1(\theta)] \leq \int_a^\theta u_1(z) dF_2(z) + u_1(\theta)[F_2(b) - F_1(\theta)]$$

$$\Rightarrow \int_a^\theta [u_1(z) dF_1(z) + u_1(\theta)[1 - F_1(\theta)] \leq \int_a^\theta [u_1(z) dF_2(z) + u_1(\theta)[1 - F_2(\theta)]$$

$$\Rightarrow \int_a^\theta u_1(z) d[F_1(z) - F_2(z)] - u_1(\theta)[F_1(\theta) - F_2(\theta)] \leq 0$$

Integrating by parts,

$$\Rightarrow u_1(\theta)[F_1(\theta) - F_2(\theta)] - \int_a^\theta [F_1(z) - F_2(z)] u_1'(z) dz - u_1(\theta)[F_1(\theta) - F_2(\theta)] \leq 0$$

$$\Rightarrow \int_a^\theta F_1(z) u_1'(z) dz \geq \int_a^\theta F_2(z) u_1'(z) dz$$