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Practices and Problems in Teaching Authentic Mathematics

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What is the synthesis of effective and responsible teaching of mathematics? If one were interested only in deciding whether the teaching of this subject is effective, then the question to be answered would be whether what is taught is what is learned. But if one is also interested in whether the teaching of mathematics is responsible, then the question of whether what is being taught and learned is authentic mathematics becomes pertinent. Teachers may teach and learners may do well on tests or be observed to have "target" skills without the learning having much to do with real mathematics. In order to judge whether mathematics is being taught and learned responsibly, one must look at whether the skills and knowledge being acquired contribute to students' ability to actually *do* mathematics. And one must consider not only outcomes but pedagogical methods as well, for students learn what it means to know something from the interactions that they have with the subject and the teacher.

In this chapter, I examine what can be learned about responsible pedagogical methods for teaching mathematics from looking at mathematical practice. I come at the question of how to make mathematics teaching both effective and responsible from the perspective of a fifth-grade teacher in a public school in a diverse community. My goal as a teacher is to have my students learn to do authentic mathematics. From this perspective, being effective and responsible means constructing curriculum and instruction in ways that make it possible for my students to participate in activities that are genuinely mathematical and to learn from those activities.

Investigating Effective and Responsible Teaching in a Lesson

In order to give the reader a sense of what it might be like to be a teacher who wants to do the right thing in the classroom, I want to begin by telling you about some assertions that were made by my fifth-grade students during a discussion of a problem that the class was working on. I played a role in this discussion as the teacher, but here I want to recount primarily what the students said and leave open for now the question of what an effective and responsible teacher of mathematics might do in response. Later in the chapter, after some examination of what it might mean to know and do mathematics in practice, I will return to the discussion in which these assertions were made and analyze the role that I played as the teacher.

In an introductory lesson on functions, I had asked my fifth-grade class to figure out how one might characterize the relationship between the x and y values in this chart:

x	y
8	4
4	2
2	1
0	0

Ellie, a fifth-grade student, made these observations: "Um, well, there were a whole bunch of . . . a whole bunch of rules you could use, use, um, divided by two . . . And you could do, um, minus one-half."* I followed Ellie's final assertion with a question: "And eight minus a half is?" to which she answered, "Four."

At this point, a gasp arose from the class, and several other students made a bid to enter the conversation. They either agreed or disagreed with Ellie.

Karim: Well, see, I agree with Ellie because you can have eight minus one-half and that's the same as eight divided by two or eight minus four.

Charlotte: I think eight minus one-half is seven and a half because one-half's a fraction and it's half of one whole and so when you subtract you aren't even subtracting one whole number so you can't get even a smaller number that's more than one whole [away from eight].

Suran: I think, um, I would agree with Ellie if she had said eight minus one-half of eight, because half of eight would be four because four plus four would be eight.

Sam: Um, I agree with Charlotte and, um, I don't agree with Ellie. Because, um, like one-half is not even one, so if, so when Ellie said that people would like, um, a really good mathematician would probably, like, would

*The student-teacher conversations in this chapter are excerpted from a transcript of a discussion that occurred in my fifth grade mathematics class. See the appendix of this chapter for the complete transcript.

probably write seven and a half, not four because they would have to know what the one half was meaning, half of a number to, um, to understand it.

Lev: I think, um, I would agree with Ellie if she had said eight minus one-half of eight because half of eight would be four because four plus four would be eight.

Tyrone: I agree with Charlotte and Sam and I disagree with Ellie and like I think Ellie meant, like, because four is half of eight, like one-half would be a half, but, and I agree with Lev when he said if she meant one-half, uh, equals, wait, eight, equals half of eight and I agree with Sam, and, uh, Charlotte because, um if, if, uh, four is not, uh, eight equals half of four is not right because it's seven and a half, because half of like, eight is the whole and um, one number away from that is seven and plus a half would be seven and half.

Suran: I would agree with Ellie if she had added something else to her explanation, if she said one half of the amount that you have to divide by two.

Ellie: Um, well, I agree with Suran and, um, when Charlotte said, um, she thought that, um, it should be one-half of eight, um, instead of just plain one-half, I don't agree with her because not all of them are eight. Not all of the problems are eight.

Consider what mathematics these students seem to know or not know and what the teacher's role might be in leading them toward a more refined understanding of mathematics. Should the relationship be called, as Lev asserts, "eight minus a half of *eight*"? Or should it be what Ellie says: "eight minus *just plain one half*"? What difference does it make? What difference does it make to judgments of whether the teaching that occurs here is effective and responsible if the teacher legitimates one expression and not the other—or neither or both? What do these students know—and what do they need to learn—about mathematics, or, more particularly, about functions or fractions or subtraction, or about the conventions of mathematical language and symbols? How should the teacher teach them what they need to learn? How should the teacher respond to Sam's assertion that "good mathematicians" would say that eight minus a half was seven and a half, while he himself asserts that "it is important to know what the 'half' is meaning"? Or to Charlotte's certainty about the idea that "one-half's a fraction and it's half of one whole"? Or to Karim's assertion that "eight minus one-half is the same as . . . eight minus four"? It is a simple matter to say that a teacher of mathematics should teach these students in a way that is true to the discipline of mathematics. But what does that mean in practice, when the practice occurs in a contemporary schoolroom? What implications does the goal of *doing mathematics* in school have for designing the kinds of ethical and intellectual interactions that should occur between teacher and students?

Two Kinds of Practices

In constructing a pedagogy that takes seriously both the nature of schoolwork and the nature of work in mathematics, one moves back and forth between two kinds of practices: the practice of teaching in school and the practice of doing mathematics.

Teaching involves the teacher in communicating with learners about something that the teacher knows and the students are supposed to be learning. Doing mathematics involves both teacher and learner in thinking about quantitative relationships and making and evaluating mathematical assertions. What the teacher knows could be constructed as the "findings and conclusions" of a particular domain or as a familiarity with the kinds of activities that are considered legitimate generators of findings and conclusions in that domain (or both). The most familiar way to communicate findings and conclusions in classrooms is for teachers to tell them to students or to tell students to read books in which they are written down. In order to communicate with learners about what is entailed in doing an activity in a domain, teachers can engage them in doing the activity with them, they can show learners aspects of the activity and talk about it with them, and/or they can prepare a synoptic description of what they believe people need to know to do the activity, teach it to them, and then guide their attempts to do the activity themselves (see Cohen, forthcoming). Each of these approaches to communicating teachers' knowledge to students constitutes a pedagogy or set of activities and assumptions about how the activities of teaching produce some desired learning. Which pedagogy we choose expresses our assumptions about what knowledge is, how knowledge is represented, and how usable knowledge is acquired.

Teaching practice is related to the practice of doing mathematics (or any other intellectual activity) through the question of how knowledge is justified (Tymoscko, 1986). Teachers and students make one assertion after another during lessons. The question of what makes an assertion true or understood or accepted for use is central to both pedagogy and mathematics. In conventional teaching, what makes an assertion true is the teacher's authority. But if we think about what it means to know something in mathematics, we would not accept simply repeating what an authority said or what was written in a book as "knowledge," even of facts and principles.

This way of thinking does not give the teacher clear prescriptions for practice, however. In my fifth-grade class, when one of the students asserted that "eight minus a half" could be "four," I could have resolved the disagreement in the class by telling students that $8 - \frac{1}{2} = 7\frac{1}{2}$ is correct, according to mathematical convention. And so Ellie's phrase "minus a half" could not legitimately be used to describe the relationship between the *x*'s and the *y*'s. But because I wanted my students to learn something about why mathematics needs linguistic conventions in the first place, to make my teaching a responsible representation of authentic mathematics, I chose not to intervene in that way. Instead, I acted as a guide while the students argued among themselves about the correct linguistic formulation for the relationship. In order for authentic mathematics to be taught and learned here, the decision about whether it should be said one way or another should have a justification that was situated in mathematical conversation, not wholly dependent on the authority of the teacher. Because I wanted my students to practice mathematics in the classroom, I wanted to make it possible for them to wander around in the territory of fractions and ratios and functions and explore the connections among these ideas. But I was torn—like Sam, I also thought that a "good mathematician" would never say "eight minus a half is four," yet I wanted the rest of the class to know that Ellie had a legitimate point: the numbers in the *y* column could be obtained by taking away a half from each of the numbers in the *x* column. And that is what functions are all about. Being effective and responsible here does not have a simple meaning. There are multiple and conflicting possibilities for how a teacher might act.

One way to understand these kinds of dilemmas is to relate them to a continuum of justification (Figure 19.1), since we are talking about both individual and social constructions of knowledge. When we talk about "teach-

Figure 19.1. Continuum of Justification.

Convincing yourself	Convincing the people you work with regularly	Convincing strangers
Making sense of what you are told	Arguing about what is true with people who share your language and assumptions	Making a formal deductive argument
Inventing conjectures	Establishing the plausibility of an argument in a community of discourse	Presenting empirical evidence according to accepted procedures
Constructing personally meaningful links between elements	Constructing links between elements that can be communicated to others	Constructing a synoptic representation of the structure of a domain that stands for the domain itself
Insight, intuition, "seeing"		Writing a proof

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ing for understanding," we are talking about only the individual, private end of the continuum. But if an individual "knows" something that does not match what others (either in or out of the classroom) believe to be true, and knowledge becomes part of the public discourse as it does in the classroom, a social confrontation occurs that must be resolved one way or another. When we talk about "effective teaching," we are usually talking about only the most public end of the continuum, in which assertions can or cannot be mapped directly onto disciplinary conventions, and controversy is resolved by the teacher's interpretations of those conventions. But in the classroom, except under unusual circumstances, students talk to one another about what they think is true. This discourse can be truncated by the assertion of the teacher's authority, or it can be a genuine intellectual argument in which the community collaborates to establish its shared assumptions—an argument in which the teacher plays a role as a clarifier and supplier of information rather than a judge.

The dilemmas that I faced as a teacher in this situation are connected to an epistemological controversy within mathematics that has been expressed in various ways throughout history and has sometimes spilled over into the realm of mathematical pedagogy. An analysis of how knowledge has been considered and communicated within mathematics thus has the potential to enlighten the way in which one might think about effective and responsible mathematics teaching. Looking at mathematical practice will not provide us with right answers to questions about what constitutes good teaching, but it will help us to better understand the nature of pedagogical practice and its problems.

Knowing in and of Mathematics

In a recent exposition on the nature of knowledge in mathematics, Philip Kitcher (1984) sought to examine the links between how knowledge valued and used in the discipline grows and changes and how practice in the discipline proceeds. He identified five characteristics of mathematical practice that provide a useful framework both for thinking about how doing and knowing are related in the discipline and for generating some hypotheses about what learning authentic mathematics might look like in school. He argues that mathematical practice is distinguished from other practices by (1) the questions that are understood as meaningful and legitimate; (2) the methods of reasoning that are accepted as supporting conclusions; (3) the goals and structures of mathematical knowledge; (4) the language that is meaningful to practitioners; and (5) statements of findings and conclusions that are accepted and established. Several researchers and reformers in mathematics education (for example, Greeno, 1991; Romberg, 1983; Bell, 1979) have complained that school mathematics seems to attend only to the last of these five characteristics. These contemporary critiques of mathematics teaching echo earlier concerns within mathematics about how findings and conclusions in mathematics are to be learned by newcomers to the field.

There is a famous project in mathematical theory building associated with the name Bourbaki. Bourbaki is not a person but the pseudonym under which a group of French mathematicians wrote about mathematics (Cartan, 1980). In 1934, this group convened and decided to write a new university textbook that would capture the substantial changes in their field around the turn of the century, now known as "modern mathematics." They were concerned that the changes in mathematical practice that had caused what they were doing to be called "modern"—changes in how one might reason about mathematical questions and what counted as evidence in a mathematical argument—were not being reflected in the material taught to university students. In the process of producing their Éléments de mathématique, the Bourbaki group came to define what was meant by "axiomatics" and reified the nature of mathematical structures by formalizing the process of establishing abstract mathematical certainty. They came to be known as "formalists" because they identified the knowledge of mathematics with knowledge of its formal structures. The structures of logic made it possible to relate mathematical entities in formulas and to transform those formulas by following deductive rules. Thus, the process whereby new truths are generated and their legitimacy secured could be described in terms of an intellectually mechanical process; this process would decrease reliance on "insight" or "genius," thereby (the Bourbaki group assumed) making practice in the discipline available to a wider range of participants.

The Bourbaki group's purpose was pedagogical. In their project, so-called modern mathematics became a body of knowledge represented in books rather than a social institution with human beings involved cooperatively in the production, organization, and changing of knowledge. In the words of a member of the group, Henri Cartan, "While the members of Bourbaki considered it their duty to elaborate all of mathematics according to a new approach, they did this with the hope and expectation of putting into the hands of future mathematicians an instrument which would ease their work and enable them to make further advancements" (quoted in Steiner, 1988, p. 9). Their fundamental intention was communication: in order to teach others about mathematics, they assumed, it was appropriate to collect everything that was known and organize it into a coherent whole, making the connections among pieces of that whole logical and coherent. And why? Because they believed that the professors who were to teach modern mathematics to future mathematicians were not as gifted as the creators of these new ideas; that is, not everyone who wanted to learn about the findings and conclusions of modern mathematics could appreciate the intuitive connections among different ideas that led to new discoveries. As Jean Alexandre Dieudonné (another Bourbaki group member) commented,

> Communication between mathematicians by means of a common language *must* be maintained . . . and the transmission of knowledge cannot be left exclusively to geniuses. In most cases it will be entrusted to professors. . . . As most of them will not

> be gifted with the exceptional "intuition" of the creators, the only way they can arrive at a reasonably good understanding of mathematics and pass it on to their students will be through a careful presentation of the material, in which definitions, hypotheses, and arguments are precise enough to avoid any misunderstanding, and possible fallacies and pitfalls are pointed out whenever the need arises. . . . It is this kind of expository writing that has been, I think, the goal of those mathematicians [called] "formalists" from Dedekind and Hilbert to Bourbaki and his successors [Steiner, 1988, p. 10].

What the Bourbaki authors were worried about was something like the knowledge base for teaching and the relationship between mathematical practice and mathematical communication and its relationship to pedagogy. These writers made a distinction between mathematics as it is known by practicing mathematicians (the ones with genius and intuitions) and mathematics as it is known by mathematics teachers (the ones who could understand what had been produced by those intuitions and communicate it to others). This distinction is rooted in the difficulties of communicating about a practice to those who are not yet or never will be a part of it, and it remains a problem for all of us as we try to understand the nature of effective and responsible teaching.

Within mathematical philosophy, there is much current writing that is critical of the images of mathematical knowledge and communication that the Bourbaki group perpetrated. But the tension between the dynamics of practice and the need to codify knowledge so that it can be passed on to novices did not begin with reactions to the Bourbaki group. Arguments about whether one should engage learners in messy and creative disciplinary activities as a method of teaching them about the discipline are at least as old as the foundations of university education in the sixteenth century. At this time, instruction began to move away from having novices engage in disciplinary discourse as a method of education and toward lecturers preparing and publishing synoptic representations of knowledge in their fields and delivering them to learners (Ong, 1958). Throughout history, questions about pedagogy—that is, questions about how to communicate what is known to the uninitiated—have been deeply tied to questions about the relationship between doing mathematics and the nature of the knowledge that results from the doing. Within the discipline, there have been many variations on how this relationship was conceived and sometimes raging controversies over it (Koerner, 1960; Davis, 1988; Kline, 1985; Tymoczko, 1986).

I want to review a bit of that history here because it suggests the many ways in which we might think about teaching and learning and knowing mathematics in classrooms. Until recently, the purpose of most philosophical scholarship focusing on mathematics was to conceptualize the nature of mathematical knowledge and to examine the characteristics of the formal language in which mathematical truths are asserted. The Platonists, who include not only the ancient Greeks but such contemporary and distinguished mathematicians as G. H. Hardy and Paul Erdos, believe that there is a definite, supernatural reality of mathematical objects and that the relationships that pertain among these objects determine the truth or falsity of any mathematical proposition. The practicing mathematician recognizes these relationships in the act of "mathematical intuition." These intuitions put knowers in touch with the world of mathematical objects and suggest the axioms, or basic assumptions, on which their logical arguments are based. Descartes, among others, broke with the idea that mathematical knowledge was derived from a kind of congruence between the mathematician's mind and "the mind of God" and posited that truth was obtained by correct reasoning alone. The Cartesians see mathematical truth as objective and believe that the individual mathematician knows it through the power of his or her own logic. In this view, axioms, or basic truths, are derived from fundamental principles of logic, and they can be known by anyone who has the capacity to reason logically.

Unfortunately, attempts to set down these principles by mathematicians such as Bertrand Russell, Gottlieb Frege, and Georg Cantor turned up some basic and irresolvable inconsistencies. So another way of thinking about what it means to know mathematics developed, in which it was asserted that mathematicians arbitrarily choose which first principles they will use and follow the rules of logic from there. With this development, what it meant to know mathematics became a mix of the individual and the social: knowledge was acquired by individuals through a process of reasoning from the axioms, but the axioms themselves were a set of assumptions that were agreed on by a given discourse community. This philosophy was too arbitrary for a group of Dutch mathematicians who practiced early in the twentieth century and called themselves "intuitionists": they asserted that the axioms were constructed by mathematicians out of intuitions derived from intellectual experience. They would not assert, as the Platonists did, that mathematical reality was there to be discovered; instead, they suggested that it is created by the act of thinking mathematically. (The intuitionists are considered a "fringe" group within mathematics, but I am continually struck with the parallels between their epistemology and some of the tenets of contemporary cognitive psychology.)

All of these ways of thinking about mathematical knowing focus on the powers of the mind of the individual mathematician. They all portray mathematical knowledge as infallible and atemporal, to the extent that the mathematician is able to think correctly about the mathematical reality. They focus more on what kind of knowledge mathematics is once it is known and on the rules of the language of mathematical representations than on the processes by which that knowledge came to be acquired and that language came to have meaning in the first place. The theme of relating intuition and reasoning that runs through all of these philosophies has undoubtedly influenced the development of the popular notion that "only geniuses can do mathematics," that doing it is something of a superhuman, antisocial endeavor. And that notion certainly influences how mathematics gets taught in school and how learners think about themselves in relation to the activity of doing mathematics (see Schoenfeld, 1985). When we think about connections between the discipline and the classroom, we need to think about these sorts of potential influences as well as the ones that currently seem more appealing.

More recently, and partly in reaction to the kind of epistemology perpetrated by the Bourbaki group, philosophers of mathematics have turned away from examining the conceptual and linguistic foundations of mathematics and have begun to try to understand more about the human social activities that are involved in producing and refining mathematical knowledge. Thus, within analyses of disciplinary knowledge, there are more and more analyses of both contemporary and historical mathematical practice. Contemporary criticisms of the equation of mathematical knowledge with mathematical formalisms focus on what of mathematics is lost when the discipline is presented in terms of finished, logical structures that can be clearly, if abstractly, communicated rather than in terms of dynamic processes for discovering and arriving at mathematical assertions. They contrast the messiness of the doing of mathematics with the polished, structured character of the form in which the results of the doing are communicated and complain that those who are not directly engaged in mathematical practice cannot appreciate what is involved in arriving at the formal structures.

Those who would challenge the formalist view of mathematical truth and the Bourbaki writers' "deductivist" approach to mathematical communication assert that the practice of mathematics is ill structured; that is, that the search for truth in this discipline is as much an attempt to make fallible and tentative sense out of a tangled web as it is in other human scholarship. Steiner (1988) calls the counterpoint to the deductivist approach "analytic-genetic" and goes back to Diderot's *Encyclopédie* for a definition: "Analysis consists in returning to the origin of our ideas, developing their order, decomposing and composing them in a variety of ways, comparing them from all points of view and making apparent their mutual interrelations. . . . In searching for truth, it does not make use of general theorems, rather it operates like a kind of 'calculus' by decomposing and composing knowledge and comparing this with intended discoveries" (translation of Steiner, 1988, p. 10).

Steiner has also uncovered French mathematics textbooks from the eighteenth century that advocate the approach to mathematical knowledge embraced in the *Encyclopédie.* Alexis Clairaut, for example, rejects the classical "theorem-proof" presentation that we are all familiar with from studying plane geometry and asserts, in the preface to his text on geometry,

> If the first originators of mathematics presented their discoveries by using the 'theorem-proof' pattern, then doubtlessly they did this in order to give their work an excellent shape or to avoid

> the hardship of reproducing the train of thought they followed in their own investigations. Be that as it may, to me it looked much more appropriate to keep my readers continuously involved with solving problems, i.e., with searching for means to apply some operation or discover some unknown truth by determining a relation between entities being given and those unknown and to be found. In this way, with every step they take, beginners learn to know the motive of the inventor; and thereby they can more easily acquire the essence of discovery [translation in Steiner, 1988, p. 12].

Clairaut's emphasis here is on the genesis of knowledge and the flexible and dynamic process of linking ideas that supports it. In using ideas, the mathematician does not structure them in the same formal way that they would be structured for communication. In the process of the mathematician's learning something new, process and content are inextricably linked. From a pedagogical point of view, it is notable that Clairaut recognizes the potential "hardship" involved in following another's train of thought, even as he advocates that teaching and learning should attend to that process. (The metaphor of a "train" that Clairaut [or perhaps Steiner in translating Clairaut] uses here seems somewhat inconsistent with Clairaut's purposes; that is, the inventor's thoughts might not be as linearly organized as a "train" but might be organized more like a web or a traffic jam at the Place de la Concorde.)

Along the same lines, Imre Lakatos more recently criticized the form that was assumed to communicate mathematical reasoning to students in textbooks. I quote at some length here from his diatribe because of the information he gives us about the alternative to the synoptic presentation of results:

> Euclidean methodology has developed a certain obligatory style of presentation. I shall refer to this as "deductivist style." This style starts with a painstakingly stated list of *axioms, lemmas,* and/or *definitions.* The axioms and definitions frequently look artificial and mystifyingly complicated. One is never told how these complications arose. The list of axioms and definitions is followed by carefully worded *theorems.* These are loaded with heavy-going conditions; it seems impossible that anyone should ever have guessed them. The theorem is followed by the *proof.*
>
> The student of mathematics is obliged, according to the Euclidean ritual, to attend to this conjuring act without asking questions either about the background or about how this sleight-of-hand is performed. If the student by chance discovers that some of the unseemly definitions are proof generated, if he simply wonders how these definitions, lemmas, and the theorem can possibly precede the proof, the conjurer will ostracize him for this display of mathematical immaturity.

> Deductivist style hides the struggle, hides the adventure. The whole story vanishes, the successive tentative formulations of the theorem in the course of the proof-generated definitions of their "proof ancestors," presents them out of the blue, in an artificial and authoritarian way [Lakatos, 1976, pp. 142, 144].

When Lakatos speaks of axioms and definitions being "proof generated," what he means is that the mathematician figures out what he or she is talking about in the process of trying to talk about it, not beforehand by some magical intuition. Setting out to prove something, the practitioner sees that the original terms of the argument were unclear and usually even changes what it was that was being asserted in the first place. The activity of developing a proof is not the straightforward series of logical steps that are portrayed to support assertions in textbooks but a "zigzag path" between conjectures and refutations. And the zigzag has much to do with trying to create a plausible argument and communicate it at the same time. The distinguished contemporary mathematician Henry Pollack talks about the importance in his own education of being exposed to what he calls "cross-country mathematics"; in contrast to a well-marked path, the cross-country terrain is jagged and uncertain. Pollack said of his teacher, Ed Begle:

> As a student, I had a very interesting time watching him struggle, inventing proofs and trying to think about the right way to do it. I learned a lot more mathematics that way than I might have if it had been a perfectly polished lecture and I think already at that time I developed my feeling that I like cross-country mathematics. Mathematics, as we teach it, is too often like walking on a path that is carefully laid out through the woods; it never comes up against any cliffs or thickets; it is all nice and easy [quoted in Albers and Alexanderson, 1985, p. 231].

In deductive syntheses of mathematical discoveries, what is learned from practice is separated from learning *about* practice. The syntheses also exclude disciplinary conversation from the epistemological picture. Recent studies (Polya, 1954; Davis and Hersch, 1981; Wang, 1986; Hersch, 1985; Thom, 1985) emphasize the fact that when a mathematician makes an assertion that is assumed to be plausible, he or she is trying to convince some audience that it is plausible. What it takes to do that, in any particular instance of mathematical practice, is not the functional equivalent of a formal deductive proof—such a proof would put a stop to the conversation and impede rather than further the process of discovery.

School Learners Doing Mathematics: What Is Effective and Responsible Teaching?

If teaching means that the teacher has some knowledge and students are supposed to acquire some knowledge that they did not previously have, one way to construe pedagogical practice is the way the Bourbaki writers (and many other curriculum developers) have done: as the logical presentation of well-formulated ideas. But as Lakatos complained, although such formal presentations are clearer and easier to communicate than the adventure of practice, they hide some essential aspect of what needs to be appreciated by learners: "What kind of knowledge is it that I am getting here, anyway?"

What I have taken into my fifth-grade classroom from these arguments in mathematical epistemology is the idea that the questions of where mathematical knowledge comes from and what makes it true ought to be an explicit part of the agenda. This means that we do not proceed as if whatever the teacher says, or whatever is in the book, is what is assumed to be true. It also means that lessons must be structured to pursue the mathematical questions that have meaning for students in the context of the problems that they are trying to solve. And this means that lessons are more like messy conversations than like synoptic logical presentations of conclusions.

I began the discussion of the table of relationships referred to at the beginning of this chapter with a question to the class. I said that the relationship in this table was a function and that it "did something" to the values on the left to obtain the values on the right, and I asked, "What did it do to the numbers in every case?" Ellie was one of the first to raise her hand, and she made an assertion: "Um, well, there are a whole bunch of rules you could use." After she stated one of those rules, "divided by two," I tested it out to see whether it indeed related each of the ordered pairs. I did not explicitly judge Ellie to be correct or incorrect, but I modeled a process whereby the truth of such an assertion would be assessed within mathematics. No one in the class challenged either me or Ellie at that point. Then Ellie asserted a second rule that could define the relationship among the same set of ordered pairs: "You could do minus one half." When she said that, several students gasped and began bidding for attention. Again, I did not judge Ellie's assertion to be correct or incorrect but began to test it against the ordered pairs in the exercise. Using her own language, I asked, "What would eight minus one half be?" When she answered "four," a gasp again went up from the class, and students' bids for the right to speak became even more aggressive.

In this situation, I took it as my responsibility to protect Ellie's right to practice mathematics by monitoring the discourse so that she would have the opportunity to explain her thinking and justify her assertion. Before calling on any of the students who were eager to argue with Ellie, I set the terms of the conversation: you can express an idea that is different from Ellie's, but you also need to make an attempt to take her position seriously. In mathematics, the legitimacy of an assertion cannot be judged without considering the assumptions and the reasoning that are supplied to justify the assertion. In the course of trying to prove that "eight minus one half is four" or the counterassertion that "eight minus one half is seven and one-half," the students and I together became clearer about the assumptions and definitions that underlay our assertions. My role was to participate in the conversation, raising questions when the terms were not clear and making it safe for students to raise questions about assertions made by their peers.

Ellie's assertion that the rule could be either "divide by two" or "minus one half" might be thought of as the result of a mathematical intuition. She "saw" both relationships in the set of ordered pairs that had been given in the exercise. In more formal terms, we might say that Ellie made a *conjecture* that each of two different mathematical operations would have the same effect on a given set of independent variables. Conjecturing about such relationships is at the heart of mathematical practice. Once a conjecture is made, the practitioner sets out to prove it and in doing so becomes clearer about the assumptions that led to the conjecture in the first place. The precision of mathematical language develops out of the process of seeking clearer and clearer assertions for which deductive proofs can be produced. Here the teacher's role is to legitimate the process, to accept mathematical intuitions, half-formed as they may be, as an essential part of the lesson.

Ellie and her classmates were embarking on the adventure of "cross-country mathematics" about which Pollack, Lakatos, and Polya write so eloquently. More particularly, they were doing what Polya, in his exposition on *Patterns of Plausible Inference,* calls "Examining a Possible Ground" to ascertain the plausibility of a proposition. As he begins this section of his book on the practice of generating knowledge in mathematics, Polya quotes Descartes: *"When we have intuitively understood some simple propositions . . . it is useful to go through them with a continuous, uninterrupted motion of thought, to meditate upon their mutual relations, and to conceive distinctly several of them, as many as possible, simultaneously. In this manner, our knowledge will grow more certain, and the capacity of the mind will notably increase"* (Polya, 1954, vol. 2, p. 18; italics in the original).

The process that Polya describes here is what Steiner (1988), following Jean Le Rond d'Alembert, calls "analysis": Ellie and the rest of the class were composing and decomposing ideas—about subtraction and fractions and functions—and making apparent their mutual interrelationships. For several turns in the discussion after Ellie's conjecture (and her assertion that it implied "eight minus one half is four"), the question of concern was whether "eight minus one half" should be "four" or "seven and one-half." The disposition of these assertions would determine whether Ellie's conjecture that "divide by two" and "minus one-half" are equivalent function rules should stand as true or be judged false. Judgments about the effectiveness of this kind of teaching, to use Polya's terms, need to be related to whether students have the opportunity to go through their mathematical ideas in a "continuous uninterrupted motion of thought" (Polya, 1954, p. 18). If the questions that are important to them in the discussion are questions that would also be considered mathematically important, they could be said to be engaged in mathematical practice, as defined by Kitcher (1984) and others.

Part of the problem in the discussion had to do with language; Ellie's conjecture was not stated very clearly, and so much of the talk was an attempt to say what she might have been meaning and to formulate it in more explicit terms. Lakatos (1976) gives a great deal of attention to this aspect of mathematical practice, as does Kitcher (1984). Both are concerned about the way in which mathematical terms come to have meaning in a discourse and how the resultant meaning affects the community's judgment about the verity of the proposition.

The underlying issue of concern to my students was how to interpret the meaning of "minus one-half": did "one-half" as it was being used here mean "half of one," or did it mean "half of the original number, whatever that number might be"? Several of the students who spoke said that you could look at it either way. But their acceptance of either point of view was not simple relativism, nor was it merely a social routine to avoid embarrassing a peer. They all spoke about *the conditions under which* one or the other assertion could be considered true. In my contribution to the discussion, I supported this method of reasoning. In response to Sam's challenge to Ellie, for example, I said, "You know when Charlotte was talking she said that she thought one-half meant half of a whole. And it sounds like that's the way you are interpreting it. But Ellie might be interpreting one-half to mean something else." I pointed out that the question of how we define "the unit" is important when we are talking about fractions and said, "We have to have some kind of agreement here if it's a fraction of eight or if it's a fraction of a whole"; I said that "it would be important to clarify" which of these interpretations we were using when we judged Ellie's original conjecture to be true or false. Although I did not strictly impose it on the class, I introduced the idea that it was convention to interpret the symbol $\frac{1}{2}$ as "half of one whole" in situations "when we just talk about numbers and we don't associate them with any objects or groups of objects." I also reformulated what Ellie was asserting in more precise mathematical language and tested out with her whether what I said was equivalent to the assertion that she had been trying to make. By the end of the lesson, we had collaboratively constructed a conjecture that she could live with and that other members of the class agreed was true. Ellie and the other members of the discussion came to address the importance of distinguishing between how things work in the domain of functions—where, as Ellie reminds us, "they are not all eight"—and the domain of arithmetic, where relationships and procedures determine specific ordered pairs (that is, once *eight* is assumed as the input, the operation determines whether the output will be seven and a half or four).

Problems of Doing Authentic Mathematics in School

If we are willing to think of this as "authentic mathematics," what problems does doing it in school raise for the teacher? And what kinds of pedagogical practices might be invented to support this kind of activity in school? I only briefly mention these questions here as markers for work in progress (see also Ball, 1990).

One problem is how we think about the relationship between individual understanding and the public justification of knowledge. Within mathematics and in the classroom, it probably makes sense to think in terms of a continuum and to consider every act of "knowing" as occurring somewhere on this continuum. The activities that I am calling "authentic mathematics" occur in a discourse community, but it is a community made up of individual learners who will go their separate ways with whatever knowledge they have acquired. Teaching and learning need to take account both of what is accomplished by individuals and what is understood to be "true" within the classroom discourse.

A second problem has to do with communication. Classroom discourse in "authentic mathematics" has to bounce back and forth between being authentic (that is, meaningful and important) to the immediate participants and being authentic in its reflection of a wider mathematical culture. The teacher needs to live in both worlds, in a sense belonging to neither but being an ambassador from one to the other.

A third problem has to do with establishing a culture of inquiry. This endeavor, too, is paradoxical, because school is supposed to be about learning to be a competent adult in our society. But as things now stand, not very many "competent" adults would appreciate the kind of mathematical discourse that occurs in my classroom, let alone be able to participate in it themselves. So the classroom here is a world apart, while at the same time perhaps reflecting some of the ideals that we publicly embrace.

A fourth problem has to do with the messiness of it all. Synoptic presentations of findings and conclusions based on chains of formal deductive argument would be easier and more efficient, as the Bourbaki writers believed, and would not rely so heavily on the teacher's capacity to move with the flow, supplying new tools and information as they are called for in the problem-posing and problem-solving process rather than as a series of neat packages of information. The extent to which the practice of mathematics in school lessons can mirror what is most exciting and admirable in the practice of mathematics in the discipline will depend on whether teaching practice can proceed in a way that takes these problems seriously.

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Appendix 19.1
Transcript of Large Group Discussion

Lampert: But let's look at this one. This is number 6. As I was walking around and I asked people which ones they thought were hard or easy and which ones they had to revise their thinking on and so on. . . . In number 6 the function machine takes 8 and it gives out 4. If it takes in 4, it gives out 2. If it takes in 2, it gives out 1. If it takes in 0, it gives out 0. What does it do to the numbers in every case? Ellie?

Ellie: Um, well, there were a whole bunch of . . . a whole bunch of rules you could use, use, um, divided by two. . . .

Lampert: Okay, so one rule you think could be divided by two. You could try eight divided by two is four, four divided by two is two, two divided by two is one, zero divided by two is zero?

Ellie: And you could do, um, minus one half. [Several hands go up around the class and students talk privately to one another.]

Lampert: Minus one half?

Ellie: Um. . . .

Lampert: Okay. What would eight minus one half be?

Ellie: Four. [More hands go up, more talking.]

Lampert: Eight minus one half. [Pause]

Ellie: Um, four.

Lampert: You think that would be four? What does somebody else think? I started raising a question because a number of people have a different idea about that. So let's hear what your different ideas are and see if you can take Ellie's position into consideration and try to let her know what your position is. Karim?

Karim: Well, see, I agree with Ellie because you can have eight minus one half and that's the same as eight divided by two or eight minus four.

Lampert: Eight divided by two is four, eight minus four is four? Okay, so Karim thinks he can do all of those things to eight and get four. Okay? Charlotte?

Charlotte: Um, I think eight minus one half is seven and a half because . . .

Lampert: Why?

Charlotte: Um, one half's a fraction and it's half of one whole and so when you subtract you aren't even subtracting one whole number so you can't get even a smaller number that's more than one whole. But I see what Ellie's doing, she's taking half the number she started with and getting the answer.

Lampert: So, you would say one half of eight? Is that what you mean?

Charlotte: Yeah, one half of eight equals four.

Lampert: How do you know that?

Charlotte: Because, um, eight and one half is, um, eight and half of eight is four, so if you have two groups of four you would, is eight.

Lampert: Ellie, what do you think?

Ellie: Um, I still think, I mean, one half, it would be eight minus one half, they would probably say oh, eight minus one half equals four.

Lampert: Who would say that?

Ellie: I don't know. Well, well if if I saw something like that, like if we were having something and the answer was missing . . . .

Lampert: Um-hmm.

Ellie: Um, and it was eight minus one half I would probably say four.

Lampert: What do other people think? Sam?

Sam: Um, I agree with Charlotte and, um, I don't agree with Ellie. Because, um, like one half is not even one, so if, so when Ellie said that people, that people would like, um, a really good mathematician would probably, like, would probably write seven and a half, not four because they would have to know what the one half was meaning, half of a number to, um, to understand it.

Lampert: You know when Charlotte was talking she said that she thought one half meant half of a whole. And it sounds like that's the way you are interpreting it. But Ellie might be interpreting one half to mean something else. Lev, what do you think?

Lev: I think, um, I would agree with Ellie if she had said eight minus one half of eight, because half of eight would be four because four plus four would be eight.

Lampert: So, in your case, you're saying one half, if Ellie meant one half was half of eight wholes, then it would work. Okay, Tyrone?

Tyrone: I agree with Charlotte and Sam and I disagree with Ellie and, like I think Ellie meant, like, because four is half of eight, like one half would be a half, but, and I agree with Lev when he said if she means one half, uh, equals, wait, eight, equals half of eight and I agree with Sam and, uh, Charlotte because, um, if . . . if, uh, four is not, uh . . . eight equals half of four is not right because it's seven and a half because half of like, eight is the whole and, um, one number away from that is seven and plus a half would be seven and half.

Lampert: Uh huh. So um, that reminds me of some of the discussion that we were having yesterday, which is that if you, if you use addition, it helps you to understand what it means to take away something on the other side. Okay? Let's hear from Suran.

Suran: I would agree with Ellie if she had added something else to her explanation, if she said one half of the amount that you have to divide by two.

Lampert: Okay. You guys are on to something really important about fractions, which is that a fraction is a fraction of something. And we have to have some kind of agreement here if, if it's a fraction of eight or if it's a fraction of a whole. Let's, let's just hear from a couple more people, Ellie, and then I'll come back and hear from you, okay? *Uh, Mina?*

Mina: Well, I think, um, I disagree with Ellie because if she means that one half of the whole you would get seven and then if you add that half on again you would get seven and a half.

Lampert: Okay. Well, that's quite similar to what Tyrone was saying, that addition is a way of helping me think about the meaning of subtraction. But again, both you and Tyrone are assuming that one half means one of one. Okay? Alexander.

Alexander: [Inaudible] . . . if you are assuming it is one half of the original number it would be different than if it is half of a whole. So it depends on what your point of view is.

Lampert: Right. And that, that would be very important to clarity. Okay. Ellie, what do you think?

Ellie: Um, well, I agree with Suran and . . . when Charlotte said she thought that it should be one half of eight . . . um . . . instead of just plain one half, I don't agree with her because not all of them are eight. Not all of the problems are eight.

Lampert: Okay. Let's . . . one of the things that is kind of a convention in mathematics is that when we just talk about numbers and we don't associate them with any objects or groups of objects, that this symbol means half of one whole. So if, if you were gonna communicate with the rest of the world who uses mathematics, they would take this to mean eight wholes minus one half of a whole. Okay? Ellie?

Ellie: Um, well, I . . . I think that eight if, you had, all you . . . all these numbers are that are going into the, um . . .

Lampert: Function machine.

Ellie: Um, on number six, they're all, they can all be divided into halves and four minus . . . well, two is one half of four.

Lampert: Okay, so the number that comes out is one half of the number that went in. Okay. And in this case is that true?

Ellie: Um . . .

Lampert: Is one one half of two? Is zero one half of zero?

Ellie: Um, yes.

Lampert: So, what do you think about that? We could write this in words, you know, we don't have to use these equations, but it's more efficient. You, you feel that . . .

Ellie: One half is . . .

Lampert: . . . if, if you said that the number that comes out is half the number that goes in it, it would be easier for you to understand?

Ellie: That's what I meant but I just couldn't put it in there, but that's what was in my mind.

Lampert: Okay. But I think you raised a lot of interesting questions by your idea of taking away a half. Okay? Alexander?

Alexander: Um, what Charlotte meant was that a half of the original number so, the original number was eight and so half of eight is four. So, if it was a different number, you would use a different number.