Lecture 11

The following characterization of injective maps is often useful:

Proposition Let $f: V \longrightarrow W$ be a linear map between finite dimensional vector spaces.
The following are equivalent:

(i) f is injective

(ii) For every linearly indep. $v_1, \dots v_m \in V$, $f(v_1), \dots, f(v_m) \in W$ are lin. indep

(iii) There is a basis $e_1, \dots e_n \in V$ s.t $f(e_1), \dots f(e_n) \in W$ are lin. indep.

Pf: (i) $\Rightarrow$ (ii)

Suppose $a_1 f(v_1) + \dots + a_m f(v_m) = 0$

$$\underbrace{a_1 f(v_1) + \dots + a_m f(v_m)}_{= f(a_1 v_1 + \dots + a_m v_m)} = 0$$

f inj $\Rightarrow$ null$(f) = \{0\} \Rightarrow a_1 v_1 + \dots + a_m v_m = 0$

$v_1, \dots v_m$ are lin indep $\Rightarrow a_1 = \dots = a_m = 0$

Hence $f(v_1), \dots, f(v_m)$ are lin. indep.

(ii) $\Rightarrow$ (iii) clear

(iii) $\Rightarrow$ (i) Suppose $u \in \text{null}(f)$ and write

$$u = a_1 e_1 + \dots + a_n e_n$$

$$\Rightarrow \underset{\substack{\| \\ 0}}{f(u)} = \underset{\substack{\| \\ a_1 f(e_1) + \dots + a_n f(e_n)}}{f(a_1 e_1 + \dots + a_n e_n)}$$

$f(e_1), \dots, f(e_n)$ lin indep

$$\Rightarrow a_1 = \dots = a_n = 0$$

$$\Rightarrow u = 0$$

Hence $\text{null}(f) = \{0\} \Rightarrow f$ injective.

Linear maps and matrices

The goal is to describe linear maps when we choose bases.

Proposition If $e_1, \dots e_n$ is a basis of the finite-dim. vector space V and if W is any vector space, then for every $w_1, \dots w_n$, there is a unique linear map $f: V \to W$ such that $f(e_i) = w_i$ for $1 \le i \le n$

Pf: By def. of a basis we know that every $v \in V$ can be written as

$$v = a_1 e_1 + \dots + a_n e_n \quad \text{for } \underline{\text{unique}} \ a_1, \dots a_n \in F$$

If f as above, then

$$\begin{aligned} f(v) &= f(a_1 e_1 + \dots + a_n e_n) \\ &= a_1 f(e_1) + \dots + a_n f(e_n) \\ &= a_1 w_1 + \dots + a_n w_n \end{aligned}$$

Hence we have at most one such map.

In order to show such a linear map exists for every $v \in V$ we write $v = a_1 e_1 + \cdots + a_n e_n$ and define $f(v) = a_1 w_1 + \cdots + a_n w_n$

It is clear that $f(e_i) = w_i$.

We only need to check f is a linear map

① Let $v, w \in V$

write $v = \sum_{i=1}^{n} a_i e_i$, $w = \sum_{i=1}^{n} b_i e_i$

$$\Rightarrow v + w = \sum_{i=1}^{n} (a_i + b_i) e_i$$

$$\Rightarrow f(v) = \sum_{i=1}^{n} a_i w_i \ , \ f(w) = \sum_{i=1}^{n} b_i w_i$$

$$f(v+w) = \sum_{i=1}^{n} (a_i + b_i) w_i$$

$$\Rightarrow f(v+w) = f(v) + f(w)$$

② Exer: check that $f(av) = a f(v)$ for all $a \in F$, $v \in V$

Remarks. 1) Note that f is injective iff $w_1, \dots w_n$ are linearly independent (use previous prop).

2) It follows from the pf. of the proposition that $\text{range}(f) = \text{span}(w_1, \dots, w_n)$. In particular, f is surjective iff $\text{span}(w_1, \dots w_n) = W$

3) Combine 1) + 2): f invertible iff $w_1, \dots w_n$ basis of W.

Corollary If V and W are finite-dimensional vector spaces and $\dim V = \dim W$, then V and W are isomorphic. In particular, if $\dim V = n$, then $V \cong F^n$.

Pf. If $e_1, \dots e_n$ basis of V and $w_1, \dots w_n$ basis of W, consider the linear map $f: V \to W$ such that $f(e_i) = w_i$ for $1 \leq i \leq n$.

This is invertible by Rmk. 3 above

Suppose now that V, W are finite dimensional vector spaces and $f: V \to W$ is a linear map

Choose bases $e_1, \ldots, e_n$ of V,
$w_1, \ldots, w_n$ of W

We have seen: f is completely described by the elem $u_1 = f(e_1), \ldots, u_n = f(e_n)$

In turn, we can write $u_j = \sum_{i=1}^{n} a_{ij} w_j$

for unique $a_{ij} \in F$.

Recall that a <u>matrix with entries in F</u> is a rectangular array with entries in F, labeled

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

hence in a_{ij}

i: row index

j: column index

This is an $m \times n$ matrix

We thus see that once we choose bases, we have a bijection

$$\left\{ \begin{matrix} \text{linear maps} \\ V \to W \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} m \times n \\ \text{matrices with entries} \\ \text{in } F \end{matrix} \right\}$$

where $m = \dim W$, $n = \dim V$

Advanced linear algebra (II)

We have seen that if V and W are finite-dimensional vector spaces and we choose bases

$$e_1, \dots e_n \text{ in } V$$

and $w_1, \dots, w_m$ in W, we have a bijection

$$\mathcal{L}(V,W) \simeq M_{m,n}(F)$$

set of matrices of size $m \times n$ with entries in F.

We will show now that familiar operations on matrices correspond to operations on linear maps that we discussed.

<u>Note first:</u> $\mathcal{L}(V,W)$ is a vector space over F

Let's see what addition and scalar multiplication corresponds to for matrices.

Say $f, g \in \mathcal{L}(V,W)$ corresp to $\overset{(a_{ij})}{\overset{\shortparallel}{A}}, \overset{(b_{ij})}{\overset{\shortparallel}{B}} \in M_{m,n}(F)$.

Hence $f(e_j) = \sum_{i=1}^{m} a_{ij} w_i$, $g(e_j) = \sum_{i=1}^{m} b_{ij} w_j$

$$(f+g)(e_j) = f(e_j) + g(e_j)$$

$$= \sum_{i=1}^{m} (a_{ij} + b_{ij}) w_j$$

If $\lambda \in F \Rightarrow (\lambda f)(e_j) = \lambda f(e_j) = \sum_{i=1}^{m} (\lambda a_{ij}) w_i$

We define on $M_{m,n}(F)$

addition of matrices:

$$A = (a_{ij}), \quad B = (b_{ij}) \quad \Rightarrow \quad A+B = (a_{ij} + b_{ij})$$

scalar multiplication:

$$\lambda A = (\lambda a_{ij})$$

Easy to check directly:

with these operations, $M_{m,n}(F)$ is a vector space over F

We have seen

Prop 1 If $e_1, \dots e_n$ is a basis of V and $w_1, \dots w_m$ is a basis of W, by mapping

$$\varphi \in \mathcal{L}(V, W) \longrightarrow (a_{ij})$$

$$\text{where } \varphi(e_j) = \sum_{i=1}^{m} a_{ij} w_i$$

we have an isomorphism of vector spaces.

Prop 2 If we put for every i and j, with $1 \le i \le m$ and $1 \le j \le n$,

$$A_{ij} = \begin{array}{c c} & \begin{array}{ccc} & j & \end{array} \\ i & \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} \end{array}$$

with 1 on spot i,j
0 everywhere else

$\Rightarrow$ These matrices give a basis of $M_{m,n}(F)$

In particular, $\dim M_{m,n}(F) = mn$

Pf. Exercise

Cor. If V, W are finite-dimensional vector spaces, then $\mathcal{L}(V,W)$ is finite dimensional and

$$\dim \mathcal{L}(V,W) = \dim V \cdot \dim W.$$

Let's see now how to describe composition of maps via matrices.

Suppose we have
$$\begin{cases} V & \text{basis } e_1, \dots e_n \\ V' & \text{basis } e'_1, \dots, e'_m \\ V'' & \text{basis } e''_1, \dots, e''_p \end{cases}$$

Let $f : V \to V'$ lin. map corresp to $A = (a_{ij}) \in M_{m,n}(F)$

$g : V' \to V''$ — " — $B = (b_{kj}) \in M_{p,m}$

Let C be the matrix corresp to $gf : V \to V''$

" (c_{kj})

What are the c_{kj}?

By definition, $(gf)(e_j) = \sum_{k=1}^{p} c_{kj} e''_k$

On the other hand

$$gf(e_j) = g(f(e_j)) = g\Big(\sum_{i=1}^{m} a_{ij} e'_j\Big)$$

$$= \sum_{i=1}^{m} a_{ij}\, g(e'_j)$$

$$= \sum_{i=1}^{m} a_{ij} \cdot \left(\sum_{j=1}^{p} b_{jk} \, e''_k \right)$$

$$= \sum_{k=1}^{p} \left(\sum_{j=1}^{m} a_{ij} \, b_{jk} \right) e''_k$$

Conclusion : $\boxed{c_{ik} = \sum_{j=1}^{m} a_{ij} \, b_{jk}}$ $*$

In other words $\boxed{C \text{ is the product of the matrices } A \text{ and } B}$

Recall: If $A \in M_{m,n}$, $B \in M_{n,p}$ $\Rightarrow$ define AB s.t the ik entry is given by the formula $(*)$.

Properties of matrix multiplication :

① Associativity : if $A_1 \in M_{m,n}$, $A_2 \in M_{n,p}$, $A_3 \in M_{p,q}$

$$\Rightarrow A_1 (A_2 A_3) = (A_1 A_2) A_3$$

② Distributivity with respect to addition ;

$$A (B_1 + B_2) = AB_1 + AB_2$$

③ Identity elem: if $I_n = \begin{pmatrix} 1 & & 0 \\ & 1 \ddots & \\ 0 & & 1 \end{pmatrix} \in M_n(F)$

$\Rightarrow$ for every $A \in M_{m,n}(F)$, we have

$$A \cdot I_n = A \quad \text{and} \quad I_m \cdot A = A$$

Pfs: easy (exercise).

Def. A matrix $A \in \overset{M_{n,n}(F)}{\overset{\|}{M_n(F)}}$ is <u>invertible</u> if there is $B \in M_n(F)$ such that

$$AB = I_n = BA$$

<u>Caveat</u>: In general, multiplic. of matrices is <u>not</u> commutative. This is why we have to put both conditions above.

e.g.: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ while $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

<u>Rmk</u>: B as above, if it exists, is unique:

if B' is another such matrix $\Rightarrow$

$$B' = B' \cdot I_n = B'(AB) = (B'A)B = I_n B = B$$

Typically, one writes A^{-1} for B.