

MEASURES ON ITERATED DELANNOY WREATH PRODUCTS

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Introduction and Background

In [1], Harman and Snowden introduced a notion of measure which can be used to create interesting tensor categories, for example:

- $S_\infty \curvearrowright \mathbb{N}$ recovers Deligne's $\text{Rep}(S_t)$,
- $\text{Aut}(\mathbb{Q}, <) \curvearrowright \mathbb{Q}$ gives rise to (new!) Delannoy Categories [2, 3]:

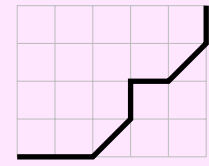
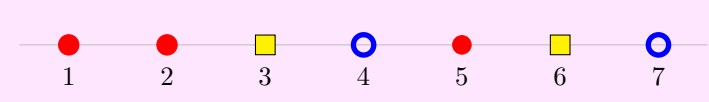


Fig. 1: Relation to Delannoy Paths, from [2].

- **Oligomorphic Group:** $G \curvearrowright \Omega$ faithful; finitely many orbits of the action on Ω^n .
- **G(A):** Subgroup of G pointwise fixing $A \subset \Omega$.
- **Topology on G:** generated by $G(A)$, A finite.
- **(Finitary, Smooth) G-set:** X with action $G \curvearrowright X$ such that $\text{stab}(x)$ is open and it has finitely many orbits. In practice, $Y/\sim$, with Y an orbit of some Ω^n and G -equivariant $\sim$.

In a **Boxed linear order**, elements are in boxes that respect the order:

$$\boxed{123} \boxed{4} \boxed{5} \quad \boxed{1} \boxed{36} \boxed{8} \boxed{9}$$

Formally: wreath product of class of linear orders with itself. (n -fold product has $n-1$ layers:)

$$n=3: \quad \boxed{\boxed{123} \boxed{4} \boxed{5}} \quad \boxed{\boxed{1} \boxed{36} \boxed{8} \boxed{9}}$$

Automorphism group: $G_n = \text{Aut}(\mathbb{Q}) \wr \dots \wr \text{Aut}(\mathbb{Q})$

Definitions of Measures [1]

A **measure for G-maps** $X \rightarrow Y$, Y transitive, with values in k is a rule ν such that:

- $\nu(f) = 1$ for isomorphisms f ,
- If $X = X_1 \sqcup X_2$, $f_i = f|_{X_i}$, $\nu(f) = \nu(f_1) + \nu(f_2)$,
- $\nu(g \circ f) = \nu(g)\nu(f)$,
- If f' is a base change of f , $\nu(f') = \nu(f)$.

In the case of G_1 , Y is of the form $\{x_1 < x_2 < \dots < x_m\}$, meaning the set of all such possible choices.

A **measure for embeddings of structures** with values in k is a rule η such that:

- $\eta(\text{isom.}) = 1$ and $\eta(i \circ j) = \eta(i) \cdot \eta(j)$,
- If $(X'_\alpha, i'_\alpha, j'_\alpha)$, are the amalgamations of (i, j) , then $\eta(i) = \sum_\alpha \eta(i'_\alpha)$:

$$\begin{array}{ccc} acd & \hookrightarrow & a(b=c)d \text{ or } abcd \text{ or } acbd \\ \uparrow & & \uparrow \\ ad & \hookrightarrow & abd \end{array}$$

These notions of measures can be related:

$$\nu(\{x < y < z\} \rightarrow \{x < z\}) = \eta(13 \hookrightarrow 123).$$

But! This relation only reaches orbits of Ω^n , not all $Y/\sim$.

In all our cases, we show $(Y/\sim) = Y'$ by showing $(Y/\sim) = (\tilde{Y}/\sim')$ for 'smaller' $\tilde{Y}$. For example, in the case of G_1 , we mean Y having a smaller m .

This gives us an equivalence between ν s and η s: we can compute measures using structures!

Nekrasov's Presentation with Minimal Structures [4]

- Main idea: do embeddings one point at a time.
 $\eta(1 \hookrightarrow 134) = \eta(1 \hookrightarrow 13) \cdot \eta(13 \hookrightarrow 134)$

- Notation: $\overset{\bullet}{123} := \eta(13 \hookrightarrow 123)$

- Can reduce marked structures (MS):

$$\begin{array}{ccc} acd & \hookrightarrow & acde \\ \uparrow & & \uparrow \\ ad & \hookrightarrow & ade \end{array} \quad \overset{\bullet}{acde} = \overset{\bullet}{acd}$$

We say e is separated from c .

- Focus on minimal MS. For G_1 : $\overset{\bullet}{1}, \overset{\bullet}{12}, \overset{\bullet}{12}, \overset{\bullet}{123}$

Q-data ensures that η is well-defined:

$$\overset{\bullet}{123} := \overset{\bullet}{123} \cdot \overset{\bullet}{13} = \overset{\bullet}{123} \cdot \overset{\bullet}{12}$$

Q-data is trivial if:

- The marked points are separated. In this case, this means they can be assumed consecutive.
- Another point is separated from all four marked points. In this case, this means they can be assumed consecutive to one of the marked points.

Relevant Q-data for G_1 : $\overset{\bullet}{12}, \overset{\bullet}{123}, \overset{\bullet}{123}, \overset{\bullet}{1234}$

L-data ensures that axiom b) holds:

$\overset{\bullet}{acd} = 1 + \overset{\bullet}{abcd} + \overset{\bullet}{acbd}$ is the L-data associated to the amalgamation above, using $\eta(\text{isom.}) = 1$.

L-data is trivial if:

- The marked and new point are 'separated'.
- In all our cases, the embedding i is not minimal.

Relevant L-data for G_1 : $(\overset{\bullet}{a}, \overset{\bullet}{b}), (\overset{\bullet}{ab}, \overset{\bullet}{ac}), (\overset{\bullet}{ab}, \overset{\bullet}{cb}), (\overset{\bullet}{abc}, \overset{\bullet}{adc})$.

For example, $(\overset{\bullet}{a}, \overset{\bullet}{b}) : \overset{\bullet}{a} = 1 + \overset{\bullet}{ab} + \overset{\bullet}{ba}$.

Computing Measures of Boxed Linear Orders

Two separation cases, marked a, b :

- a, b are not neighbors Any element between them 'witnesses' their position.
- a, b are in different boxes, at least one is not alone Any element sharing a box with one 'witnesses' their boxes.

Minimal MS: $\mathcal{A} : \overset{\bullet}{1}, \mathcal{B}_0, \mathcal{B}_1 : \overset{\bullet}{12}, \overset{\bullet}{1} \overset{\bullet}{2},$

$$\mathcal{D}_0, \mathcal{D}_1 : \overset{\bullet}{12}, \overset{\bullet}{1} \overset{\bullet}{2}, \quad \mathcal{C}_0, \mathcal{C}_1 : \overset{\bullet}{123}, \overset{\bullet}{1} \overset{\bullet}{2} \overset{\bullet}{3}.$$

General n : $\mathcal{B}_i, \mathcal{D}_i, \mathcal{C}_i$ with $n-i$ boxes in common.

Q-data: very similar. For example,

$$\overset{\bullet}{1} \overset{\bullet}{23} : \overset{\bullet}{1} \overset{\bullet}{23} \cdot \overset{\bullet}{1} \overset{\bullet}{3} = \overset{\bullet}{1} \overset{\bullet}{23} \cdot \overset{\bullet}{1} \overset{\bullet}{2} \leftrightarrow \mathcal{B}_0 \mathcal{D}_1 = \mathcal{D}_0 \mathcal{D}_1$$

L-data also similar, for example:

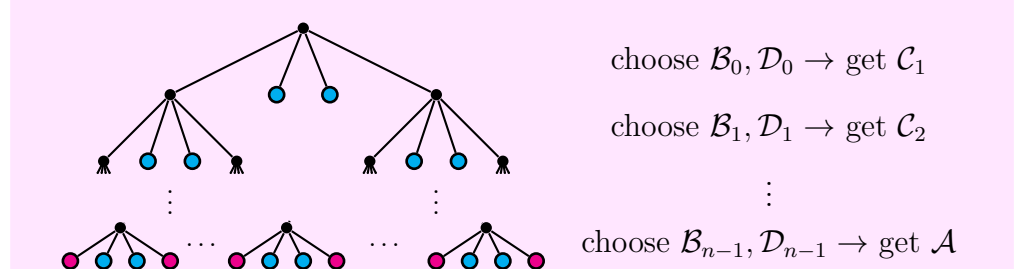
$$\begin{array}{ccc} ab & \hookrightarrow & a(b=c) \text{ or } abc \text{ or } acb \\ \uparrow & & \uparrow \\ a & \hookrightarrow & ac \end{array}$$

(Here, a, b and a, c need to share the same number of boxes, otherwise L-data is trivial) L-data $(\mathcal{D}_k, \mathcal{D}_k)$:

$$\mathcal{D}_k = 1 + \sum_{i=0}^k \mathcal{D}_i + \sum_{i=0}^{k-1} \mathcal{B}_i + \mathcal{C}_k \text{ since } b \text{ and } c \text{ share between } n-k \text{ and } n \text{ boxes and, for example, } \boxed{\overset{\bullet}{1} \overset{\bullet}{23}} = \mathcal{B}_0 \text{ but } \boxed{\overset{\bullet}{1} \overset{\bullet}{2} \overset{\bullet}{3}} = \mathcal{C}_1.$$

Equations: $\mathcal{C}_0 = -1, \quad \mathcal{C}_k \mathcal{B}_k = \mathcal{B}_k^2, \quad \mathcal{C}_k \mathcal{D}_k = \mathcal{D}_k^2,$
 $\mathcal{C}_{k+1} = \mathcal{C}_k - \mathcal{B}_k - \mathcal{D}_k, \quad \mathcal{A} = \mathcal{B}_{n-1} + \mathcal{D}_{n-1} - \mathcal{C}_{n-1}$
(The remaining equations are always satisfied.)

The $3 \cdot 2^n - 2$ measures:



Measures of $\text{Aut}(\mathbb{Q}) \wr \dots \wr \text{Aut}(\mathbb{Q})$

Let $G_2 \curvearrowright \mathbb{Q}^2 \sqcup \mathbb{Q}$. (similar for larger n .)

Example of $(Y/\sim) \cong (Y'/\sim)$:

- $Y : \{x_1, (x_2, y_1), (x_2, y_2)\}; Y' : \{x_1, (x_2, y_1)\}$
- $(1, (2, 4)(2, 9)) \sim (1, (3, 1), (3, 2))$. Apply g, h :
 $(a, (b, c), (b, X)) \sim (a, (b+1, 0), (b+1, 1))$
 $(a, (b, c), (b, Y)) \sim (a, (b+1, 0), (b+1, 1))$
So, any $(a, (b, c), (b, X)) \sim (a, (b, c), (b, Y))$!

There are new structures: $\boxed{\overset{\bullet}{12}} \boxed{\overset{\bullet}{3}}$

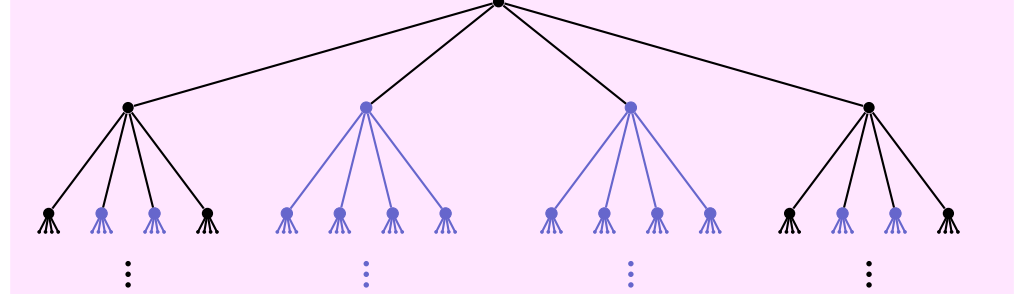
- Reduction 1: all labels $\boxed{\overset{\bullet}{12}} \rightarrow \boxed{\overset{\bullet}{12}}$
- Reduce minimal MS:

$$\begin{array}{ccc} \boxed{\overset{\bullet}{12}} \boxed{\overset{\bullet}{3}} & \simeq & \boxed{\overset{\bullet}{1}} \boxed{\overset{\bullet}{2}} \boxed{\overset{\bullet}{3}} \simeq \boxed{\overset{\bullet}{123}} \\ \boxed{\overset{\bullet}{1}} \boxed{\overset{\bullet}{12}} & \simeq & \boxed{\overset{\bullet}{1}} \boxed{\overset{\bullet}{23}}, \quad \boxed{\overset{\bullet}{1}} \boxed{\overset{\bullet}{2}} \boxed{\overset{\bullet}{3}} \simeq \boxed{\overset{\bullet}{1}} =: \mathcal{T} \end{array}$$

So, a measure in G_n consists of a measure for boxed linear orders for each $1, \dots, n$ which respect transition maps such as $\mathcal{T}$:

$$\mathcal{A}_2 = \boxed{\overset{\bullet}{1}} = \boxed{\overset{\bullet}{1}} \boxed{\overset{\bullet}{2}} \boxed{\overset{\bullet}{3}} = \mathcal{A}_0 \mathcal{T}_0 \mathcal{T}_1$$

In the end, get the 'completed' tree:



So, 4^n measures!
(The naïve conjecture, given that $\mathbb{Q}$ has 4.)