Lecture 5: Colorings of graphs

September 15, 2020
Proper colorings

In this lecture we will discuss various questions and results having to do with colorings of graphs. Fix a finite graph $G$.

**Definition.** A proper coloring of $G$ is a function from $V(G)$ to a set $C$ of "colors" such that the ends of every edge in $G$ have different colors. If $\#C = k$, we say that $G$ is $k$-colored.

**Remark.** Note that if $G$ has a loop, then it has no proper colorings.

**Convention.** Because of the above remark, in this lecture we assume that all graphs have no loops. Also, since multiple edges do not impose any extra restriction, after possibly removing some edges, we may and will assume that $G$ has no multiple edges. Therefore $G$ is simple.

**Definition.** The chromatic number $\chi(G)$ of $G$ is the minimal number $k$ such that $G$ can be $k$-colored.
First examples of chromatic numbers

**Remark.** Note that the chromatic number is well-defined. In fact, we have $\chi(G) \leq \#V(G)$.

**Examples.**
1) We have $\chi(G) = 1$ if and only if $E(G) = \emptyset$.
2) We have $\chi(G) \leq 2$ if and only if $G$ is bipartite (recall that this means that there is a decomposition $V(G) = A \sqcup B$ such that every edge in $E(G)$ has one end in $A$ and one end in $B$).
3) $\chi(K_n) = n$.
4) If $n \geq 2$, then $\chi(P_n) = 2$ if $n$ is even and $\chi(P_n) = 3$ if $n$ is odd.
5) If $H$ is a subgraph of $G$, then $\chi(H) \leq \chi(G)$.

**Remark.** It is an important result that for every simple graph $G$, the number of $m$-colorings of $G$ is given by a polynomial function of $m$. We will prove this later in the course.
What is the chromatic number of a tree on $n \geq 2$ vertices?
Further examples

What is the chromatic number of a tree on $n \geq 2$ vertices?

Answer: 2. Exercise: prove this!
Consider the Petersen graph:

What is its chromatic number?

Answer: 3.
Consider the Petersen graph:

What is its chromatic number?

Answer: 3.
Consider the Grötzsch graph:

What is its chromatic number?
Further examples, cont’d

Answer: 4
Theorem (Brooks). If \( d \geq 3 \) and \( G \) is a finite graph with no loops such that

1) \( K_{d+1} \) is not isomorphic with a subgraph of \( G \), and
2) All vertices of \( G \) have degree \( \leq d \),

then \( \chi(G) \leq d \).

Remark 1. If condition 2) holds, then condition 1) fails if and only if \( G \) has a connected component that is isomorphic to \( K_{d+1} \).

Remark 2. The assertion in the theorem can fail for \( d = 2 \): consider, for example, \( G = P_5 \).

As we will see, the proof of the theorem is quite intricate. However, there is a slightly weaker assertion that is easier to prove.
A bound in terms of the largest degree

For a finite graph $G$, let $\Delta(G) = \max_{x \in V(G)} \deg(x)$.

**Remark.** With this notation, for every $G$, we have

$$\chi(G) \leq \Delta(G) + 1.$$  

This follows easily from Brooks’ theorem (the case $\Delta(G) \leq 1$ being trivial), but it has a direct proof. Namely, let’s order the vertices of $G$ as $v_1, \ldots, v_n$. We color the vertices recursively with colors from $\{1, \ldots, N\}$, where $N = \Delta(G) + 1$.

At Step 1, we color $v_1$ with color 1. At Step $i$, we color $v_i$ with the smallest $j$ that has not been used to color the vertices $v_m$, with $m < i$, that are adjacent to $v_i$ (this is well-defined since $N > \deg(v_i)$).

It is clear that after Step $n$, we have a proper coloring of $G$ with $\leq N$ colors.
The bound in terms of $\Delta(G)$ can be achieved:

- If $n$ is odd, then $\Delta(P_n) = 2$ and $\chi(G) = 3$.
- For every $n$, $\Delta(K_n) = n - 1$ and $\chi(K_n) = n$.

A consequence of Brooks’ theorem: these are, essentially, the only cases when we have equality. More precisely:

**Corollary.** If $G$ is a connected graph such that $G$ is not isomorphic to some $P_n$, with $n$ odd, or to some $K_n$, then $\chi(G) \leq \Delta(G)$.

**Proof.** Let $d = \Delta(G)$. Note that if $d = 0$, then $G \simeq K_1$ and if $d = 1$, then $G \simeq K_2$. Also: if $d = 2$, then $\chi(G) \leq 2$, unless $G \simeq P_n$, with $n$ odd. Hence we may assume $d \geq 3$.

By Brooks’ theorem and since $G$ is connected, if $\chi(G) = d + 1$, then $G \simeq K_{d+1}$. This proves the corollary.
The proof of Brooks’ theorem

Proof of Brooks’ theorem. Suppose that the theorem fails and consider a counterexample $G$ with a minimal number of vertices. For a vertex $x$ of $G$, consider the set $\Gamma(x)$ of vertices adjacent to $x$

$$\Gamma(x) = \{x_1, \ldots, x_\ell\}.$$ 

Note that $\ell \leq d$ by condition 2).

Let $H$ be the graph obtained from $G$ by removing the vertex $x$ and removing all edges joining $x$ with one of the $x_i$. It is clear that $H$ still satisfies conditions 1) and 2). Since $\# V(H) < \# V(G)$, by the minimality condition in the choice of $G$, we see that $H$ has a proper $d$-coloring, with colors $1, \ldots, d$. We may assume that all these colors are used in the coloring of $\Gamma(x)$: otherwise, we could use the left-over color to color $x$, leading to a $d$-coloring of $G$, a contradiction.

Since $\ell \leq d$, it follows that we have $\ell = d$. After relabeling the vertices, we may assume that $x_i$ is colored with color $i$. 
The proof of Brooks’ theorem, cont’d

For \( i \neq j \), let \( H_{i,j} \) be the subgraph of \( H \) consisting of the vertices colored in either \( i \) or \( j \) (together with the edges in \( E(G) \) between these vertices). If \( x_i \) and \( x_j \) lie in different connected components of \( H_{i,j} \), then we can recolor the vertices in the connected component of \( x_i \) by switching the \( i \) and \( j \) colors; this will give a \( d \)-coloring of \( H \) such that \( x_i \) and \( x_j \) have the same color. As we have seen, this leads to a contradiction.

Therefore \( x_i \) and \( x_j \) lie in the same connected component of \( H_{i,j} \) that we denote by \( C_{i,j} \).

Since \( \deg(x_i) \leq d \) by condition 2), it follows that \( x_i \) has at most \( d - 1 \) neighbors in \( H \). On the other hand, for every \( j \neq i \), \( x_i \) has \( \deg_{C_{i,j}}(x_i) \) neighbors of color \( j \). Note that \( \deg_{C_{i,j}}(x_i) \geq 1 \) since \( C_{i,j} \) is connected and contains at least one more vertex besides \( x_i \) (namely \( x_j \)). We thus conclude that for every \( i \neq j \), \( x_i \) has precisely one neighbor in \( C_{i,j} \) (and the same holds true, of course, for \( x_j \)).
Since $C_{i,j}$ is connected, we can find a path $\gamma_{i,j}$ from $x_i$ to $x_j$. We may and will assume that $\gamma_{i,j}$ passes through each vertex only once.

Claim 1. Every vertex on this path different from $x_i$ and $x_j$ has degree 2 in $C_{i,j}$.

Arguing by contradiction, suppose that we have such a point $y$ of degree $\geq 3$ in $C_{i,j}$ and we chose the one that is closest to $x_i$.

Since $y$ has 3 neighbors in $C_{i,j}$ (all of which are colored in the same color), it follows that the neighbors of $y$ in $H$ are covered in $\leq (d - 2)$ colors. We can then recolor $y$ with a color different from $i$ and $j$ to get a proper $d$-coloring for $H$. Of course, after our recoloring, $y$ does not lie in $H_{i,j}$ anymore.
Since \( x_i \) has degree 1 in \( H_{i,j} \) and the vertices on \( \gamma_{i,j} \) strictly between \( x_i \) and \( y \) all have degree 2 in \( H_{i,j} \), it follows that every walk from \( x_i \) to \( x_j \) in \( H_{i,j} \), which passes through every vertex only once, has to agree with \( \gamma_{i,j} \) between \( x_i \) and \( y \). This implies that after recoloring, \( x_i \) and \( x_j \) lie in different connected components of \( H_{i,j} \), a contradiction. This proves Claim 1.

Since \( C_{i,j} \) is connected, we deduce from Claim 1 that all vertices in \( C_{i,j} \) lie on \( \gamma_{i,j} \). Moreover, \( E(C_{i,j}) \) is the set of edges in \( \gamma_{i,j} \).

**Claim 2.** For every distinct \( i, j, \) and \( k \), we have \( C_{i,j} \cap C_{i,k} = \{x_i\} \).

Indeed, suppose that \( z \neq x_i \) lies in \( C_{i,j} \cap C_{i,k} \). In this case \( z \) has color \( i \); in particular, \( z \neq x_j \) and \( z \neq x_k \).
The proof of Brooks’ theorem, cont’d

Therefore $z$ has two neighbors in $C_{i,j}$ and 2 neighbors in $C_{i,k}$. The vertices in each of these two pairs are colored in the same color, hence the neighbors of $z$ in $H$ are colored in $\leq d - 2$ colors. We may thus change the color of $z$ to a color not in \{i, j, k\} to get a proper coloring of $H$. Using what we know about the vertices and the edges of $C_{i,j}$, we conclude that with the new coloring, $x_i$ and $x_j$ do not lie in the same connected component of $H_{i,j}$, a contradiction. This proves Claim 2.

Finally, since $G$ contains no subgraph isomorphic to $K_{d+1}$, it follows that two of $x_1, \ldots, x_d$ (say $x_1$ and $x_2$) are not adjacent. We then have the following picture:
Let $a$ be the neighbor of $x_1$ in $C_{1,2}$ (different from $x_2$ by assumption); the color of $a$ is 2. We switch colors 1 and 3 in $C_{1,3}$ and we get a proper $d$-coloring of $H$.

Let $C'_{1,2}$ (resp. $C'_{2,3}$) be the new connected components containing $x_2$ and $x_1$ (resp. $x_2$ and $x_3$). Since $a$ and $x_1$ are adjacent, it follows that $a \in C'_{1,2}$. On the other hand, the only vertex in $C_{1,2}$ that changed its color is $x_1$; using the path between $a$ and $x_2$ in $C_{1,2}$, we see that $a \in C'_{2,3}$.

Since $a \in C'_{1,2} \cap C'_{2,3}$, this contradicts Claim 2. This completes the proof of the theorem.
Let’s take a break and discuss in groups the following problem:

We will prove later in the course that for every finite graph $G$, there is a polynomial $Q_G$ (the chromatic polynomial of $G$) such that for every $m \geq 1$, the number of proper colorings of $G$ with $m$ colors is equal to $Q_G(m)$.

Show that if $Q_n$ is the chromatic polynomial of the polygon $P_n$ (where $n \geq 1$), then

$$Q_n(x) = (x - 1)^n + (-1)^n(x - 1).$$
Kneser’s example

We now discuss of interesting example of a coloring problem, due to Kneser. Let’s first introduce the relevant graphs.

For a positive integer $n$, we write $[n] = \{1, \ldots, n\}$. For $1 \leq k \leq n$, let

$$\binom{[n]}{k} := \{ A \subseteq [n] \mid \#A = k \}.$$ 

This is a set with $\binom{n}{k}$ elements.

For every $n, k$ as above, let $KG_{n,k}$ be the graph with vertex set $\binom{[n]}{k}$, in which $A$ and $B$ are joined by an edge if and only if $A \cap B = \emptyset$. 
Examples of graphs $KG_{n,k}$:

1) If $k > \frac{n}{2}$, then $KG_{n,k}$ has no edges (not interesting). From now on, we always assume $2k \leq n$.

2) If $2k = n$, then $KG_{n,k}$ has the property that for every vertex $x$, there is a unique other vertex adjacent to it. In particular, we have $\chi(KG_{2k,k}) = 2$.

3) $KG_{5,2}$ is the Petersen graph:

![Diagram of the Petersen graph](image-url)
Kneser’s bound for $\chi(KG_{n,k})$

**Proposition** (Kneser). For every $n \geq 1$ and every $k$, with $1 \leq k \leq \frac{n}{2}$, we have

$$\chi(KG_{n,k}) \leq n - 2k + 2.$$ 

**Proof.** We define $c: \binom{[n]}{k} \to [n - 2k + 2]$ by

$$c(A) = \min \left\{ \min \{x \mid x \in A\}, n - 2k + 2 \right\}.$$ 

Suppose that $A, B \in \binom{[n]}{k}$ are such that $A \cap B = \emptyset$, but $c(A) = c(B)$. If this common value is $\leq n - 2k + 1$, then there is $x \in A \cap B$, a contradiction.

Therefore the common value must be $\geq n - 2k + 2$. In this case, both $A$ and $B$ are contained in $\{n - 2k + 2, \ldots, n\}$, a set with $2k - 1$ elements. Since both $A$ and $B$ have $k$ elements, we again get $A \cap B \neq \emptyset$, a contradiction. Hence $c$ is a proper coloring.
Kneser’s conjecture for $\chi(KG_{n,k})$

**Conjecture** (Kneser). For every $n \geq 1$ and every $k$, with $0 \leq k \leq \frac{n}{2}$, we have

$$\chi(KG_{n,k}) = n - 2k + 2.$$ 

This is now a theorem, with several known proofs. We give a proof due to Greene (after Baranyi also gave a similar one).

The proof will make use of the following result in topology.

**Theorem** (Borsuk-Ulam). If $f: S^n \to \mathbb{R}^n$ is a continuous function, then there is $x \in S^n$ such that $f(x) = f(-x)$.

Here $S^n \subseteq \mathbb{R}^n$ is the $n$-dimensional sphere

$$S^n = \{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \mid a_0^2 + \ldots + a_n^2 = 1\}.$$
A consequence of Borsuk-Ulam

We will make use of the following consequence of Borsuk-Ulam:

**Corollary.** If $A_0, \ldots, A_n \subseteq S^n$ are open or closed subsets such that $S^n = A_0 \cup \ldots \cup A_n$, then there is $x \in S^n$ and $0 \leq i \leq n$ such that $x, -x \in A_i$.

**Proof.** We only give the proof when all $A_i$ are closed. Consider the continuous function

$$f : S^n \to \mathbb{R}^n, \quad f(x) = (d(x, A_1), \ldots, d(x, A_n))$$

It follows from the Borsuk-Ulam theorem that there is $x \in S^n$ such that $f(x) = f(-x)$.

Note that since the $A_i$ are closed, we have $d(y, A_i) = 0$ iff $y \in A_i$.

If there is $i$, with $1 \leq i \leq n$, such that $x \in A_i$ or $-x \in A_i$, then by above the other condition also holds. If this is not the case, since $X = A_0 \cup \ldots \cup A_n$, we have $x, -x \in A_0$. This completes the proof.
Proof of Kneser’s conjecture

Greene’s proof of Kneser’s conjecture:
Suppose \( n \) and \( k \) with \( 2k \leq n \) are such that \( \chi(KG_{n,k}) \leq n - 2k + 1 =: d \) and we have a proper coloring \( c : \left( \binom{n}{k} \right) \to [d] \).

Consider the sphere \( S^d \subseteq \mathbb{R}^{d+1} \) and choose \( n \) points on \( S^d \) such that any \( d + 1 \) of these are linearly independent (note that \( n \geq d + 1 \)). We will identify \( [n] \) with the set \( X \) of these points on \( S^d \).

For \( 1 \leq i \leq d \), let \( U_i \subseteq S^d \) consist of those \( x \in S^d \) for which there is a \( k \)-set \( S \subseteq X \) such that \( c(S) = i \) and \( S \) is contained in the half-space \( \{y \mid \langle x, y \rangle > 0\} \). Note that \( U_i \) is open in \( S^d \).

We also put \( A = S^d \setminus (U_1 \cup \ldots \cup U_d) \), which is closed in \( S^d \). Applying the consequence of Borsuk-Ulam to \( A, U_1, \ldots, U_n \), we obtain \( x \in S^d \) such that

1) Either there is \( i \), with \( 1 \leq i \leq d \), such that \( x, -x \in U_i \),
2) Or \( x, -x \in A \).
Proof of Kneser’s conjecture, cont’d

**Case 1.** We have $x, -x \in U_i$. In this case, we have $k$-subsets $S_1, S_2 \subseteq X$, with $c(S_1) = c(S_2)$ and

$$S_1 \subseteq \{ y \mid \langle x, y \rangle > 0 \} \quad \text{while} \quad S_2 \subseteq \{ y \mid \langle x, y \rangle < 0 \}.$$

This implies $S_1 \cap S_2 = \emptyset$, contradicting the fact that $c$ gives a proper coloring.

**Case 2.** We have $x, -x \in A$. In this case, for every $k$-subset $S \subseteq X$, there are $w, w' \in S$ such that $\langle x, w \rangle \leq 0$ and $\langle x, w' \rangle \geq 0$.

This implies that

$$\#(X \cap \{ y \mid \langle x, y \rangle > 0 \}) \leq k - 1 \quad \text{and} \quad \#(X \cap \{ y \mid \langle x, y \rangle > 0 \}) \leq k - 1.$$

This implies that

$$\#(X \cap \{ y \mid \langle x, y \rangle = 0 \}) \geq n - 2k + 2 = d + 1,$$

contradicting the fact that any $d + 1$ elements of $X$ are linearly independent. This completes the proof of the theorem.
Further connections to algebraic topology

In fact, the first proof of Kneser’s theorem was due to Lovász. His proof was very interesting since he made a more general connection to algebraic topology.

More precisely, Lovász associated to any finite simple graph $G$ a simplicial complex $\mathcal{N}_G$. Recall that a simplicial complex $\Delta$ on a finite set $S$ consists of a collection of subsets of $S$ with the property that if $A \in \Delta$ and $B \subseteq A$, then $B \in \Delta$.

Given a finite simple graph $G$, the neighborhood complex $\mathcal{N}_G$ of $G$ is the simplicial complex on the set $V(G)$ such that a subset $A \subseteq V(G)$ is in $\mathcal{N}_G$ if there is $b \in V(G)$ that is a neighbor of all elements of $A$.

Why is this useful: attached to a simplicial complex $\Delta$ there is a topological space $|\Delta|$. Various topological invariants (such as homology or homotopy groups) provide interesting invariants for the simplicial complex. In the case of a graph $G$, one tries to use such invariants for $|\mathcal{N}_G|$ to get information about $G$. 
Lovász’s approach to Kneser’s conjecture

Recall that a nonempty, path connected topological space $X$ is $k$-connected if $\pi_i(X) = 0$ for $1 \leq i \leq k$ (convention: $(-1)$-connected means “nonempty” and 0-connected means “path connected”).

**Theorem** (Lovász). If $G$ is a finite simple graph such that $|\mathcal{N}_G|$ is $k$-connected, then

$$\chi(G) \geq k + 3.$$  

**Example.** Consider the case when $G = K_n$. In this case $\mathcal{N}_G$ is the simplicial complex on $\{1, \ldots, n\}$ such that $A \subseteq \{1, \ldots, n\}$ lies in $\mathcal{N}_G$ iff $A \neq \{1, \ldots, n\}$. Therefore $|\mathcal{N}_G|$ is homeomorphic to the sphere $S^{n-2}$. It is known that $S^{n-2}$ is $k$-connected iff $k \leq n - 3$. Since $\chi(K_n) = (n - 3) + 3$, we see that we have equality in the bound in the theorem.
Lovász also showed: if $2k \leq n$, then $|\mathcal{N}_{KG_{n,k}}|$ is homotopically equivalent to a bouquet of $S^{n-2k}$.

In particular, this implies that $|\mathcal{N}_{KG_{n,k}}|$ is $(n - 2k - 1)$-connected. The theorem then implies

$$\chi(KG_{n,k}) \geq n - 2k + 2.$$  

For details and further connections between graph theory and combinatorics, see

*Mark de Longueville, A course in topological combinatorics.*