

Let $\mathcal{C}$ be a class of finite structures:

e.g.: * sets

* linear orders

* partial orders (hard!!)

* graphs

We define a measure on embeddings of finite structures valued in k such that

$$\mu(\text{isom}) = 1$$

$$\mu(i \circ j) = \mu(i) \mu(j)$$

$$\begin{array}{ccc} \cdot B & & X_1, \dots, X_n \text{ amalgamations} \\ i \uparrow & i_1 \uparrow \dots i_n \nearrow & \\ A \xrightarrow{j} C & & \mu(i) = \sum_{\alpha} \mu(i_{\alpha}) \end{array}$$

Amalgamation X, i', j' such that

$$\begin{array}{ccc} B & \xrightarrow{j'} & X \\ i \uparrow & \square & \uparrow i' \\ A & \xrightarrow{j} & C \end{array}$$

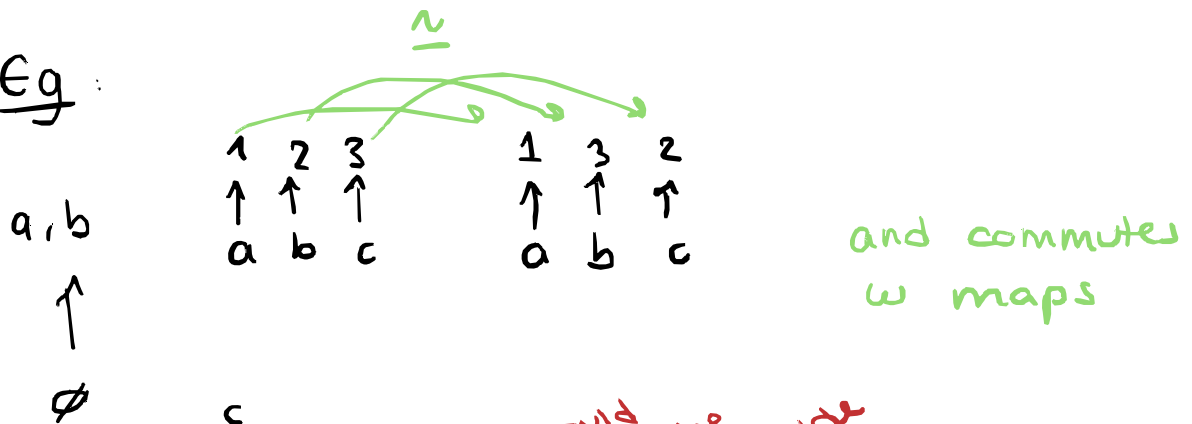
i', j' embeddings, jointly surjective.

①

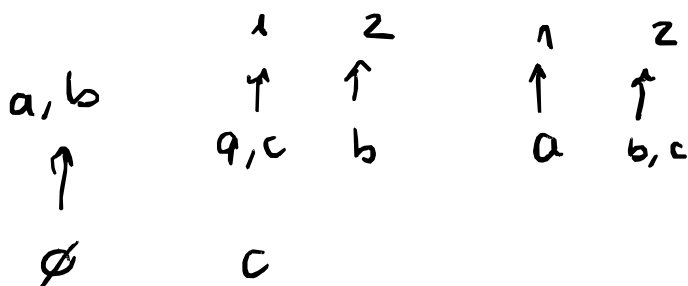
up to isomorphism:



Eg:



any $\cong$ would still have a, c coincide

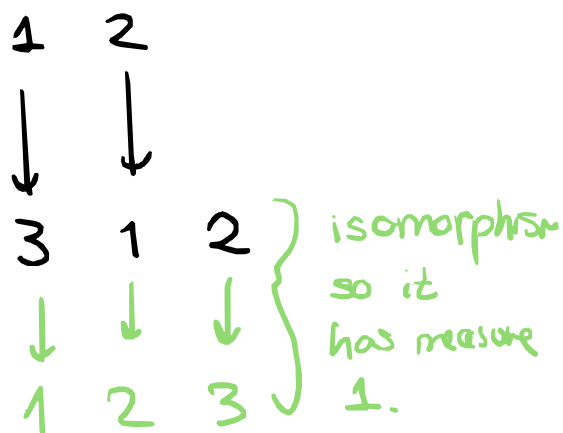
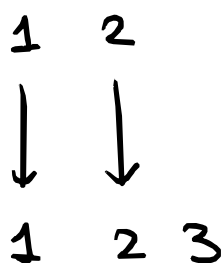


We will see a few examples of finite structures and compute the measures there.

Measures on sets:

We want to find measures for all the embeddings $[n] \rightarrow [m]$.

Note that all embeddings of $[n] \rightarrow [m]$ are isomorphic:



We also have by the composition rule

$$\mu([n] \rightarrow [m]) = \mu([n] \rightarrow [n+1]) \cdots \mu([m-1] \rightarrow [m]).$$

So we want to find $\mu([n] \rightarrow [n+1])$.

$$\text{let } \mu(\emptyset \rightarrow [1]) = t.$$

3

[1]



$\emptyset \rightarrow [1]$

options: they can collapse or not.

$$\mu(\emptyset \rightarrow [1]) = \underbrace{\mu([1] \rightarrow [1])}_1 + \mu([1] \rightarrow [2])$$

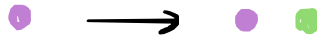
$1 \approx$

$$\mu([1] \rightarrow [2]) = t-1.$$

[2]



$[1] \rightarrow [2]$



options: blue & green collapse or not

$$\mu([1] \rightarrow [2]) = \mu([2] \rightarrow [2]) + \mu([2] \rightarrow [3])$$

$$t-1-1 = \mu([2] \rightarrow [3])$$

[n]



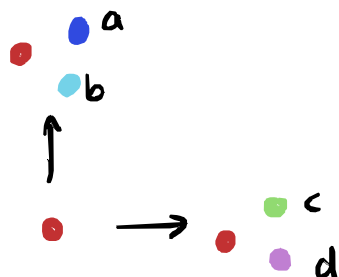
$[n-1] \rightarrow [n]$

$$\mu([n-1] \rightarrow [n]) = 1 + \mu([n] \rightarrow [n+1])$$

$$\mu(\emptyset_n) = t-n.$$

(4)

Rmk: In general, amalgamations can be more complicated:



options:

• all a, b, c, d are diff

$$[3] \rightarrow [5]$$

• one pair collapses

$$a=c \quad a=d \quad b=c \quad b=d$$

$$4 \text{ times } [3] \rightarrow [4]$$

• two pairs collapse

$$a=c \rightarrow b=d \quad \text{or} \quad a=d \rightarrow b=c$$

$$2 \text{ times } [3] \rightarrow [3]$$

$$\phi_1 \phi_2 = \phi_3 \phi_4 + 4 \phi_3 + 2$$

$$(t-3)(t-4) + 4(t-3) + 2$$

$$= t(t-3) + 2 = t^2 - 3t + 2 = (t-1)(t-2)$$

Doing general inclusions,

$$\begin{array}{c} [a+k] \\ \uparrow \\ [a] \end{array} \rightarrow [a+l]$$

$$\sum_{i=0}^{\min(l,k)} \underbrace{\binom{l}{i} \binom{k}{i} i!}_{\text{how to choose the collapses}} \mu([a+l] \rightarrow [a+l+k-i])$$

$\nwarrow$ number of collapses

(5)

Doing general inclusions,

how to choose the collapses

$$\begin{array}{c} [a+k] \\ \uparrow \\ [a] \rightarrow [a+l] \end{array} \quad \sum_{i=0}^{\min(l,k)} \binom{l}{i} \binom{k}{i} i! \mu([a+l] \rightarrow [a+l+k-i])$$

number of collapses

$$\mu([n] \rightarrow [m]) = (t-n)(t-n-1)\dots(t-m) = \binom{t-n}{m-n} (m-n)!$$

$$\binom{t-a}{k} k! = \sum_{i=0}^{\min(l,k)} \binom{l}{i} \binom{k}{i} i! \binom{t-a-l}{k-i} (k-i)!$$

$$\binom{x}{k} = \sum_{i=0}^{\min(l,k)} \underbrace{\binom{x-l}{k-i}}_{\text{choose}} \underbrace{\binom{l}{i}}_{\text{choose } i \text{ from } \{1, \dots, l\} \text{ remaining from the rest.}}$$

$k-i \geq 0$ $i \leq l$

Measures of linear orders

Same reasoning holds: we can consider adding one element at the time

But now elements are different.

We need to ensure

$$X \rightarrow X, a \rightarrow X, a, b = X \rightarrow X, b \rightarrow X, a, b \text{ 'Q-data'}$$

Marked structure: $1 \overset{*}{2} 3 4$ means the embedding $1 2 4 \rightarrow 1 2 3 4$

But in this case there are a lot of amalgamations that are just too simple:

$$1 \boxed{2} 4 7 \rightarrow 1 2 4 6 7$$

$\uparrow$
 $\uparrow$

only one amalgamation!

$$1 \ 4 \ 7 \rightarrow 1 \ 4 \ 6 7$$

$$\overset{*}{1} 2 4 6 7 = \overset{*}{1} 2 4 7$$

the 6 does not matter! Neither does the 7

$$1 2 4 \rightarrow 1 2 4 7$$

$\uparrow$

$\uparrow$

$$\rightsquigarrow \overset{*}{1} 2 4 = \overset{*}{1} 2 4 7$$

$$1 \ 4 \rightarrow 1 \ 4 7$$

Does the 4 matter?

12
 $\uparrow$
 1

1

$2=4$
 $2<4$
 $4<2$

yes it does!

$1 \longrightarrow 14$

General rule:

$\star$
 $1 \ 4 \ 5 \ 6 \ 7 = \star \ 1 \ 4$

$1 \ 2 \ \star \ 4 \ 5 \ 7 \ 9 = 2 \ \star \ 4 \ 5$

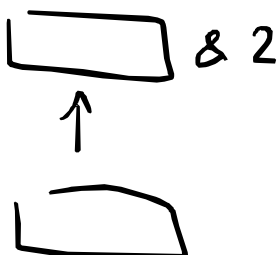
$1 \ 2 \ 4 \ 7 \ \star \ 10 = 7 \ \star \ 10$

everything boils down to $\star \ 1$, $\star \ 12$, $\star \ 12$ or

$\star \ 123$.

For amalgamations, we may assume the left map is minimal.

1234
 $\uparrow$
 $1 \ 3 \ 4 \longrightarrow$



$\left. \begin{array}{l} 4 \text{ is apart in all} \\ \text{these so we} \\ \text{can just remove it} \end{array} \right\}$

Similar thing happens for Q, but

$$(X \rightarrow X, a)(X, a \rightarrow X, a, b) =$$

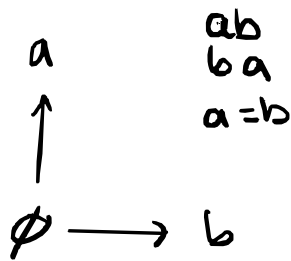
$$(X \rightarrow X, b)(X, b \rightarrow X, a, b)$$

Can ignore

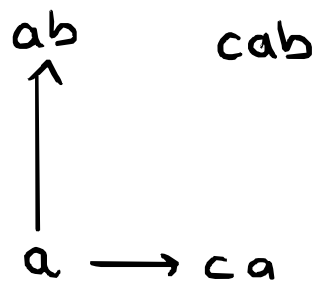
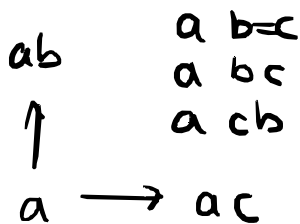
→ a sep from b

→ c sep from relevant point IN ALL 4

Now it is kind of easy to find all measures:



$$\dot{1} = 1 + \dot{1}2 + \dot{1}\dot{2}$$



$cab = ab$
already knew

$$\dot{1}\dot{2} = 1 + \dot{1}\dot{2}3 + \dot{1}\dot{2}$$

$$\dot{1}\dot{2}3 = -1$$

$a \rightarrow ab$ is the same

$$\begin{array}{ccc} acb & \rightarrow & dacb \\ \uparrow & & \uparrow \\ ab & \rightarrow & dab \end{array} \quad \text{triv}$$

$$\begin{array}{ccc} acb & \rightarrow & acbd \\ \uparrow & & \uparrow \\ ab & \rightarrow & abd \end{array} \quad \text{tr}$$

$$\begin{array}{ccc} acb & & ac=db \\ \uparrow & & acdb \\ ab & \rightarrow & adcb \\ ab & \rightarrow & adb \end{array}$$

$$\begin{aligned} 1\dot{2}3 &= 1 + 1\bar{2}3 + 1\dot{2}\bar{3} \\ 1\dot{2}3 &= -1 \end{aligned}$$

Q-data a b neighbors. So can have at most 4 elements:

$$\begin{array}{ll} ab & (1)(1\dot{2}) = (1)(1\dot{2}) \\ cab & (1\dot{2}3)(1\dot{2}) = (1\dot{2})(1\bar{2}) \\ abd & (1\dot{2}3)(1\dot{2}) = (1\dot{2})(1\dot{2}) \\ cabd & (1\dot{2}3)(1\dot{2}3) = (1\dot{2}3)(1\dot{2}3) \end{array}$$

$$AB = AD \quad B^2 = BC \quad D^2 = DC$$

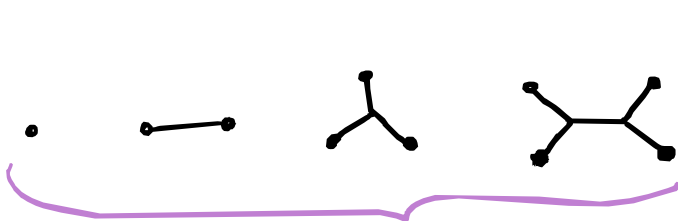
$$C = -1, \quad A = B + D + 1.$$

$$\begin{array}{l|l|l} B^2 = -B & B = D = 0 & B = D = -1 \\ D^2 = -D & A = 1 & A = 1 \end{array} \quad \begin{array}{l} B \neq D \\ A = 0 \end{array}$$

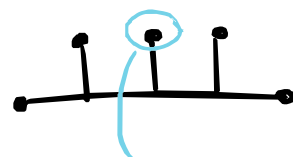
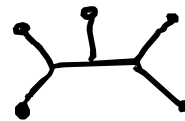
$$AB = AD \quad \text{holds!}$$

- Two linear orders $\Leftrightarrow$ permutations
 ↳ 37, unsure why
- Three linear orders
 ↳ much bigger already
 ↳ L-data needs 7 elements
 ↳ Q-data needs a-b neighbors,
 then each has 5 more so could
 have as much as 12 elements
- Some words on my research
- Circular measures on n colors $2^n(n+1)^{n-2}$

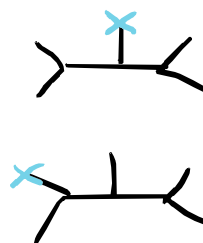
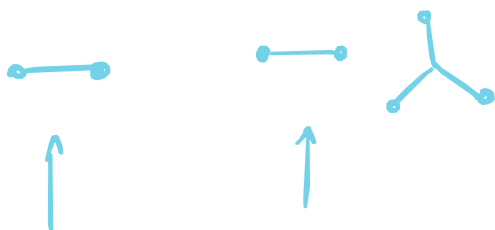
Measures on Boron Trees



you have automorphisms
permuting all the vertices
so all are the same.



this guy is
special



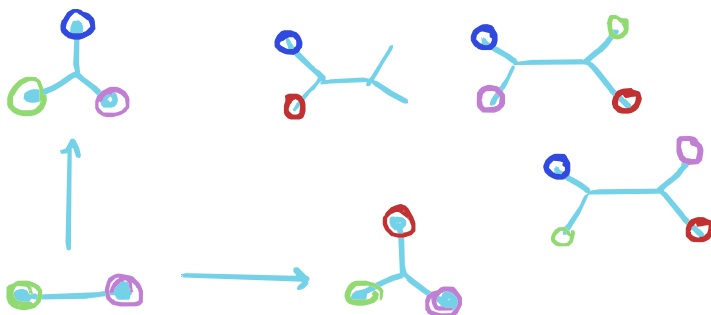
$$X = 1 + \text{two-vertex path with X on the right vertex}$$

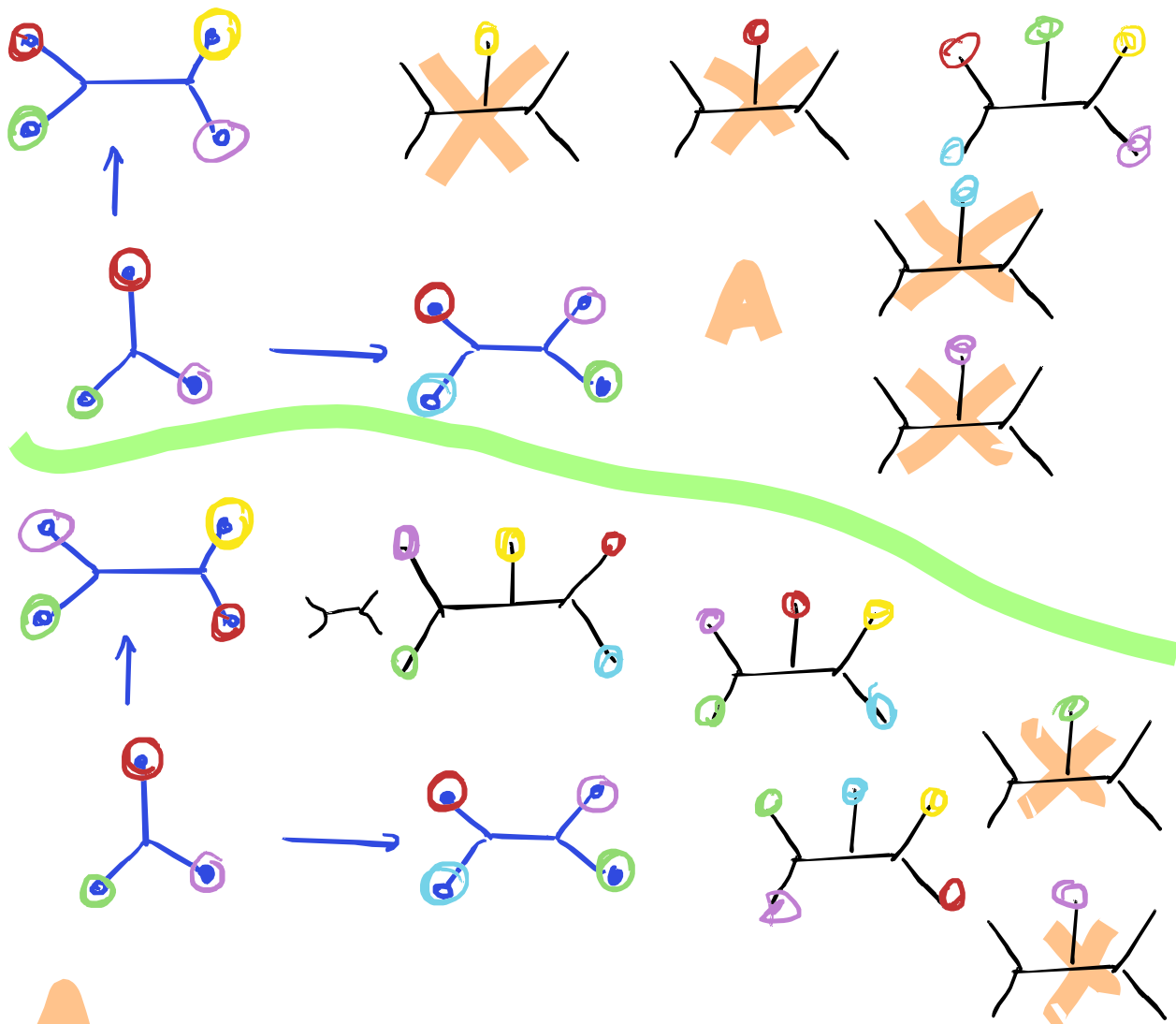


$$\text{two-vertex path with X on the right vertex} = 1 + \text{three-vertex star with X on the central vertex}$$

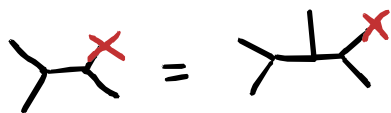


$$\text{three-vertex star with X on the central vertex} = 1 + 3 \times \text{four-vertex star with X on one of the outer vertices}$$





A



B



$$X = 1 + \text{diagram}$$

$$\text{diagram} = 1 + \text{diagram}$$

$$\text{diagram} = 1 + 3 \text{ diagrams}$$

$$\text{diagram} = \text{diagram}$$

$$\text{diagram} = 1 + \text{diagram} + 2 \text{ diagrams}$$

$$0 = 1 + \boxed{} + \frac{t-3}{3}$$



Composition :

$$\text{diagram} \text{ diagram} = \text{diagram} \text{ diagram}$$

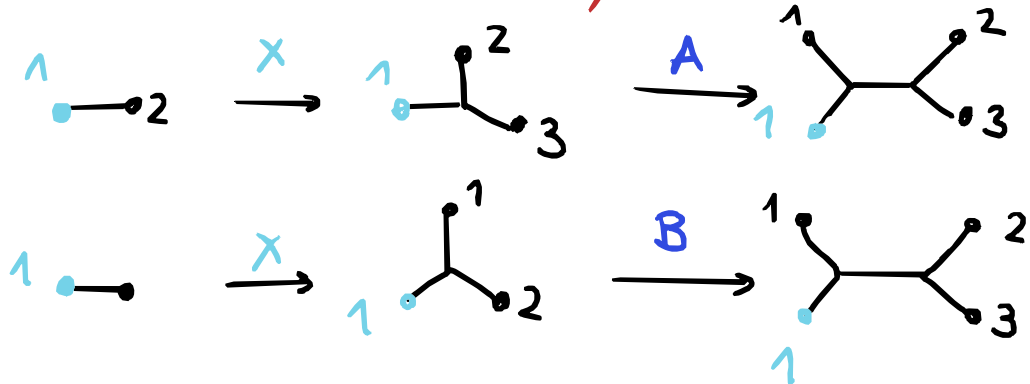
two cases: $\text{diagram} = \text{diagram} \quad \frac{t-3}{3} = -\frac{t}{3} \rightarrow t = \frac{3}{2}$

$$\text{diagram} = 0 \quad \frac{t-3}{3} = 0 \quad t = 3$$

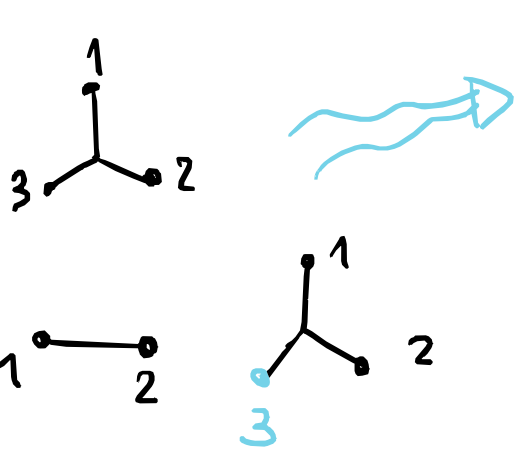
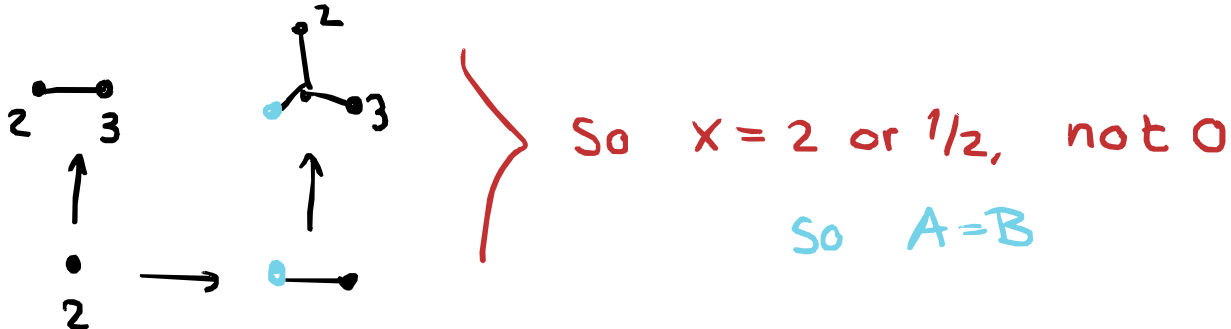
	t	$3/2$	3
	$t-1$	$1/2$	2
	$t-2$	$-1/2$	1
	$\frac{t-3}{2}$	$-1/2$	0
	$\frac{t-3}{3}$	$-1/2$	0
	$-t/3$	$-1/2$	-1

Colored becomes impossible:

1. Composing / Amalgamating monochromatic gives monochromatic, so $\mu(\bullet \rightarrow \bullet) = 2$ or $1/2$



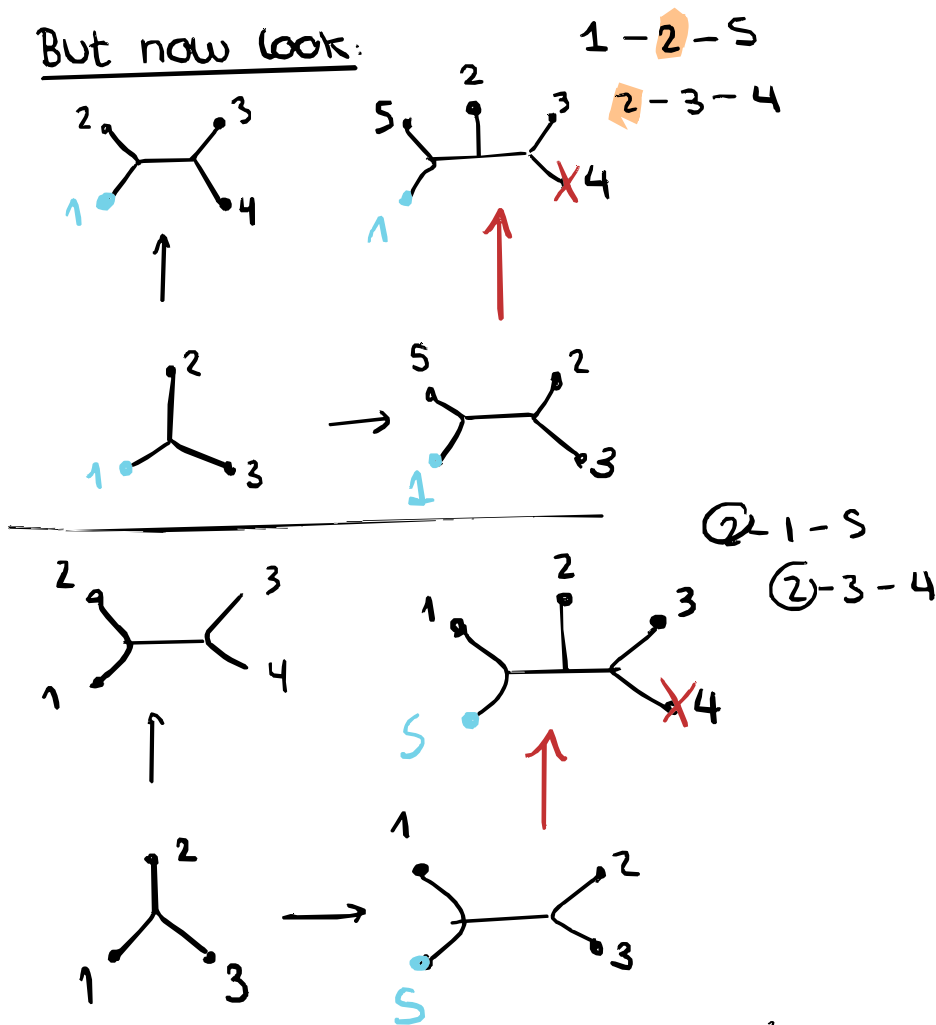
$\bullet \times A = \bullet \times B$



The blue can have each of the 3 as neighbors

So $A/B = \frac{t-2}{3}$

But now look:



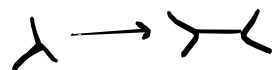
so the measure of
as the measure of

which is $\frac{t-3}{3}$

But

$$\frac{t-2}{3} \neq \frac{t-3}{3}$$

is the same



No measures!

Final Remarks:

The $\mathcal{G}$ -category construction:

- Objects ' Vec_X ' := certain functions $X \rightarrow K$
G-equiv
- Morphisms $\text{Vec}_X \rightarrow \text{Vec}_Y$:= certain functions
 $Y \times X \rightarrow K$ G-equiv
'matrices'

• Composition:

$$AB(x, z) := \int A(x, y) B(y, z) dy$$

need notion of integral for these functions $\rightarrow$

need notion of measure for $\hat{G}$ -sets.

we need fibers, but
fibers don't have
actions on the full G ,
just $\text{stab}(x)$

Eg $1 \dot{2} 3 \leftarrow x < y < z$, but forget y . $y \in (x, z)$

$1 \dot{2} \leftarrow x < y$ but forget y $y \in (x, \infty)$

$\dot{1} 2 \leftarrow x < y$ but forget x $x \in (-\infty, y)$

$\dot{1} \leftarrow x$ but forget x $x \in (-\infty, \infty)$.

On these, these do look like measures

$$\overbrace{(-\infty \quad \infty)} = \overbrace{(-\infty \quad \infty)} \cdot \overbrace{(-\infty \quad \infty)} \\ a \quad a \quad a$$

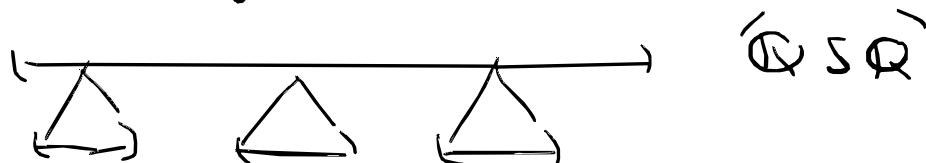
$$1 = 12 + 1 + 12$$

$$\overbrace{a \quad b} = \overbrace{a \quad c} \cdot \overbrace{c \quad b} \leadsto 123 = 123 + 1 + 123 \\ \Rightarrow 123 = -1.$$

So we can use fibers to think of this as an actual measure on the Fraïssé limit.

My work: this sees measures on $\mathbb{Q}$

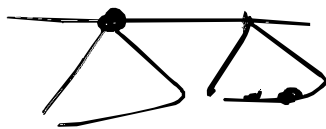
I am studying



This is like $\mathbb{Q} \times \dots \times \mathbb{Q}$ so we expect 4^n measures. If you look at structures here they are like, a linear order w boxes



This is not enough to see all measures,

Since  $\leadsto \boxed{12}$ is also relevant.

No empty boxes $\rightarrow 3 \cdot 2^n - 2$
empty boxes $\rightarrow 4^n$