Lecture 7

- Recall that on Friday we proved the following propositio

if $w_1, \dots, w_n$ are linearly dependent elements of a vector space V, then there is j, with $1 \leq j \leq n$, such that

$$w_j \in \text{span}(w_1, \dots w_{j-1})$$

This is often applied when we know $w_1, \dots w_r$ linearly indep. for some $r < n$ (in which case we can't have $w_i \in \text{span}(w_1, \dots w_{i-1})$ for $i \leq r$). Hence the above j satisfies $j \geq r+1$.

This is how we applied this several times on Monday.

- Recall from Monday:

a finite subset $T \subseteq V$ basis if $\left\{ \begin{array}{c} \text{linearly independent} \\ + \\ \text{span}(T) = V \end{array} \right.$

We showed:

- every finite set T with $\text{span}(T) = V$ contains a basis of V

- every finite subset T in a finite-dim vector space V can be completed to a basis.

- Thm: if $\underbrace{v_1, \dots v_r}_{\text{linearly indep}} \in \text{span}(w_1, \dots w_n)$, then $r \leq n$

<u>Theorem 1</u> If V is a finite-dimensional vector space, then any two bases of V have the same number of elements.

<u>Def.</u> The <u>dimension</u> $\dim V$ of V is the number of elements in a basis of V.

<u>Pf. of thm:</u> let $B: u_1, \dots u_r$ and $B': u'_1, \dots u'_n$ be two bases of V.

$$\underbrace{u_1, \dots u_r}_{\text{linearly indep}} \in V = \operatorname{span}(u'_1, \dots u'_n)$$

Above thm. from Monday $\Rightarrow r \leq n$.

By symmetry, $n \leq r$, hence $r = n$ $\square$

<u>Examples</u>

1) $\dim(\mathbb{F}^n) = n$

2) $\mathcal{P}(d) = \{ P \in \mathbb{F}[x] \mid \deg(P) \leq n \}$

$\Rightarrow \dim \mathcal{P}(d) = n+1$

3) $\dim V = 0$ iff $V = \{0\}$

4) $\dim V = 1$ iff $V = \{\lambda u \mid \lambda \in \mathbb{F}\}$ for some $u \neq 0$, that is, V is a line.

The following criterion for a basis is useful when we know beforehand the dimension of a vector space.

Prop 1 If V is a finite-dimensional vector space with $n = \dim V$ and $B \subseteq V$ is a finite subset, the following are equivalent:

i) B is a basis of V

ii) B is linearly indep and $\# B = n$

iii) $\operatorname{span}(B) = V$ and $\# B = n$

Pf. i) $\Rightarrow$ ii) and i) $\Rightarrow$ iii) are clear.

ii) $\Rightarrow$ i) We know that B must be contained in a basis of V, say B'

$n = \# B$ and $\# B' = n \Rightarrow B = B'$

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iii) $\Rightarrow$ i) Similarly, we know that there is $B' \subseteq B$ basis.

$\# B' = n$ and $\# B = n \Rightarrow B = B'$.

Remark If V is a finite-dimensional vector space and $T \subseteq V$ finite subset, then

a) If T linearly indep $\Rightarrow \# T \leq \dim V$

(since T is contained in a basis)

b) If $\text{span}(T) = V \Rightarrow \# T \geq \dim V$
(since T contains a basis of V)

<u>Theorem 2</u> If V is a finite-dimensional vector space and $W \subseteq V$ is a linear subspace, then W is finite-dimensional and $\dim W \leq \dim V$. Moreover, we have $\dim W = \dim V$ iff $V = W$.

<u>Proof.</u> Note first that if $v_1, \ldots, v_r \in W$ are lin. indep, they are also lin. indep in V.
Prev. remark $\Rightarrow r \leq \dim V$.
Since the number of such elements is bounded above, we may choose such a set $v_1, \ldots, v_r \in W$ with r <u>maximal</u>. We will show that

$\text{span}(v_1, \ldots v_r) = W$.

This implies that W is finite-dimensional and $v_1, \ldots, v_r$ is a basis of W, hence

$$r = \dim W \leq \dim V.$$

Since $v_1, \ldots, v_r \in W$, we have $\mathrm{span}(v_1, \ldots v_r) \subseteq W$. To see that this is an equality, we argue by contradiction: suppose we have

$w \in W \setminus \mathrm{span}(v_1, \ldots, v_r)$.

By the maximality in the choice of $r \Rightarrow$

$$\left.\begin{array}{l} v_1, \ldots v_r, w \text{ are \underline{not} linearly indep} \\ \text{Since } v_1, \ldots v_r \text{ are linearly indep} \end{array}\right\} \Rightarrow$$

$w \in \mathrm{span}(v_1, \ldots v_r)$ (we use the result discussed at the beginning of the lecture).

This is a contradiction.

In order to prove the last assertion in the theorem, note that if $W = V$, we clearly have $\dim W = \dim V$. Conversely, suppose $\dim W = \dim V$ and let $\mathcal{B}$ be a basis of W.

In particular, $\mathcal{B}$ is lin. indep (in W, hence in V)

$$\# \mathcal{B} = \dim W = \dim V$$

Prop 1 $\Rightarrow$ $\mathcal{B}$ is a basis of V, hence

$$W = \mathrm{span}(\mathcal{B}) = V \qquad \square$$

Example We can use Thm. 2 to prove that a certain vector space is not finite-dimensional if we can produce subspaces of arbitrarily large dimension.

For example, we know that

$$\dim \{ P \in F[x] \mid \deg(P) \leq d \} = d+1$$

If $F[x]$ is finite-dimensional, then Thm 2. $\Rightarrow$

$$d+1 \leq \dim(V) \text{ for all } d, \text{ a contradiction.}$$

Theorem 3 Let V be a finite-dimensional vector space over F. If W_1 and W_2 are linear subspaces of V, then

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2).$$

Pf. Note that by Thm. 2, all vector spaces in the above formula are finite-dimensional.

Choose a basis $u_1, \dots u_r$ for $W_1 \cap W_2$.

These elem. are lin. indep also in $W_1 \Rightarrow$ we can complete them to a basis

$$u_1, \dots, u_r, w_1, \dots, w_p \text{ of } W_1$$

Similarly, we have a basis

$u_1, \dots, u_r, w'_1, \dots w'_q$ for W_2.

Claim: $u_1, \dots, u_r, w_1, \dots w_p, w'_1, \dots w'_q$ form a basis of $W_1 + W_2$.

This is enough: then $\dim W_1 + \dim W_2 = (r+p) + (r+q)$

$$\|$$

and $\dim(W_1 \cap W_2) + \dim(W_1 + W_2) = r + (r+p+q)$

We now prove the claim.

First: all these elements lie in either W_1 or W_2, hence they all lie in $W_1 + W_2$.

Therefore $\mathrm{span}(u_1, \dots, u_r, w'_1, \dots w'_q, w_1, \dots w_p) \subseteq W_1 + W_2$

For the opposite inclusion, note that if $v \in W_1 + W_2$

$\Rightarrow$ can write $v = v_1 + v_2$, with $v_1 \in W_1$, $v_2 \in W_2$

$\Rightarrow$ v_1 is a linear comb. of $w_1, \dots w_p, u_1, \dots u_r$

v_2 —— " —————— $w'_1, \dots w'_q, u_1, \dots u_r$.

$\Rightarrow v = v_1 + v_2$ is a linear combination of $u_1, \dots u_r, w_1, \dots w_p, w'_1, \dots w'_q$

We conclude that $\mathrm{span}(u_1, \dots u_r, w_1, \dots w_p, w'_1, \dots w'_q) = W_1 + W_2$

We now prove that these elements are linearly independent. Suppose we have

$$*) \quad \underbrace{a_1 u_1 + \dots + a_r u_r + b_1 w_1 + \dots + b_p w_p}_{\in W_1} + \underbrace{b_1' w_1' + \dots + b_q' w_q'}_{\in W_2} = 0$$

$$\Rightarrow \quad b_1' w_1' + \dots + b_q' w_q' = -(a_1 u_1 + \dots + a_r u_r + b_1 w_1 + \dots + b_p w_p)$$

$$\in W_1 \cap W_2$$

$\Rightarrow b_1' w_1' + \dots + b_q' w_q'$ is a lin. combination of $u_1, \dots u_r$

Since $u_1, \dots u_r, w_1', \dots w_q'$ are lin. indep $\Rightarrow$

$$b_1' = \dots = b_q' = 0$$

We now deduce from $*)$ using the fact that $u_1, \dots, u_r, w_1, \dots w_p$ are lin. indep. that $a_1 = \dots = a_r = b_1 = \dots = b_p = 0$

This completes the proof.

Corollary Suppose V is a finite-dimensional vector space and $W_1, W_2 \subseteq V$ are linear subspaces such that $V = W_1 + W_2$. Then $V = W_1 \oplus W_2$ iff $\dim V = \dim W_1 + \dim W_2$.

Pf. Since $V = W_1 + W_2$, we know that $V = W_1 \oplus W_2$ iff $W_1 \cap W_2 = \{0\}$, which is the case iff $\dim(W_1 \cap W_2) = 0$.

By Thm 3, $\dim(W_1 \cap W_2) = \dim(V) - \dim W_1 - \dim W_2$

$\Rightarrow$ QED