What is extremal combinatorics

Extremal combinatorics deals with questions of the following form: given a graph $G$ on $n$ vertices, how big certain global invariants can be when $G$ does not exhibit certain local substructures? What are the graphs without such local substructures that have maximal global invariants?

The questions we will discuss: how many edges can $G$ have if $G$ does not have certain types of subgraphs.

We start with the easiest such result. All graphs in these two lectures will be simple.

**Proposition 1.** If $G$ is a bipartite graph on $n$ vertices, then $G$ can have $\leq \frac{n^2}{4}$ edges.

**Remark.** This falls under the previous paradigm since $G$ is bipartite if and only if $G$ does not contain any polygon of odd length.
The maximal number of edges of a bipartite graph

Proof of the proposition. If $G$ is a bipartite graph on $n$ vertices, then we have disjoint subsets $A$ and $B$ of $V(G)$, with $k$ and $n - k$ elements, such that every edge of $G$ joins a vertex in $A$ with a vertex in $B$. It follows that

$$\#E(G) \leq k(n - k) \leq \left( \frac{k + (n - k)}{2} \right)^2 = \frac{n^2}{4},$$

where the inequality is given by the inequality between the arithmetic and geometric means.

Proof of the remark. The fact that a bipartite graph contains no polygon of odd length is clear. Suppose now that $G$ contains no such polygon and let’s show that $G$ is bipartite. By treating each connected component of $G$ separately, we may assume $G$ is connected. Fix a vertex $v_0$ of $G$ and let $A$ and $B$ be the subsets of $V(G)$ that can be connected to $v_0$ via a path of even (respectively, odd) length. We clearly have $A \cup B = V(G)$ and the hypothesis implies $A \cap B = \emptyset$. Moreover, it implies that no two vertices in $A$ (or $B$) can be neighbors in $G$. Hence $G = G(A, B)$ is bipartite.
Graphs with no triangles

With a bit more work, we can prove a stronger version of the previous result.

**Proposition 2** (Mantel, 1907). If $G$ is a graph on $n$ vertices that does not contain any triangle, then $\#E(G) \leq \frac{n^2}{4}$.

**Proof.** Let $m = \#E(G)$. For every $x \in V(G)$, let $d_x = \text{deg}(x)$. Given any two adjacent vertices $x, y \in V(G)$, the hypothesis implies that $x$ and $y$ have no common neighbors. We thus have $d_x + d_y \leq n$. Therefore

$$\sum_{\{x,y\} \in E(G)} (d_x + d_y) \leq mn.$$ 

On the other hand, we have

$$\sum_{\{x,y\} \in E(G)} (d_x + d_y) = \sum_{x \in V(G)} d_x^2.$$
Recall now that the Cauchy-Schwarz inequality says that for every $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, we have

$$\left( \sum_{i=1}^{n} a_i^2 \right) \cdot \left( \sum_{i=1}^{n} b_i^2 \right) \geq \left( \sum_{i=1}^{n} a_i b_i \right)^2.$$ 

By taking $b_1 = \ldots = b_n = 1$, we obtain

$$n \cdot \left( \sum_{i=1}^{n} a_i^2 \right) \geq \left( \sum_{i=1}^{n} a_i \right)^2.$$

Applying this and our inequalities, we obtain

$$mn^2 \geq n \cdot \sum_{x \in V(G)} d_x^2 \geq \left( \sum_{x \in V(G)} d_x \right)^2 = (2m)^2 = 4m^2,$$

hence $m \leq n^2/4$. 

Remark. It follows from the proof of Proposition 2 that if $G$ is as in the proposition and $m = \frac{n^2}{4}$, then for every edge $\{x, y\}$ in $E(G)$, every vertex of $G$ is a neighbor of precisely one of $x$ and $y$. If $A$ consists of the neighbors of $x$ and $B$ consists of the neighbors of $y$, since $G$ contains no triangles, we see that $G$ is the complete bipartite graph on $A$ and $B$. If $a = \#A$ and $b = \#B$, then $n = a + b$ and $\#E(G) = ab = (a + b)^2 / 4$, hence $a = b$.

Remark. If $n$ is odd and $G$ is a graph on $n$ vertices that does not contain any triangles, then it follows from Proposition 2 that $\#E(G) \leq \frac{n^2 - 1}{4}$. This is sharp, as can be seen by taking the complete bipartite graph $K_{k+1,k}$, where $k = \frac{n-1}{2}$.

Exercise. Show that in this case, too, if $\#E(G) = \frac{n^2 - 1}{4}$, then $G$ is isomorphic to $K_{k+1,k}$. 
Graphs with no $K_{t+1}$

Next: we want to extend the result in Proposition 2, by asking what is the maximum number of edges for a graph on $n$ vertices that does not contain any $K_{t+1}$, for some $t \geq 2$ (recall: $K_{t+1}$ is the complete graph on $t + 1$ vertices).

How to construct graphs without any $K_{t+1}$: let $V = V_1 \sqcup \ldots \sqcup V_t$. If we consider the graph $G$ with $V(G) = V$ and where the edges in $G$ are given by all edges that connect vertices in different $V_i$, then $G$ does not contain any $K_{t+1}$ (for any $t + 1$ vertices of $G$, two of them must lie in the same $V_i$ by the pigeonhole principle).

Note: in this example, if $d_i = \# V_i$, then $\# V = d_1 + \ldots + d_t$ and $\# E(G) = \sum_{i<j} d_i d_j$.

Easy fact: if $n = d_1 + \ldots + d_t$ is fixed, then $\sum_{i<j} d_i d_j$ is maximal precisely when $d_i - d_j \in \{0,1,-1\}$ for all $i,j$. 
Graphs with no $K_{t+1}$, cont’d

Let’s prove this fact when $t$ divides $n$. In this case, we have

$$\sum_{i<j} d_id_j = \frac{(d_1 + \ldots + d_t)^2 - (d_1^2 + \ldots + d_t^2)}{2}.$$ 

The Cauchy-Schwarz inequality gives

$$t \cdot (d_1^2 + \ldots + d_t^2) \geq (d_1 + \ldots + d_t)^2$$

(it also says that equality holds iff $d_1 = \ldots = d_t$). We thus conclude that

$$\sum_{i<j} d_id_j \leq \frac{n^2 - \frac{n^2}{t}}{2} = \frac{n^2}{2} \left(1 - \frac{1}{t}\right)$$

(with equality iff $d_i = n/t$ for all $i$).
The proof of the fact in the general setting (when $t$ might not divide $n$ is similar). Note that the condition $d_i - d_j \in \{0, 1, -1\}$ for all $i$ and $j$ uniquely determines the graph we constructed, up to isomorphism. Indeed, if $n = tk + r$, with $r \in \{0, 1, \ldots, t - 1\}$, then after reordering the $V_i$, we may assume that $d_i = k + 1$ for $1 \leq i \leq r$ and $d_i = k$ for $r + 1 \leq i \leq t$. The corresponding graph is the Turán graph $T_{n,t}$.

**Remark.** The number of edges of the Turán graph $T_{n,t}$ is equal to

\[
\frac{n^2}{2} \left(1 - \frac{1}{t}\right) + \text{lower degree terms in } n. \tag{1}
\]

This is immediate if $t$ divides $n$: in this case every $V_i$ has $\frac{n}{t}$ elements, hence the number of edges is

\[
\frac{1}{2} t \cdot \frac{n}{t} \cdot \left(n - \frac{n}{t}\right) = \frac{n^2}{2} \left(1 - \frac{1}{t}\right).
\]

**Exercise.** Prove the formula in (1) in general.
Pál Turán is another famous Hungarian mathematician, who lived 1910-1976. Worked mostly in number theory, but also in analysis and graph theory. He collaborated extensively with Paul Erdős. As a Jew, he could not get a university job for several years and was sent to labour service at various times between 1940-44. Peter Franks: “Mathematicians have only paper and pen, he doesn’t have anything in camp. So he created extremal combinatorics for which he did not need either.”
Turán’s theorem

**Theorem** (Turán). For every $t \geq 2$, among all graphs on $n$ vertices that contain no $K_{t+1}$, the graph $T_{n,t}$ has the most edges; moreover, it is unique with this property.

**Proof.** We argue by induction on $t \geq 2$. For $t = 2$, we have seen that among the graphs on $n$ vertices that contain no triangles, the unique one with the maximal number of edges is $K_{k,k}$ (if $n = 2k$ is even) or $K_{k+1,k}$ if $n = 2k + 1$ is odd. We now consider $t \geq 3$ and assume the theorem known for $t - 1$.

Let $G$ be a graph on $n$ vertices with no $K_{t+1}$ subgraphs. Choose $v \in V(G)$ such that its degree $d_v := \deg_G(v)$ is maximal. Let $S_v \subseteq V(G)$ be the subset consisting of the neighbors of $v$ (hence $\#S_v = d_v$) and $T_v = V(G) \setminus S_v$.

Clear: since $v$ is a common neighbor of all vertices in $S_v$, the assumption on $G$ implies that the subgraph of $G$ spanned by $S_v$ contains no $K_t$ subgraph.
Turán’s theorem

We now modify $G$ to get a new graph $G'$ with $V(G') = V(G)$, as follows:
- We keep all edges in $E(G)$ between the vertices in $S_v$.
- We add edges between the vertices in $S_v$ and the vertices in $T_v$.
- We remove all edges in $E(G)$ between the vertices in $T_v$.

Claim. For every vertex $w$ in $V(G)$, we have $\deg_{G'}(w) \geq \deg_G(w)$.
- This is clear if $w \in S_v$ (we have only added some new edges incident to $w$).
- If $w \in T_v$, then

$$\deg_{G'}(w) = \#S_v = d_v \geq \deg_G(w),$$

by the maximality in our choice of $v$.

The claim, together with the formula relating the sum of the degrees with the number of edges gives

$$\#E(G') \geq \#E(G).$$
Turán’s theorem

By induction hypothesis for the subgraph $\langle S_v \rangle$ of $G$ spanned by $S_v$:

$$\#E(\langle S_v \rangle) \leq \#E(T_{d_v, t-1}).$$

Since $G'$ contains no edges between vertices in $T_v$ and contains all edges between vertices in $S_v$ and vertices in $T_v$, we conclude that

$$\#E(G) \leq \#E(G') = (#T_v) \cdot d_v + \#E(\langle S_v \rangle)$$

$$\leq (#T_v) \cdot d_v + \#E(T_{d_v, t-1}) \leq \#E(T_{n, t}).$$

Furthermore, if $\#E(G) = \#E(T_{n, t})$, then we see that in $G$ already every vertex in $S_v$ was adjacent to every vertex in $T_v$ (otherwise, by the previous computation $\#E(G') > \#E(G)$). Moreover, $S_v$ has to be isomorphic to $T_{d_v, t-1}$. Since $G$ contains no $K_{t+1}$, it follows that no vertices of $T_v$ are adjacent in $G$. Hence $G$ is constructed out of subsets $V_1, \ldots, V_t$. Maximality in the number of edges implies that $G$ is isomorphic to $T_{n, t}$. 
The Erdős-Stone theorem

We now consider the following general problem. Given a fixed graph $H$, estimate for large $n$ the number $\text{ex}(n, H)$ consisting of the maximum number of edges of a graph on $n$ vertices that does not contain any graphs isomorphic to $H$. The answer is provided by the Erdős-Stone theorem. Of course, this is not interesting if $E(H) = \emptyset$ (in which case every graph with $n \geq \#V(H)$ vertices contains a subgraph isomorphic to $H$).

Recall that the chromatic number $\chi(H)$ is the smallest number $c$ such that the vertices of $H$ can be colored with $c$ colors such that no two adjacent vertices get the same color. Note that $\chi(H) = 1$ if and only if $\#E(G) = \emptyset$.

**Theorem** (Erdős-Stone). For every (finite simple) graph $H$ with $E(G) \neq \emptyset$ and for every $\epsilon > 0$, for $n \gg 0$, we have

$$\frac{1}{2} \left( 1 - \frac{1}{\chi(H) - 1} - \epsilon \right) n^2 < \text{ex}(n, H) < \frac{1}{2} \left( 1 - \frac{1}{\chi(H) - 1} + \epsilon \right) n^2.$$
The Erdös-Stone theorem

**Example.** If \( H = K_{t+1} \) (when \( \chi(H) = t + 1 \)), this matches the assertion given by Turán’s theorem, which gives

\[
\text{ex}(n, K_{t+1}) = \frac{1}{2} \left( 1 - \frac{1}{t} \right) n^2 + \text{lower order terms in } n
\]

(depending on the residue of \( n \) when divided by \( t \)).

**Remark.** The formula in the Erdös-Stone theorem implies that if \( \chi(H) \geq 3 \), then

\[
\lim_{n \to \infty} \frac{\text{ex}(n, H)}{n^2} = \frac{1}{2} \left( 1 - \frac{1}{\chi(H) - 1} \right) > 0.
\]

Hence we understand the behavior of \( \text{ex}(n, H) \) for \( n \to \infty \). The situation is more delicate when \( \chi(H) = 2 \) (that is, for bipartite graphs), when we can only conclude that for every \( \epsilon > 0 \), we have \( \text{ex}(n, H) \leq \epsilon n^2 \) for \( n \gg 0 \). We will discuss some examples in the next lecture.
This part is easy: say $\chi(H) = k + 1$. Since the Turán graph $T_{n,k}$ has vertices in $k$ disjoint sets, with neighbors being precisely vertices in different such subsets, it follows that $\chi(T_{n,k}) = k$. Since $\chi(H) = k + 1$, we deduce that $T_{n,k}$ has no subgraph isomorphic to $H$. We thus conclude that $\text{ex}(n, H) \geq \#E(T_{n,k})$. We have seen that

$$\#E(T_{n,k}) = \frac{1}{2} \left( 1 - \frac{1}{k} \right) n^2 + \text{lower order terms in } n$$

(with the precise formula depending on the residue of $n$ divided by $t$).

It thus follows that given $\epsilon > 0$, we have

$$\text{ex}(n, H) \geq \#E(T_{n,k}) > \frac{1}{2} \left( 1 - \frac{1}{k} - \epsilon \right) n^2 \quad \text{for } n \gg 0.$$
Lemma. Given positive integers $k$ and $t$, and $0 < \epsilon < 1/k$, for every graph $G$ on $n$ vertices, with $n$ large enough (depending on $k$, $t$, and $\epsilon$) and with $m \geq \frac{1}{2} \left( 1 - \frac{1}{k} + \epsilon \right) n^2$ edges, there are disjoint subsets $A_1, \ldots, A_{k+1}$ of $V(G)$, all of size $t$, such that any two vertices in two different $A_i$ and $A_j$ are neighbors in $G$.

Proof. We first prove the following Claim:
Given any $p$ and every $\epsilon'$, with $0 < \epsilon' < \epsilon$, if $n$ is large enough, then for every graph $G$ on $n$ vertices, with $\#E(G) \geq \frac{1}{2} \left( 1 - \frac{1}{k} + \epsilon \right) n^2$, we can find a subgraph $G'$ of $G$ on $\geq p$ vertices, such that for every $v \in V(G')$, we have
\[
\deg_{G'}(v) \geq \left( 1 - \frac{1}{k} + \epsilon' \right) \cdot \#V(G').
\]
We construct $G'$ by successively removing the “bad” vertices.
A lemma for the upper bound in Erdős-Stone, cont’d

More precisely, we construct a sequence of subgraphs $G_1, G_2, \ldots$ of $G = G_0$ as follows: if $G_i$ has been constructed and it contains a vertex $v_i$ with $\deg_{G_i}(v_i) < (1 - \frac{1}{k} + \epsilon') \cdot \#V(G_i)$, then we take $G_{i+1}$ to be the subgraph of $G_i$ spanned by $V(G_i) \setminus \{v_i\}$; if there is no such $v_i$, then we stop.

We need to show that if $n \gg 0$, we can’t reach a subgraph $G_\ell$ such that $\#V(G_\ell) < n'$. If this is the case, note that $q := \#V(G_\ell) = n - \ell < n'$. In this case, we have $\#E(G_\ell) \leq \frac{q(q-1)}{2} < \frac{n'^2}{2}$.

On the other hand, for every $i \geq 0$, we have $\#E(G_i) \leq \#E(G_{i+1}) + \left(1 - \frac{1}{k} + \epsilon'\right) \cdot \#V(G_i) = \#E(G_{i+1}) + \left(1 - \frac{1}{k} + \epsilon'\right) \cdot (n - i)$.

By combining all these, we get $\#E(G) \leq \frac{n'^2}{2} + \sum_{i=0}^{\ell-1} \left(1 - \frac{1}{k} + \epsilon'\right) \cdot (n - i) = \frac{n'^2}{2} + \left(1 - \frac{1}{k} + \epsilon'\right) \cdot \left(\frac{n(n+1)}{2} - \frac{q(q+1)}{2}\right)$.
Since $q$ is bounded above and $\epsilon' < \epsilon$, this implies that for $n$ large enough, we get

$$\#E(G) < \frac{1}{2} \left(1 - \frac{1}{k} + \epsilon\right)n^2,$$

a contradiction. This completes the proof of the claim.

The claim implies that from now on (after possibly replacing $\epsilon$ by a smaller value), we may assume that we have a graph on $n$ vertices, such that every vertex has degree $\geq \left(1 - \frac{1}{k} + \epsilon\right)n$. We show by induction on $q$, with $1 \leq q \leq k + 1$, that for all $t$, if $n \gg 0$, then we can find disjoint subsets $A_1, \ldots, A_q$ of $V(G)$, all of size $t$, such that any two vertices in two different $A_i$ and $A_j$ are neighbors in $G$. For $q = k + 1$, we obtain the assertion in the lemma.
The assertion to prove is clear for $q = 1$, hence we may assume $q \geq 2$. We apply the induction hypothesis to get disjoint subsets $A'_1, \ldots, A'_q$ of $V(G)$, all of size $s = \lceil t/\epsilon \rceil$, such that any two vertices in two different $A'_i$ and $A'_j$ are neighbors in $G$.

Let $U = V(G) \setminus (A'_1 \cup \ldots \cup A'_q)$ and consider

$$W = \{v \in U \mid v \text{ has } \geq t \text{ neighbors in each } A'_i\}.$$ 

We first show that we can make $\#W$ arbitrarily large by taking $n \gg 0$.

We bound in two ways the number $N$ of missing edges in $G$ between $U \setminus W$ and $A'_1 \cup \ldots \cup A'_q$. Since every vertex in $U \setminus W$ has $< t$ neighbors in some $A'_i$, it follows that

$$N \geq \#(U \setminus W) \cdot (qs - t) = (n - qs - \#W) \cdot (qs - t) \geq (n - qs - \#W)s(q - \epsilon).$$
On the other hand, since every vertex in $G$ has degree $\geq \left( 1 - \frac{1}{q} + \epsilon \right) n$ neighbors, we have

$$N \leq \#(A'_1 \cup \ldots \cup A'_q) \cdot \left( \frac{1}{q} - \epsilon \right) n = qsn \left( \frac{1}{q} - \epsilon \right) = sn(1 - q\epsilon).$$

We now combine the two inequalities involving $N$ to get a lower bound for $\#W$:

$$(q - \epsilon)\#W \geq (n - qs)(q - \epsilon) - n(1 - q\epsilon) = n(\epsilon + 1)(q - 1) - qs(q - \epsilon).$$

Since $q$, $\epsilon$, and $s$ are fixed, we see that when we make $n$ large enough, we may assume that $\#W$ is as large as we want.

In particular, we may and will assume that $\#W > \binom{s}{t}^q(t - 1)$. 

A lemma for the upper bound in Erdös-Stone, cont’d
Proof of the upper bound in Erdös-Stone

We now make the following construction: for every \( w \in W \), we choose \( t \) neighbors of \( w \) in each of the \( A'_i \) and let \( \Gamma_w \) be their union.

Clearly, there are at most \( \binom{s}{t}^q \) such subsets \( \Gamma_w \). By our condition on \( \#W \) and the pigeonhole principle, we can find distinct \( w_1, \ldots, w_t \in W \) such that \( \Gamma_{w_1} = \ldots = \Gamma_{w_t} = \Gamma \).

If we put \( A_{q+1} = \{w_1, \ldots, w_t\} \) and let \( A_i = \Gamma \cap A'_i \) for \( 1 \leq i \leq q \), we see that \( A_1, \ldots, A_{q+1} \) all have size \( t \), and any two vertices in two different \( A_i \) and \( A_j \) are neighbors in \( G \).

This completes the proof of the induction step for the assertion depending of \( q \) and thus the proof of the lemma.
We can now prove the upper bound in the Erdös-Stone theorem. Let \( k = \chi(H) - 1 \) and \( t = \#V(H) \). It follows the lemma that if \( n \gg 0 \) and \( G \) is a graph on \( n \) vertices and with \( m \geq \frac{1}{2} \left( 1 - \frac{1}{k} + \epsilon \right) n^2 \) edges, we have disjoint subsets \( A_1, \ldots, A_{k+1} \) of \( V(G) \) such that any two vertices in two different \( A_i, A_j \) are neighbors in \( G \).

Since \( \chi(H) = k + 1 \), if we consider a coloring of \( H \) with colors \( 1, \ldots, k + 1 \) and if we map the vertices colored by \( i \) to distinct vertices in \( A_i \), then we see that \( G \) has a subgraph isomorphic to \( H \). We thus have

\[
\text{ex}(n, H) < \frac{1}{2} \left( 1 - \frac{1}{k} + \epsilon \right) n^2 \quad \text{for} \quad n \gg 0.
\]

This completes the proof of the Erdös-Stone theorem.
In the next lecture, we will consider in some examples the asymptotic behavior of $\text{ex}(n, H)$ for certain bipartite graphs $H$.

For more on extremal graph theory, see Jacob Fox’s MIT course, available at http://math.mit.edu/~fox/MAT307