CHAPTER 5. PROPER, FINITE, AND FLAT MORPHISMS

In this chapter we discuss an algebraic analogue of compactness for algebraic varieties, *completeness*, and a corresponding relative notion, *properness*. As a special case of proper morphisms, we have finite morphisms, which we have already encountered in the case of morphisms of affine varieties. Finally we discuss an algebraic property, *flatness*, that is very important in the study of families of algebraic varieties.

1. Proper morphisms

We will define a notion that is analogous to that of *compactness* for usual topological spaces. Recall that the Zariski topology on algebraic varieties is quasi-compact, but not Hausdorff. As we have seen, *separatedness* is the algebraic counterpart to the Hausdorff property. A similar point of view allows us to define the algebraic counterpart of compactness. The key observation is the following.

Remark 1.1. Let us work in the category of Hausdorff topological spaces. A topological space X is compact if and only if for every other topological space Z, the projection map $p\colon X\times Z \to Z$ is closed. More generally, a continuous map $f\colon X \to Y$ is proper (recall that this means that for every compact subspace $K \subseteq Y$, its inverse image $f^{-1}(K)$ is compact) if and only if for every continuous map $g\colon Z \to Y$, the induced map $X \times_Y Z \to Z$ is closed.

Definition 1.2. A morphism of varieties $f\colon X \to Y$ is *proper* if for every morphism $g\colon Z \to Y$, the induced morphism $X \times_Y Z \to Z$ is closed. A variety X is *complete* if the morphism from X to a point is proper, that is, for every variety Z, the projection $X \times Z \to Z$ is closed.

Remark 1.3. Note that if $f\colon X \to Y$ is a proper morphism, then it is closed (simply apply the definition to the identity map $Z = Y \to Y$.

We collect in the next proposition some basic properties of this notion.

Proposition 1.4. *In what follows all objects are algebraic varieties.*

i) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are proper morphisms, then $g\circ f$ is a proper morphism.*
ii) *If $f\colon X \to Y$ is a proper morphism, then for every morphism $g\colon Z \to Y$, the induced morphism $X \times_Y Z \to Z$ is proper.*
iii) *Every closed immersion $i\colon X \hookrightarrow Y$ is proper.*
iv) *If X is a complete variety, then any morphism $f\colon X \to Y$ is proper.*
v) *If $f\colon X \to Y$ is a morphism and Y has an open cover $Y = U_1 \cup \ldots \cup U_r$ such that each induced morphism $f^{-1}(U_i) \to U_i$ is proper, then f is proper.*

Proof. Under the assumption in i), given any morphism $h\colon W \to Z$, consider the commutative diagram with Cartesian squares:

$$\begin{array}{ccccc}
X \times_Y Y \times_Z W & \xrightarrow{p} & Y \times_Z W & \xrightarrow{q} & W \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z.
\end{array}$$

In this case, the big rectangle is Cartesian. The assumption implies that the morphisms p and q are closed, hence the composition $q \circ p$ is closed. This gives i).

For ii), we argue similarly: given a morphism $h\colon W \to Z$, consider the commutative diagram with Cartesian squares:

$$\begin{array}{ccccc}
X \times_Y Z \times_Z W & \longrightarrow & X \times_Y Z & \longrightarrow & X \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle f} \\
W & \longrightarrow & Z & \xrightarrow{g} & Y.
\end{array}$$

Since the big rectangle is Cartesian, it follows from the hypothesis that p is closed. This proves that q is proper.

If $i\colon X \hookrightarrow Y$ is a closed immersion, then for every morphism $g\colon Z \to Y$, the induced morphism $X \times_Y Z \to Z$ is a closed immersion, whose image is $g^{-1}\big(i(X)\big)$ (see Example 4.8 in Chapter 3). Since every closed immersion is clearly closed, it follows that i is proper.

Suppose now that X is a complete variety and $f\colon X \to Y$ is an arbitrary morphism. We can factor f as

$$X \overset{i_f}{\hookrightarrow} X \times Y \xrightarrow{p} Y,$$

where i_f is the graph morphism associated to f and p is the projection. The map p is proper, by property ii), since X is complete, and i_f is proper by iii), being a closed immersion, since X and Y are separated. Therefore the composition $f = p \circ i_f$ is proper, proving iv).

Under the assumptions in v), consider a morphism $g\colon Z \to Y$ and let $p\colon X \times_Y Z \to Z$ be the induced morphism. We have an induced open cover $Z = \bigcup_{i=1}^r g^{-1}(U_i)$ and for every i, we have an induced morphism

$$p_i\colon p^{-1}\big(g^{-1}(U_i)\big) = f^{-1}(U_i) \times_{U_i} g^{-1}(U_i) \to g^{-1}(U_i).$$

Since $f^{-1}(U_i) \to U_i$ is proper, it follows that p_i is closed, which easily implies that p is closed. □

Remark 1.5. It follows from property ii) in the proposition that if $f\colon X \to Y$ is a proper morphism, then for every $y \in Y$, the fiber $f^{-1}(y)$ is a complete variety (possibly empty).

Exercise 1.6. Show that if X is a connected, complete variety, then $\Gamma(X, \mathcal{O}_X) = k$. Deduce that a complete variety is also affine if and only if it is a finite set of points.

Exercise 1.7. Show that if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of algebraic varieties, with $g \circ f$ proper, then f is proper. Show that the same holds if we replace "proper" by "closed immersion" or "locally closed immersion.

The following is the main result of this section.

Theorem 1.8. *The projective space* $\mathbf{P}^n$ *is a complete variety.*

Proof. We need to show that given any variety Y, the projection morphism $p\colon \mathbf{P}^n \times Y \to Y$ is closed. If we consider an affine open cover $Y = \bigcup_{i=1}^r U_i$, it is enough to show that each projection $\mathbf{P}^n \times U_i \to U_i$ is closed. Therefore we may and will assume that Y is affine, say $Y = \mathrm{MaxSpec}(A)$ and we need to show that the canonical morphism

$$f\colon X = \mathrm{MaxProj}\big(A[x_0, \ldots, x_n]\big) \to Y$$

is closed.

Let $W = V(I)$ be a closed subset of X. Recall that if

$$I' = \{f \in A[x_0, \ldots, x_n] \mid f \cdot (x_0, \ldots, x_n) \subseteq \mathrm{rad}(I)\},$$

then $V(I') = V(I)$. We need to show that if $\mathfrak{m} \not\in f(W)$, then there is $h \in A$ such that $\mathfrak{m} \in D_Y(h)$ and $D_Y(h) \cap f(W) = \emptyset$. For this, it is enough to find $h \in A$ such that $h \in I'$ and $h \not\in \mathfrak{m}$. Indeed, in this case, for every $\mathfrak{q} \in W = V(I')$, we have $h \in \mathfrak{q} \cap A$, hence $\mathfrak{q} \cap A \not\in D_Y(h)$.

For every i, with $0 \leq i \leq n$, consider the affine open subset $U_i = D_X(x_i)$ of X. Since U_i is affine, with $\mathcal{O}(U_i) = A[x_0, \ldots, x_n]_{(x_i)} = A[x_0/x_i, \ldots, x_n/x_i]$, and $W \cap U_i$ is the open subset defined by

$$I_{(x_i)} = \{g/x_i^m \mid m \geq 0, g \in I \cap A[x_0, \ldots, x_n]_m\},$$

the condition that $\mathfrak{m} \not\in f(U_i)$ is equivalent to the fact that

$$\mathfrak{m} \cdot A[x_0/x_i, \ldots, x_n/x_i] + I_{(x_i)} = A[x_0/x_i, \ldots, x_n/x_i].$$

By putting the condition that 1 lies on the left-hand side and by clearing the denominators, we conclude that

$$x_i^m \in \mathfrak{m} \cdot A[x_0, \ldots, x_n] + I \quad \text{for some} \quad m \in \mathbf{N}.$$

Since such a condition holds for all i, we conclude that if $N \gg 0$ then

$$(x_0, \ldots, x_n)^N \subseteq \mathfrak{m} \cdot A[x_0, \ldots, x_n] + I.$$

This implies

$$A_{\mathfrak{m}}[x_0, \ldots, x_n]_N \subseteq \mathfrak{m} \cdot A_{\mathfrak{m}}[x_0, \ldots, x_n]_N + (I \cdot A_{\mathfrak{m}}[x_0, \ldots, x_n])_N$$

and we deduce from Nakayama's lemma that

$$A_{\mathfrak{m}}[x_0, \ldots, x_n]_N \subseteq (I \cdot A_{\mathfrak{m}}[x_0, \ldots, x_n])_N.$$

This implies that there is $h \in A \smallsetminus \mathfrak{m}$ such that $h \cdot (x_0, \ldots, x_n)^N \subseteq I$, hence $h \in I'$. This completes the proof of the theorem. $\square$

Corollary 1.9. *Every projective variety is complete. Moreover, every morphism of varieties* $f\colon X \to Y$*, with* X *projective, is proper; in particular, it is closed.*

Proof. This follows from the theorem, using various assertions in Proposition 1.4. Since X is a projective variety, there is a closed immersion $i\colon X \hookrightarrow \mathbf{P}^n$ for some n. Note that i is proper by assertion iii) in the proposition and $\mathbf{P}^n$ is complete by the theorem, hence we conclude that X is complete, using assertion i) in the proposition. The fact that every morphism $X \to Y$ is proper now follows from assertion iv) in the proposition. □

Corollary 1.10. *If S is a reduced, $\mathbf{N}$-graded, finitely generated k-algebra, generated as an S_0-algebra by S_1, then the canonical morphism $f\colon \mathrm{MaxProj}(S) \to \mathrm{MaxSpec}(S_0)$ is proper.*

Proof. The morphism f factors as

$$\mathrm{MaxProj}(S) \overset{i}{\hookrightarrow} \mathrm{MaxSpec}(S_0) \times \mathbf{P}^n \xrightarrow{p} \mathrm{MaxSpec}(S_0),$$

where i is a closed immersion and p is the projection. Since $\mathbf{P}^n$ is complete, we deduce that p is proper by assertion ii) in Proposition 1.4 and i is a closed immersion by assertion iii) in the proposition. We thus conclude that f is proper by assertion i) in the proposition. □

For the sake of completeness, we mention the following embedding theorem. Its proof is more involved (see, for example, [Con07]).

Theorem 1.11. *(Nagata, Deligne) For every algebraic variety X, there is an open immersion $i\colon X \hookrightarrow Y$, where Y is complete. More generally, every morphism of algebraic varieties $f\colon X \to Z$ factors as a composition*

$$X \overset{i}{\hookrightarrow} Y \xrightarrow{p} Z,$$

with i an open immersion and p a proper morphism.

The next exercise deals with an important example of a proper, birational morphism: the *blow-up* of the affine space at the origin.

Exercise 1.12. Thinking of $\mathbf{P}^{n-1}$ as the set of lines in $\mathbf{A}^n$, define the *blow-up of $\mathbf{A}^n$ at 0* as the set

$$\mathrm{Bl}_0(\mathbf{A}^n) := \{(P, \ell) \in \mathbf{A}^n \times \mathbf{P}^{n-1} \mid P \in \ell\}.$$

1) Show that $\mathrm{Bl}_0(\mathbf{A}^n)$ is a closed subset of $\mathbf{A}^n \times \mathbf{P}^{n-1}$.
2) Show that the restriction of the projection onto the first component gives a morphism $\pi\colon \mathrm{Bl}_0(\mathbf{A}^n) \to \mathbf{A}^n$ that is an isomorphism over $\mathbf{A}^n \smallsetminus \{0\}$.
3) Show that $\pi^{-1}(0) \simeq \mathbf{P}^{n-1}$.
4) Show that π is a proper morphism.

2. Chow's lemma

In this section we discuss a result that is very useful in reducing statements about complete varieties to the case of projective varieties. More generally, it allows reducing statements about proper morphisms to a special case of what we will later define as projective morphisms. In order to make things more transparent, we begin with the statement in the absolute case.

Theorem 2.1. *(Chow's lemma) If X is a complete variety, then there is a projective variety Y and a morphism $g\colon Y \to X$ that induces an isomorphism between dense open subsets of Y and X.*

Here is the relative version of the above result:

Theorem 2.2. *(Chow's lemma, relative version) If $f\colon X \to Z$ is a proper morphism of algebraic varieties, then there is a morphism $g\colon Y \to X$ that satisfies the following conditions:*

i) *The morphism g induces an isomorphism between dense open subsets of Y and X.*
ii) *The composition $f \circ g$ factors as*

$$Y \overset{i}{\hookrightarrow} Z \times \mathbf{P}^N \overset{p}{\longrightarrow} Z,$$

where i is a closed immersion, N is a positive integer, and p is the projection onto the first factor.

Of course, it is enough to only prove the relative statement. We give the proof following [Mum88].

Proof of Theorem 2.2. Note first that we may assume that X is irreducible. Indeed, if $X_1, \ldots, X_r$ are the irreducible components of X and if we can construct morphisms $Y_i \to X_i$ as in the theorem, then we have an induced morphism $Y = \bigsqcup_i Y_i \to X$ which satisfies the required conditions (note that if we have closed immersions $Y_i \hookrightarrow Z \times \mathbf{P}^{n_i}$, then we can construct a closed immersion $Y \hookrightarrow Z \times \mathbf{P}^d$, where $d+1 = \sum_{i=1}^r (n_i+1)$, by embedding the $\mathbf{P}^{n_i}$ in $\mathbf{P}^d$ as disjoint linear subspaces).

Suppose now that X is irreducible and consider an affine open cover $X = U_1 \cup \ldots \cup U_n$. Since each U_i is an affine variety, it admits a locally closed immersion in a projective space $\mathbf{P}^{m_i}$. We thus obtain a morphism $U_i \hookrightarrow Z \times \mathbf{P}^{m_i}$ which is again a locally closed immersion (see Exercise 1.7) and we denote its image by $\overline{U_i}$. Using the Segre embedding we see that we have a closed immersion

$$\overline{U_1} \times_Z \ldots \times_Z \overline{U_n} \hookrightarrow Z \times \mathbf{P}^{m_1} \times \ldots \times \mathbf{P}^{m_n} \hookrightarrow Z \times \mathbf{P}^N,$$

where $N + 1 = \prod_i (m_i + 1)$.

Let $U^* = U_1 \cap \ldots \cap U_n$. Since X is irreducible, U^* is a nonempty open subset of X. We consider two locally closed immersions. First, we have

$$\alpha\colon U^* \to \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$$

that on each component is given by the corresponding inclusion map. This is a locally closed immersion since it factors as the composition

$$U^* \to U^* \times_Z \times \ldots \times_Z U^* \to \overline{U_1} \times_Z \ldots \times_Z \overline{U_n},$$

with the first map being a diagonal map (hence a closed immersion) and the second being a product of open immersions (hence an open immersion). We denote by W the closure of $\alpha(U^*)$. Since W is a closed subvariety of $\overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$, we see that the canonical morphism $W \to Z$ factors as

$$W \hookrightarrow Z \times \mathbf{P}^N \to Z,$$

where the first morphism is a closed immersion and the second morphism is the projection onto the first component.

We also consider the map

$$\beta \colon U^* \to X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$$

that on each component is given by the corresponding inclusion. Again, this is a locally closed immersion, and we denote the closure of its image by Y. It is clear that the projection onto the last n components

$$X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n} \to \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$$

induces a morphism $q \colon Y \to W$, while the projection onto the first component

$$X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n} \to X$$

induces a morphism $g \colon Y \to X$. The restriction of g to U^* is the identity, hence g is birational. Note that q is a closed map, since f is proper. In particular, since its image contains the dense open subset U^*, it follows that q is surjective.

The key assertion is that q is an isomorphism. Once we know this, we see that $f \circ g$ factors as

$$Y \hookrightarrow Z \times \mathbf{P}^N \to Z,$$

with the first map being a closed immersion, and therefore g has the required properties.

In order to show that q is an isomorphism, we consider for every i the map

$$\alpha_i \colon U_i \hookrightarrow X \times_Z \overline{U_i},$$

given by the inclusion on each component. This is again a locally closed immersion. Moreover, since the maps

$$U_i \hookrightarrow X \times_Z U_i \quad \text{and} \quad U_i \hookrightarrow U_i \times_Z \overline{U_i}$$

are closed immersions (as the graphs of the inclusion maps $U_i \hookrightarrow X$ and $U_i \hookrightarrow \overline{U_i}$, respectively), it follows that

$$\overline{\alpha_i(U_i)} \cap (X \times_Z U_i) = \{(u, u) \mid u \in U_i\} = \overline{\alpha_i(U_i)} \cap (U_i \times_Z \overline{U_i}).$$

Consider the projection map

$$\pi_{1,i} \colon X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n} \to X \times_Z \overline{U_i}.$$

Since $\pi_{1,i}(Y) \subseteq \overline{\alpha_i(U^*)} = \overline{\alpha_i(U_i)}$, we deduce that

$$V_i := Y \cap (X \times_Z \overline{U_1} \times_Z \ldots \times_Z U_i \times_Z \ldots \times_Z \overline{U_n})$$
$$= Y \cap (U_i \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}) = Y \cap \{(u_0, u_1, \ldots, u_n) \mid u_0 = u_i \in U_i\}.$$

The first formula for V_i shows that $V_i = q^{-1}(V_i')$, where

$$V_i' = W \cap \overline{U_1} \times_Z \ldots \times_Z U_i \times_Z \ldots \times_Z \overline{U_n}$$

is an open subset of W. From the second formula for V_i' we deduce that $Y = V_1 \cup \ldots \cup V_n$ and since q is surjective, it follows that $W = V_1' \cup \ldots \cup V_n'$.

In order to conclude the proof, it is thus enough to show that each induced morphism $V_i \to V_i'$ is an isomorphism. We define the morphism

$$\gamma_i \colon V_i' \to X \times_Z \overline{U_1} \times_Z \ldots \times_Z \overline{U_n}$$

by

$$\gamma_i(u_1, \ldots, u_n) = (u_i, u_1, \ldots, u_n).$$

This is well-defined, and since it maps U^* to U^*, it follows that its image lies inside Y. Moreover, we clearly have $q \circ \gamma_i(u_1, \ldots, u_n) = (u_1, \ldots, u_n)$; in particular, the image of γ_i lies inside V_i. Finally, if $u = (u_0, u_1, \ldots, u_n) \in V_i$, then $u_0 = u_i$ lies in U_i, hence $u = \gamma_i\big(q(u)\big)$. This shows that γ_i gives an inverse of $q|_{V_i} \colon V_i \to V_i'$ and thus completes the proof of the theorem. □

3. Finite morphisms

We discussed in Chapter 2 finite morphisms between affine varieties. We now consider the general notion.

Definition 3.1. The morphism $f \colon X \to Y$ between algebraic varieties is *finite* if for every affine open subset $V \subseteq Y$, its inverse image $f^{-1}(V)$ is an affine variety, and the induced k-algebra homomorphism

$$\mathcal{O}_Y(V) \to \mathcal{O}_X\big(f^{-1}(V)\big)$$

is finite.

It is not clear that in the case when X and Y are affine varieties, the above definition coincides with our old one. However, this follows from the following theorem.

Proposition 3.2. *Let $f \colon X \to Y$ be a morphism of algebraic varieties. If there is an affine open cover $Y = \bigcup_{i=1}^r V_i$ such that each $U_i = f^{-1}(V_i)$ is an affine variety and the induced morphism*

$$\mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$$

is finite, then f is a finite morphism.

We begin with a lemma which is useful in several other situations.

Lemma 3.3. *If X is an algebraic prevariety and $U, V \subseteq X$ are affine open subsets, then for every $p \in U \cap V$, there is open neighborhood $W \subseteq U \cap V$ of p that is a principal affine open subset in both U and V.*

Proof. We first choose an open neighborhood $W_1 \subseteq U \cap V$ of p of the form $W_1 = D_U(f)$ for some $f \in \mathcal{O}(U)$. We next choose another open neighborhood $W \subseteq W_1$ of the form $W = D_V(g)$, for some $g \in \mathcal{O}(V)$. It is enough to show that W is a principal affine open subset also in U.

Since $\mathcal{O}(W_1) \simeq \mathcal{O}(U)_f$, it follows that there is $h \in \mathcal{O}(U)$ such that $g|_{W_1} = \frac{h}{f^m}$ for some non-negative integer m. In this case we have $W = D_U(fh)$, completing the proof. □

Proof of Proposition 3.2. Note that if W is a principal affine open subset of some of the V_i, then $f^{-1}(W)$ is affine and the induced morphism

$$\mathcal{O}_Y(W) \to \mathcal{O}_X\big(f^{-1}(W)\big) \tag{1}$$

is finite. Indeed, if $W = D_{V_i}(\varphi)$, then $f^{-1}(W) = D_{U_i}(\varphi \circ f)$ is affine and the morphism (1) is identified to

$$\mathcal{O}_Y(V_i)_\varphi \to \mathcal{O}(U_i)_{\varphi\circ f},$$

which is finite.

Let $V \subseteq Y$ be an arbitrary affine open subset. Since V is covered by the open subsets $V \cap V_i$, applying for each pair (V, V_i) Lemma 3.3, and using what we have already seen, we see that we can cover V by finitely many principal affine open subsets $W_1, \ldots, W_s$, such that each $f^{-1}(W_i)$ is affine and the induced morphism

$$\mathcal{O}_Y(W_i) \to \mathcal{O}_X\big(f^{-1}(W_i)\big) \tag{2}$$

is finite. Let us write $W_i = D_V(\varphi_i)$, for some $\varphi_i \in \mathcal{O}_Y(V)$. The condition that $V = \bigcup_{i=1}^s W_i$ is equivalent to the fact that $\varphi_1, \ldots, \varphi_s$ generate the unit ideal in $\mathcal{O}_Y(V)$. This implies that the $f^{\#}(\varphi_i) = \varphi_i \circ f$ generate the unit ideal in $\mathcal{O}_X\big(f^{-1}(V)\big)$. Since each $D_{f^{-1}(V)}(\varphi_i \circ f)$ is affine, it follows from Proposition 3.16 in Chapter 3 that $f^{-1}(V)$ is affine.

Moreover, the $\mathcal{O}_Y(V)$-module $\mathcal{O}_X\big(f^{-1}(V)\big)$ has the property that $\mathcal{O}_X\big(f^{-1}(V)\big)_{\varphi_i}$ is a finitely generated module over $\mathcal{O}_Y(V)_{\varphi_i}$ for all i. Since the φ_i generate the unit ideal in $\mathcal{O}_Y(V)$, we conclude using Corollary 3.4 in Review Sheet 3 that $\mathcal{O}_X\big(f^{-1}(V)\big)$ is a finitely generated $\mathcal{O}_Y(V)$-module. □

Remark 3.4. If $f\colon X \to Y$ is a finite morphism, then for every $y \in Y$, the fiber $f^{-1}(y)$ is finite. Indeed, if V is an affine open neighborhood of y, then $U = f^{-1}(V)$ is affine and the induced morphism $f^{-1}(V) \to V$ is finite. Applying to this morphism Remark 2.7 in Chapter 2, we deduce that $f^{-1}(y)$ is finite.

In the next proposition we collect some general properties of finite morphisms.

Proposition 3.5. *In what follows, all objects are algebraic varieties.*

i) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are finite morphisms, then $g \circ f\colon X \to Z$ is a finite morphism.*
ii) *If $f\colon X \to Y$ is a finite morphism, then for every morphism $g\colon Z \to Y$, the induced morphism $h\colon X \times_Y Z \to Z$ is a finite morphism.*
iii) *Every closed immersion $i\colon X \hookrightarrow Y$ is a finite morphism.*
iv) *If $f\colon X \to Y$ is a morphism and $Y = V_1 \cup \ldots \cup V_r$ is an open cover such that each induced morphism $f^{-1}(V_i) \to V_i$ is finite, then f is finite.*

Proof. The assertions in i) and iii) are straightforward to see and the one in iv) follows by covering each V_i by affine open subsets and then using Proposition 3.2. We now prove the assertion in ii). Let $V = V_1 \cup \ldots \cup V_r$ be an affine open cover of Y. For every i, consider an affine open cover $g^{-1}(V_i) = \bigcup_j U_{i,j}$. Note that we have

$$h^{-1}(U_{i,j}) = f^{-1}(V_i) \times_{V_i} U_{i,j}.$$

Using Proposition 3.2, we thus see that it is enough to prove the assertion when X, Y, and Z are affine varieties. In this case, $X \times_Y Z$ is affine, since it is a closed subvariety of $X \times Z$ (see Proposition 4.7 in Chapter 3). Moreover, the morphism

$$h^{\#} \colon \mathcal{O}(Z) \to \mathcal{O}(X \times_Y Z)$$

factors as

$$\mathcal{O}(Z) = \mathcal{O}(Y) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Z) \xrightarrow{f^{\#} \otimes 1} \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Z) \xrightarrow{p} \mathcal{O}(X \times_Y Z).$$

The homomorphism $f^{\#} \otimes 1$ is finite since $f^{\#}$ is finite and p is surjective (this follows, for example, from the fact that $X \times_Y Z$ is a closed subvariety of $X \times Z$, but see also Remark 4.9 in Chapter 3 for a more precise statement). This completes the proof of ii). □

The next proposition extends to arbitrary morphisms some properties that we have already proved for finite morphisms between affine varieties.

Proposition 3.6. *Let $f \colon X \to Y$ be a finite morphism.*

1) *The map f is closed.*
2) *If $Z_1 \subsetneq Z_2$ are irreducible closed subsets of X, then $f(Z_1) \subsetneq f(Z_2)$ are irreducible, closed subsets of Y.*
3) *If f is surjective, then given any irreducible, closed subset W of Y, there is an irreducible, closed subset Z in X such that $f(Z) = W$.*
4) *If Z_1 is an irreducible, closed subset of X and $W_1 \supseteq W_2$ are irreducible, closed subsets of Y, with $W_1 = f(Z_1)$, then there is $Z_2 \subseteq Z_1$ irreducible and closed such that $f(Z_2) = W_2$.*

Proof. We have already seen these properties when X and Y are affine varieties in Corollary 2.9 in Chapter 2. Let $Y = V_1 \cup \ldots \cup V_r$ be an affine open cover of Y. By definition, each $f^{-1}(V_i)$ is affine and the induced morphism $f^{-1}(V_i) \to V_i$ is finite, hence it satisfies the properties in the proposition. Since each map $f^{-1}(V_i) \to V_i$ is closed, it follows that f is closed, hence we have 1). The assertions in 2), 3), and 4) similarly follow from the corresponding ones for the morphisms $f^{-1}(V_i) \to V_i$. □

Corollary 3.7. *Every finite morphism $f \colon X \to Y$ is proper.*

Proof. Given any morphism of varieties $g \colon Z \to Y$, assertion ii) in Proposition 3.5 implies that the induced morphism $X \times_Y Z \to Z$ is finite. This is thus closed by assertion 1) in Proposition 3.6, which shows that f is proper. □

We mention the following converse to Corollary 3.7. This is a deeper result that we will only prove later.

Theorem 3.8. *If $f \colon X \to Y$ is a proper morphism with finite fibers, then f is finite.*

The following proposition gives another property of finite morphisms that we have seen for affine varieties.

Proposition 3.9. *If $f\colon X\to Y$ is a finite, surjective morphism of algebraic varieties, then for every closed subset Z of X, we have*

$$\dim\big(f(Z)\big)=\dim(Z).$$

Moreover, if Z is irreducible, then

$$\mathrm{codim}_Y\big(f(Z)\big)=\mathrm{codim}_X(Z).$$

Proof. This can be deduced from the properties in Proposition 3.6 as in the proof of Corollary 2.10 in Chapter 2. ☐

We end this section by introducing another class of morphisms.

Definition 3.10. A morphism of algebraic varieties $f\colon X\to Y$ is *affine* if for every affine open subset $V\subseteq Y$, its inverse image $f^{-1}(V)$ is affine.

The next proposition shows that, in fact, it is enough to check the property in the definition for an affine open cover of the target. In particular, this implies that every morphism of affine varieties is affine.

Proposition 3.11. *Let $f\colon X\to Y$ be a morphism of algebraic varieties. If there is an open cover $Y=V_1\cup\ldots\cup V_r$, with each V_i affine, such that all $f^{-1}(V_i)$ are affine, then f is an affine morphism.*

Proof. The argument follows as in the proof of Proposition 3.2. ☐

4. Semicontinuity of fiber dimension for proper morphisms

Our goal in this section is to prove the following semicontinuity result for the dimensions of the fibers of a proper morphism.

Theorem 4.1. *If $f\colon X\to Y$ is a proper morphism of algebraic varieties, then for every non-negative integer m, the set*

$$\{y\in Y\mid \dim\big(f^{-1}(y)\big)\geq m\}$$

is closed in Y.

This is an immediate consequence of the following more technical statement, but which is valid for an arbitrary morphism.

Proposition 4.2. *If $f\colon X\to Y$ is a morphism of algebraic varieties, then for every non-negative integer m, the set X_m consisting of those $x\in X$ such that the fiber $f^{-1}\big(f(x)\big)$ has an irreducible component of dimension $\geq m$ passing through x, is closed.*

Proof. Arguing by Noetherian induction, we may assume that the assertion in the proposition holds for every $f|_Z$, where Z is a proper closed subset of X. If X is not irreducible and $X^{(1)}, \ldots, X^{(r)}$ are the irreducible components of X, we know that each $X_m^{(j)}$ is closed in $X^{(j)}$, hence in X. Since

$$X_m = \bigcup_{j=1}^{r} X_m^{(j)},$$

we conclude that X_m is closed.

Therefore we may and will assume that X is irreducible. Of course, we may replace Y by $\overline{f(X)}$ and thus assume that Y is irreducible and f is dominant. In this case, if $m \leq \dim(X) - \dim(Y)$, then $X_m = X$ by Theorem 4.1 in Chapter 2, hence we are done. On the other hand, it follows from Theorem 4.2 in Chapter 2 that there is an open subset V of Y such that if $y \in V$, then every irreducible component of $f^{-1}(y)$ has dimension equal to $\dim(X) - \dim(Y)$. We deduce that if $m > \dim(X) - \dim(Y)$ and we put $Z = X \smallsetminus f^{-1}(V)$, then Z is a proper closed subset of X such that $X_m = Z_m$. Since Z_m is closed in Z, hence in X, by the inductive assumption, we are done. □

Proof of Theorem 4.1. With the notation in the proposition, we have

$$\{y \in Y \mid \dim\big(f^{-1}(y)\big) \geq m\} = f(X_m).$$

Since X_m is closed and f is proper, it follows that $f(X_m)$ is closed. □

Remark 4.3. If $f\colon X \to Y$ is an arbitrary morphism of algebraic varieties, we can still say that the subset

$$\{y \in Y \mid \dim\big(f^{-1}(y)\big) \geq m\}$$

is constructible in Y. Indeed, with the notation in Proposition 4.2, we see that this set is equal to $f(X_m)$. Since X_m is closed in X by the proposition, its image $f(X_m)$ is constructible by Theorem 5.3 in Chapter 2.

Note that also the set

$$\{y \in Y \mid \dim\big(f^{-1}(y)\big) = m\}$$

is constructible in Y, being the difference of two constructible subsets.

5. An irreducibility criterion

The following result is an useful irreducibility criterion.

Proposition 5.1. *Let $f\colon X \to Y$ be a morphism of algebraic varieties. Suppose that Y is irreducible and that all fibers of f are irreducible, of the same dimension d (in particular, f is surjective). If either one of the following two conditions holds:*

a) *X is pure-dimensional;*
b) *f is closed,*

then X is irreducible, of dimension $d + \dim(Y)$.

We will be using the proposition for proper morphisms f, so that condition b) will be automatically satisfied.

Proof of Proposition 5.1. We will show that in general–that is, without assuming a) or b)– the following assertions hold:

i) There is a unique irreducible component of X that dominates Y, and
ii) Every irreducible component Z of X is a union of fibers of f. Its dimension is equal to $\dim\left(\overline{f(Z)}\right) + d$.

Let $X = X_1 \cup \ldots \cup X_r$ be the irreducible decomposition of X. For every $y \in Y$, we put $X_y = f^{-1}(y)$, and $(X_j)_y = X_y \cap X_j$. Since $X_y = \bigcup_{j=1}^{r} (X_j)_y$, and since X_y is irreducible, it follows that for every y, there is j such that $X_y = (X_j)_y$.

For every i, let $U_i := X_i \smallsetminus \bigcup_{j \neq i} X_j$. This is a nonempty open subset of X. Note that if $y \in f(U_i)$, then X_y can't be contained in $(X_j)_y$ for any $j \neq i$. It follows that

$$X_y = (X_i)_y \quad \text{for all} \quad y \in f(U_i). \tag{3}$$

Note that some X_ℓ has to dominate Y: since f is surjective, we have $Y = \bigcup_j \overline{f(X_j)}$, and since Y is irreducible, we see that there is ℓ such that $Y = \overline{f(X_\ell)}$. In this case we also have $Y = \overline{f(U_\ell)}$, and Theorem 4.2 in Chapter 2 implies that there is an open subset V of Y contained in $f(U_\ell)$. We deduce from (3) that $X_y = (X_\ell)_y$ for every $y \in V$, hence for all $j \neq \ell$, we have $X_j \smallsetminus X_\ell \subseteq f^{-1}(Y \smallsetminus V)$. Therefore $X_j = \overline{X_j \smallsetminus X_\ell}$ is contained in $f^{-1}(Y \smallsetminus V)$ (which is closed). We conclude that X_j does not dominate Y for any $j \neq \ell$.

On the other hand, it follows from Theorems 4.1 and 4.2 in Chapter 2 that for every i, the following hold

α) $\dim(X_i)_y \geq \dim(X_i) - \dim\left(\overline{f(X_i)}\right)$ for every $y \in f(X_i)$ and
β) There is an open subset W_i in $\overline{f(X_i)}$ such that for all $y \in W_i$ we have $\dim(X_i)_y = \dim(X_i) - \dim\left(\overline{f(X_i)}\right)$.

Since $W_i \cap f(U_i) \neq \emptyset$, it follows from β) and (3) that $d = \dim(X_i) - \dim\left(\overline{f(X_i)}\right)$ for every i. Furthermore, for every $y \in f(X_i)$, we know by α) that $(X_i)_y$ is a closed subset of dimension d of the irreducible variety X_y of dimension d. Therefore $X_y = (X_i)_y$ for all $y \in f(X_i)$, which says that each X_i is a union of fibers of f. Therefore assertions i) and ii) hold.

In particular, it follows from i) and ii) that if $i \neq \ell$, then $\overline{f(X_i)}$ is a proper subset of Y, and

$$\dim(X_i) = d + \dim(\overline{f(X_i)}) < d + \dim(Y) = \dim(X_\ell).$$

If X is pure-dimensional, then we conclude that X is irreducible.

Suppose now that f is a closed map. Since $f(X_\ell)$ is closed, it follows that $f(X_\ell) = Y$. We have seen that X_ℓ is a union of fibers of f, hence $X_\ell = X$. Therefore X is irreducible also in this case. □

Example 5.2. Consider the incidence correspondence between points and hyperplanes in $\mathbf{P}^n$, defined as follows. Recall that $(\mathbf{P}^n)^*$ is the projective space parametrizing the hyperplanes in $\mathbf{P}^n$. We write $[H]$ for the point of $(\mathbf{P}^n)^*$ corresponding to the hyperplane H. Consider the following subset of $\mathbf{P}^n \times (\mathbf{P}^n)^*$:

$$\mathcal{Z} = \left\{ \big(p, [H]\big) \in \mathbf{P}^n \times (\mathbf{P}^n)^* \mid p \in H \right\}.$$

Note that if we take homogeneous coordinates $x_0, \ldots, x_n$ on $\mathbf{P}^n$ and $y_0, \ldots, y_n$ on $(\mathbf{P}^n)^*$, then $\mathcal{Z}$ is defined by the condition $\sum_{i=0}^n x_i y_i = 0$. It is the straightforward to see, by considering the products of the affine charts on $\mathbf{P}^n$ and $(\mathbf{P}^n)^*$, that $\mathcal{Z}$ is a closed subset of $\mathbf{P}^n \times (\mathbf{P}^n)^*$. The projections on the two components induce morphisms $\pi_1 \colon \mathcal{Z} \to \mathbf{P}^n$ and $\pi_2 \colon \mathcal{Z} \to (\mathbf{P}^n)^*$. For every $[H] \in (\mathbf{P}^n)^*$, we have $\pi_2^{-1}([H]) \simeq H$, hence all fibers of π_2 are irreducible, of dimension $n-1$. Since $(\mathbf{P}^n)^*$ is irreducible, it follows from Proposition 5.1 that $\mathcal{Z}$ is irreducible, of dimension $2n-1$. Note that the picture is symmetric: for every $p \in \mathbf{P}^n$, the fiber $\pi_1^{-1}(p)$ consists of all hyperplanes in $\mathbf{P}^n$ that contain p, which is a hyperplane in $(\mathbf{P}^n)^*$.

6. Flat morphisms

We begin by reviewing the concept of a flat module. Recall that if M is a module over a commutative ring A, then the functor $M \otimes_A -$ from the category of A-modules to itself, is right exact. The module M is *flat* if, in fact, this is an exact functor. Given a ring homomorphism $\varphi \colon A \to B$, we say that φ is flat (or that B is a flat A-algebra) if B is flat as an A-module.

Example 6.1. The ring A is flat as an A-module, since $A \otimes_A M \simeq M$ for every A-module M.

Example 6.2. A direct sum of flat A-modules is flat, since tensor product commutes with direct sums and taking a direct sum is an exact functor. It follows from the previous example that every free module is flat. In particular, every vector space over a field is flat.

Example 6.3. If $(M_i)_{i \in I}$ is a filtered direct system of flat A-modules, then $M = \varinjlim_{i \in I} M_i$ is a flat A-module. Indeed, since the tensor product commutes with direct limits, for every injective morphism of A-modules $N_1 \hookrightarrow N_2$, the induced morphism

$$N_1 \otimes_A M \to N_2 \otimes_A M$$

can be identified with the direct limit of the injective morphisms

$$N_1 \otimes M_i \to N_2 \otimes_A M_i.$$

Since a filtered direct limit of injective morphisms is injective, we obtain our assertion.

Example 6.4. If M is a flat A-module, then for every non-zero-divisor $a \in A$, multiplication by a is injective on A, and after tensoring with M, we see that multiplication by a is injective also on M. In particular, if A is a domain, then M is torsion-free.

The converse holds if A is a PID: every torsion-free A-module is flat. Indeed, M is the filtered direct limit of its finitely generated submodules, which are free A-modules, being finitely generated and torsion-free over a PID. Since every filtered direct limit of flat modules is flat, we conclude that M is flat.

Example 6.5. For every ring A and every multiplicative system $S \subseteq A$, the A-algebra $S^{-1}A$ is flat. Indeed, for every A-module N, we have a canonical isomorphism

$$S^{-1}A \otimes_A N \simeq S^{-1}N$$

and the functor taking N to $S^{-1}N$ is exact.

We do not discuss the more subtle aspects of flatness, which we do not need at this point, and whose treatment is better handled using the Tor functors. We only collect in the next proposition some very easy properties that we need in order to define flatness for morphisms of algebraic varieties.

Proposition 6.6. *Let M be an A-module.*

i) *If M is flat, then for every ring homomorphism $A \to B$, the B-module $M \otimes_A B$ is flat.*
ii) *If $B \to A$ is a flat homomorphism and M is flat over A, then M is flat over B.*
iii) *If $\mathfrak{p}$ is a prime ideal in A and M is an $A_{\mathfrak{p}}$-module, then M is flat over A if and only if it is flat over $A_{\mathfrak{p}}$.*
iv) *If $B \to A$ is a ring homomorphism, then M is flat over B if and only if for every prime (respectively, maximal) ideal $\mathfrak{p}$ in A, the B-module $M_{\mathfrak{p}}$ is flat.*

Proof. The assertion in i) follows from the fact that for every B-module N, we have a canonical isomorphism

$$(M \otimes_A B) \otimes_B N \simeq M \otimes_A N.$$

Similarly, the assertion in ii) follows from the fact that for every B-module N, we have a canonical isomorphism

$$N \otimes_B M \simeq (N \otimes_B A) \otimes_A M.$$

With the notation in iii), note that if M is a flat $A_{\mathfrak{p}}$-module, since $A_{\mathfrak{p}}$ is a flat A-algebra, we conclude that M is flat over A by ii). The converse follows from the fact that if N is an $A_{\mathfrak{p}}$-module, then we have a canonical isomorphism

$$N \otimes_{A_{\mathfrak{p}}} M \simeq N \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \otimes_A M) \simeq N \otimes_A M.$$

We now prove iv). Suppose first that M is flat over B and let $\mathfrak{p}$ be a prime ideal in A. We deduce that $M_{\mathfrak{p}}$ is flat over B from the fact that for every B-module N, we have a canonical isomorphism

$$N \otimes_B M_{\mathfrak{p}} \simeq (N \otimes_B M) \otimes_A A_{\mathfrak{p}}.$$

Conversely, suppose that for every maximal ideal $\mathfrak{p}$ in A, the B-module $M_{\mathfrak{p}}$ is flat. Given an injective map of B-modules $N' \hookrightarrow N$, we see that for every maximal ideal $\mathfrak{p}$, the induced homomorphism

$$N' \otimes_B M_{\mathfrak{p}} \simeq (N' \otimes_B M)_{\mathfrak{p}} \to (N \otimes_B M)_{\mathfrak{p}} \simeq N \otimes_B M_{\mathfrak{p}}$$

is injective. This implies the injectivity of

$$N' \otimes_B M \to N \otimes_B M$$

by Corollary 3.3 in Review Sheet 3. $\square$

Remark 6.7. If $\varphi\colon A \to B$ is a flat homomorphism of Noetherian rings and $\mathfrak{p}$ is a prime ideal in A, then for every minimal prime ideal $\mathfrak{q}$ containing $\mathfrak{p}B$, we have $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Indeed, it follows from assertion i) in Proposition 6.6 that the morphism $A/\mathfrak{p} \to B/\mathfrak{p}B$ is flat. It then follows from Example 6.4 that if $\overline{a}$ is a nonzero element in $A/\mathfrak{p}$, then its image in $B/\mathfrak{p}B$ is a non-zero-divisor, hence it can't lie in a minimal prime ideal (see Proposition 2.1 in Review Sheet 5). This gives our assertion.

We now define flatness in our geometric context. We say that a morphism of varieties $f\colon X \to Y$ is *flat* if it satisfies the equivalent conditions in the next proposition.

Proposition 6.8. *Given a morphism of varieties $f\colon X \to Y$, the following conditions are equivalent:*

i) *For every affine open subsets $U \subseteq X$ and $V \subseteq Y$ such that $U \subseteq f^{-1}(V)$, the induced homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is flat.*
ii) *There are affine open covers $X = \bigcup_i U_i$ and $Y = \bigcup_i V_i$ such that for all i, we have $U_i \subseteq f^{-1}(V_i)$ and the induced homomorphism $\mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$ is flat.*
iii) *For every point $x \in X$, if $y = f(x)$, then the homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is flat.*

Proof. We begin by showing that ii)$\Rightarrow$iii). Given x and y as in iii) and covers as in ii), we choose i such that $x \in U_i$, in which case $y \in V_i$. Note that x corresponds to a maximal ideal $\mathfrak{p}$ in $\mathcal{O}_X(U_i)$ and y corresponds to the inverse image $\mathfrak{q}$ of $\mathfrak{p}$ in $\mathcal{O}_Y(V_i)$. Since

$$B = \mathcal{O}_Y(V_i) \to A = \mathcal{O}_X(U_i)$$

is flat, we conclude that $A_{\mathfrak{q}}$ is B-flat by property iv) in Proposition 6.6. It follows that $A_{\mathfrak{p}}$ is flat over $B_{\mathfrak{q}}$ by property ii) in the same proposition.

Since the implication i)$\Rightarrow$ii) is trivial, in order to complete the proof it is enough to show iii)$\Rightarrow$i). Let U and V be affine open subsets as in i). Given the induced homomorphism

$$B = \mathcal{O}_Y(V) \to \mathcal{O}_X(U) = A,$$

it follows from iii) that for every maximal ideal $\mathfrak{p}$ in A, if its inverse image in B is $\mathfrak{q}$, then the induced homomorphism $B_{\mathfrak{q}} \to A_{\mathfrak{p}}$ is flat. Assertion iii) in Proposition 6.6 implies that $A_{\mathfrak{p}}$ is flat over B for every $\mathfrak{p}$, in which case assertion iv) in the proposition implies that A is flat over B. □

Remark 6.9. The argument for the implication ii)$\Rightarrow$iii) in the proof of the above proposition shows that more generally, if $f\colon X \to Y$ is a flat morphism, then for every irreducible closed subset $V \subseteq X$, if $W = \overline{f(V)}$, then the induced ring homomorphism $\mathcal{O}_{Y,W} \to \mathcal{O}_{X,V}$ is flat.

Example 6.10. Every open immersion $i\colon U \hookrightarrow X$ is flat: indeed, it is clear that property iii) in the above proposition is satisfied.

Example 6.11. If X and Y are varieties, then the projection maps $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are flat. Indeed, by choosing affine covers of X and Y, we reduce to the case when both X and Y are affine. In this case, since $\mathcal{O}(Y)$ is a free k-module, it follows from assertion i) in Proposition 6.6 that $\mathcal{O}(X \times Y) \simeq \mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is flat over $\mathcal{O}(X)$. This shows that p is flat and the assertion about q follows similarly.

Remark 6.12. A composition of flat morphisms is a flat morphism. Indeed, this follows from definition and property ii) in Proposition 6.6.

Remark 6.13. If $f\colon X \to Y$ is flat and $W \subseteq Y$ is an irreducible, closed subset such that $f^{-1}(W) \neq \emptyset$, then for every irreducible component V of $f^{-1}(W)$, we have $\overline{f(V)} = W$. Indeed, we may replace X and Y by suitable affine open subsets that intersect V and W, respectively, to reduce to the case when both X and Y are affine. In this case the assertion follows from Remark 6.7.

Example 6.14. A morphism $f\colon X \to \mathbf{A}^1$ is flat if and only if every irreducible component of X dominates $\mathbf{A}^1$. The "only if" part follows from the previous remark. For the converse, note that under the hypothesis, for every affine open subset U of X, the $k[x]$-module $\mathcal{O}_X(U)$ is torsion-free: if a nonzero $u \in k[x]$ annihilates $v \in \mathcal{O}_X(U)$, it follows that every irreducible component of U on which v does not vanish is mapped by f in the zero-locus of u, a contradiction. We then deduce that f is flat using Example 6.4.

Our goal is to prove two geometric properties of flat morphisms. We begin with the following generalization of Proposition 6.6 in Chapter 1.

Theorem 6.15. *If $f\colon X \to Y$ is a flat morphism between algebraic varieties, then f is open.*

The proof will make use of the following openness criterion.

Lemma 6.16. *Let W be a subset of a Noetherian topological space Y. The set W is open if and only if whenever $Z \subseteq Y$ is a closed irreducible subset of Y such that $W \cap Z \neq \emptyset$, then W contains nonempty open subset of Z.*

Proof. The "only if" part is clear, so we only need to prove the converse. Arguing by Noetherian induction, we may assume that the assertion holds for all proper closed subspaces of Y. Let $Y_1, \ldots, Y_r$ be the irreducible components of Y. We may assume that W is nonempty, and suppose that W contains a point y in some Y_i. By hypothesis, there is a nonempty open subset $U \subseteq Y_i$ such that $U \subseteq W$. After replacing U by $U \smallsetminus \bigcup_{j \neq i} Y_j$, we may assume that $U \cap Y_j = \emptyset$ for every $j \neq i$, in which case U is open in Y.

Note that $Y \smallsetminus U$ is a proper closed subset of Y. Moreover, $W \smallsetminus U \subseteq Y \smallsetminus U$ satisfies the same hypothesis as W: if $Z \subseteq Y \smallsetminus U$ is an irreducible closed subset such that $(W \smallsetminus U) \cap Z \neq \emptyset$, then W contains a nonempty open subset of Z, hence the same holds for $W \smallsetminus U$. By induction, we conclude that $W \smallsetminus U$ is open in $Y \smallsetminus U$. This implies that W is open, since

$$Y \smallsetminus W = (Y \smallsetminus U) \smallsetminus (W \smallsetminus U)$$

is closed in $Y \smallsetminus U$, hence in Y. □

Proof of Theorem 6.15. If U is an open subset of X, we may replace f by its restriction to U, which is still flat. Therefore we only need to show that $f(X)$ is open in Y and it is enough to show that $f(X)$ satisfies the condition in the lemma. Suppose that W is an irreducible closed subset of Y such that $f(X) \cap W \neq \emptyset$. If V is an irreducible component of

$f^{-1}(W)$, then V dominates W by Remark 6.13. In this case, the image of V in W contains an open subset of W by Theorem 4.2 in Chapter 2. This completes the proof. □

Our second main property of flat morphisms will follow from the following

Proposition 6.17. *(Going Down for flat homomorphisms) If $\varphi\colon A \to B$ is a flat ring homomorphism, then given prime ideals $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ in A and $\mathfrak{q}_2$ in B such that $\varphi^{-1}(\mathfrak{q}_2) = \mathfrak{p}_2$, there is a prime ideal $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ such that $\varphi^{-1}(\mathfrak{q}_1) = \mathfrak{p}_1$.*

Proof. As we have seen in the proof of Proposition 6.8, the fact that φ is flat implies that the induced homomorphism $A_{\mathfrak{p}_2} \to B_{\mathfrak{q}_2}$ is flat. After replacing φ by this homomorphism, we may thus assume that $(A, \mathfrak{p}_2)$ and $(B, \mathfrak{q}_2)$ are local rings and φ is a local homomorphism. In this case every prime ideal in B is contained in $\mathfrak{q}_2$. Since the prime ideals in B lying over $\mathfrak{p}_1$ are in bijection with the prime ideals in $(A_{\mathfrak{p}_1}/\mathfrak{p}_1 A_{\mathfrak{p}_1}) \otimes_A B$, it is enough to show that this ring is not the zero ring.

In fact, the following more general fact is true: under our assumptions, for every nonzero A-module M, the B-module $M \otimes_A B$. is nonzero. Indeed, if $u \in M$ is nonzero and $I = \mathrm{Ann}_A(u)$, then $I \subseteq \mathfrak{p}_2$ and $Au \simeq A/I$. We thus have an inclusion $A/I \hookrightarrow M$ and the flatness assumption implies that the induced morphism $B/IB = A/I \otimes B \to M \otimes_A B$ is injective. Since $IB \subseteq \mathfrak{q}_2$, it follows that B/IB is nonzero, hence $M \otimes_A B$ is nonzero. □

Proposition 6.18. *If $\varphi\colon A \to B$ is a ring homomorphism that satisfies the Going-Down property in the previous proposition, then for every prime ideal $\mathfrak{q}$, if we put $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$, then*

$$\dim(B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}) \leq \dim(B_{\mathfrak{q}}) - \dim(A_{\mathfrak{p}}).$$

Proof. Let $r = \dim(B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}})$ and $s = \dim(A_{\mathfrak{p}})$. We can choose prime ideals $\mathfrak{p}_s \subsetneq \ldots \subsetneq \mathfrak{p}_0 = \mathfrak{p}$ in A and $\mathfrak{q}_r \subsetneq \ldots \subsetneq \mathfrak{q}_0 = \mathfrak{q}$ in B, with $\mathfrak{p}B \subseteq \mathfrak{q}_r$. Applying the Going-Down property successively, we obtain a sequence of prime ideals $\mathfrak{p}'_s \subseteq \ldots \subseteq \mathfrak{p}'_0 \subseteq \mathfrak{q}_r$ such that $\varphi^{-1}(\mathfrak{p}'_i) = \mathfrak{p}_i$ for $0 \leq i \leq s$. In particular, we have $\mathfrak{p}'_i \neq \mathfrak{p}'_{i+1}$ for $0 \leq i \leq s-1$ (however, we might have $\mathfrak{p}'_0 = \mathfrak{q}_s$). From the sequence of prime ideals in B

$$\mathfrak{p}'_s \subsetneq \ldots \subsetneq \mathfrak{p}'_1 \subsetneq \mathfrak{q}_r \subsetneq \ldots \subsetneq \mathfrak{q}_0 = \mathfrak{q},$$

we conclude that $\dim(B_{\mathfrak{q}}) \geq r + s$. □

By combining the above two propositions, we obtain the following consequence in our geometric setting:

Theorem 6.19. *If $f\colon X \to Y$ is a flat morphism between two algebraic varieties, W is an irreducible closed subset of Y such that $f^{-1}(W) \neq \emptyset$, then for every irreducible component V of $f^{-1}(W)$, we have*

$$\mathrm{codim}_X(V) = \mathrm{codim}_Y(W).$$

Proof. Note first that V dominates W (see Remark 6.13). We apply Proposition 6.18 for the flat morphism

$$\mathcal{O}_{Y,W} \to \mathcal{O}_{X,V},$$

which satisfies the Going-Down property by Proposition 6.17. Since V is an irreducible component of $f^{-1}(W)$, we obtain the inequality

$$\mathrm{codim}_X(V) \geq \mathrm{codim}_Y(W).$$

In order to prove the opposite inequality, let X' be an irreducible component of X containing V and such that $\mathrm{codim}_X(V) = \mathrm{codim}_{X'}(V)$. If Y' is an irreducible component of Y that contains $\overline{f(X')}$, then X' dominates Y' by Remark 6.13. We can thus apply Theorem 4.1 in Chapter 2 to deduce

$$\mathrm{codim}_X(V) = \mathrm{codim}_{X'}(V) \leq \mathrm{codim}_{Y'}(W) \leq \mathrm{codim}_Y(W).$$

This completes the proof of the theorem. □

Example 6.20. If $f\colon X \to Y$ is a flat morphism between algebraic varieties, with X of pure dimension m and Y of pure dimension n, then for every irreducible closed subset W of Y with $W \cap f(X) \neq \emptyset$, the inverse image $f^{-1}(W)$ has pure dimension equal to $\dim(W) + m - n$. In particular, every non-empty fiber of f has pure dimension $m - n$.

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