5.1

Modeling Linear Programs

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Product Mix Models

Company can make many different products. This determines how many of each to make in a period to maximize profit.

INPUT DATA

1. List of all products that can be made.
2. List of all raw materials and other resources needed as inputs.
3. All *input-output (*or *technology*) coefficients.
4. Cost or net profit coefficients.
5. Bounds on each resource availability.
6. Bounds on amount of each product made.

Example: Fertilizer Company

Two products. 3 raw materials. No limit on amount of each product made.

Table 2.1

Item	Input (tons)/ ton of		Max. available daily (tons)
	Hi-ph	Lo-ph	
RM 1	2	1	1500
RM 2	1	1	1200
RM 3	1	0	500
Net profit ($) / ton	15	10	

Two activities, two decision variables.

Some More Definitions

Slack Variables: RM 1 Constraint $2x_1 + x_2 \leq 1500$ equivalent to $1500 - 2x_1 - x_2 \geq 0$.

Let $s_1 = 1500 - 2x_1 - x_2$, it is *Tons of RM 1 Unused in Solution x*, and called *Slack Variable for RM 1 Constraint.* By introducing it, RM 1 constraint can be written as:

$$\begin{aligned} 2x_1 + x_2 + s_1 &= 1500 \\ s_1 &\geq 0 \end{aligned}$$

1. Every inequality constraint can be written as an equation by introducing appropriate slack variable.

2. The slack variable for each inequality constraint is separate.

3. Sign of slack in equation is +1 or -1 depending on whether inequality is $\leq$ or $\geq$.

4. Slacks are always nonnegative variables.

5. LP said to be in *STANDARD FORM* if all constraints are equations and all variables are nonnegative variables. Every LP can be transformed into standard form.

Production with Environmental Protection

Firm produces chemical **C** using two main raw materials **A, B**.

1 lb.**A** + 2 lb.**B** → [Process 1] → 1 lb each of **C, LW, SW**

LW = Liquid Waste, currently dumped in river.

SW = Solid Waste, currently being taken by a fertilizer company at no charge, no payment.

From next year, direct dumping of **LW** in river will be prohibited by EPA. 3 alternatives for handling **LW**.

Alternative 1: Treat **LW** (cost $0.35/lb) to remove pollutants, then release into river.

Alternative 2: 1 lb **A** + 1 lb **LW** ⟶ [Process 2] ⟶ 2 lb of byproduct **D**

Alternative 3: 1 lb **B** + 1 lb **LW** ⟶ [Process 3] ⟶ 2 lb of byproduct **E**

Costs	
1 lb **A**	\$1.5, upto 5000 lb **A** available/day
1 lb **B**	\$1.75, upto 7000 lb **B** available/day
Process 1 labor/lb **C** made	\$0.5
Process 1 other costs/lb **C** made	\$1.6
Process 2 labor/lb of **D** made	\$0.2
Process 3 labor/lb of **E** made	\$0.15
Treatment/lb of **LW**	\$0.45

Product	Selling Price
C	\$5.95/lb
D	\$0.85/lb
E	\$0.65/lb

Required to find best plan for company while meeting EPA guidelines.

Linear Blending Models

Many applications in Petroleum, Food, Paint, Chemical, Pharmaceutical industries.

Linearity Assumptions: Suppose $p_1 + \ldots + p_k = 1$ and a mixture contains:

Materials	M_1	M_2	$\ldots$	M_k
In Proprtion	p_1	p_2	$\ldots$	p_k
Characteristic t for material	t_1	t_2	$\ldots$	t_k

Linearity holds iff Characteristic for mixture is: $\Sigma_{r=1}^{k} p_r t_r$.

Example: Mixture, 4 barrels of fuel F_1 and 6 barrels of fuel F_2. OCR of F_1, F_2 are 68, 92 respectively. What will be the OCR of mixture under linearity?

Decision Variables in Blending Models Are

Either QUANTITIES of materials blended, $w_1, \ldots, w_k$, say. In this case remember to include constraint that: Quantity of blend made, $w = w_1 + \ldots + w_k$.

Or PROPORTIONS of materials in the blend, $p_1, \ldots, p_k$, say. In this case remember to include the constraint that: $p_1 + \ldots + p_k = 1$.

Gasoline Blending Example: Specs. on many properties: Octane Rating (OCR), Viscosity, Vapor Pressure, Pour Point, Freezing Point, etc. For simplicity we consider only one property, OCR.

Company blends 4 raw gasolines RG_1 to RG_4 into 3 different grades of fuel F_1 to F_3. All quantities measured in B (Barrels).

Any RG not blended into fuels can be sold to small dealers at:

$38.95/$B$. if OCR > 90

$36.85/$B$. if OCR ≤ 90

Other data given below. Formulate best buying, blending, selling plan.

RG	OCR	Max. Availability/day	Cost $/$B$.
RG_1	68	4000 B.	$31.02
RG_2	86	5050	33.15
RG_3	91	7100	36.35
RG_4	99	4300	38.75

Fuel	Min. OCR	Price $/$B$.	Demand
F_1	95	$45.15	$\leq$10,000 B. /day
F_2	90	42.95	No limit
F_3	85	40.99	$\geq$ 15,000 B./day

The Diet Model

Nutriets	Chemicals that body needs daily to stay healthy and fit. EXAMPLES: Vitamins A, B, C, D, E, etc., different proteins, starches, etc., fiber; etc.
Foods	Available in retail stores. EXAMPLES: Milk (measured in gallons), yogurt (cups), rice (pounds), bread (slices), lettuce (heads), bean curd (packets), etc.
Diet	Combination of foods that one eats daily. Different foods measured in different units.
Food Composition	No. of units of each nutrient per unit of that food. Determined by a careful chemical analysis.
MDR of nutrient	Minimum Daily Requirement for nutrient. Determined by careful medical analysis.

AIM: Foods vary greatly in cost and composition. Aim is to find a minimum cost diet meeting MDR of all nutrients.

History: First studied by G. J. Stiegler in 1940's. His model had 77 foods at 1939 prices. Combined nutrients into groups (like vitamins, etc.), and considered 9 different nutrient groups. Taste not considered. Modeled as an LP involving 9 constraints in 77 variables. Awarded 1982 ECON. NOBEL PRIZE partly for this model.

Now-A-Days used extensively for designing feed for farm chickens, turkeys, cows, hogs, etc.

EXAMPLE: Two foods, F_1, F_2. Three nutrients: S(starches), P(proteins), V(vitamins).

Nutrient	Nutrient units/kg.		
	F_1	F_2	MDR
S	5	7	8
P	4	2	15
V	2	1	3
Cost ($/kg.)	0.60	0.35	

The Transportation Model

AIM is to find a way of transferring goods at min. transportation cost. In USA alone, transportation of goods is estimated to cost over US $900 billion/year!

SOURCES: Places where material is available.

SINKS, MARKETS, or DEMAND CENTERS: Places where material is required.

AVAILABILITY (at sources): How much material is available at each source.

REQUIREMENTS (at sinks): How much material is required at each sink.

COST COEFFs.: Cost of transporting material/unit from each source to each sink.

BALANCE CONDITION: {Sum availabilities at sources} = {Sum requirements at sinks}. If this holds, problem called BALANCED TRANSPORTATION PROBLEM.

History: Formulated in 1930's by Russian economist L. V. Kantorovitch. He & T. Koopmans received 1975 ECON. NOBEL PRIZE partly for this work.

EXAMPLE: Sources are 2 mines, sinks are 3 steel plants.

	c_{ij} (cents/ton)			Availability at mine (tons) daily
	$j = 1$	2	3	
Mine $i = 1$	11	8	2	800
2	7	5	4	300
Requirement at plant (tons) daily	400	500	200	

Array Repersentation of Transportation Model

Each row (Column) of array corresponds to a source (Sink). Cell (i, j) in array corresponds to cooresponds to transportation route from source i to sink j. Put variable x_{ij} in its center.

Each row and column of array leads to a constraint on sum of variables in it.

SPECIAL PROPERTY OF TRANSPORTATION MODEL: THE INTEGER PROPERTY: If all availabilities and requirements are positive integers, and if the problem has a feasible solution, then it has an optimum solution in which all variables take only integer values.

Array Representation

	Plant			
	1	2	3	
Mine 1	x_{11} 11	x_{12} 8	x_{13} 2	$= 800$
Mine 2	x_{21} 7	x_{22} 5	x_{23} 4	$= 300$
	$= 400$	$= 500$	$= 200$	

$x_{ij} \geq 0$ for all i, j. Min. cost.

Assignment Problem

It is a transportation problem with equality constraints in which :

no. of sources = no. of sinks = n say,

all availabilities and all requirements are = 1.

EXAMPLE: Write assignment problem with cost matrix

$$\begin{pmatrix} 5 & 9 & 10 \\ 4 & 5 & 19 \\ 6 & 8 & 8 \end{pmatrix}$$

Has many applications such as in assigning jobs to machines etc.

Marriage application of Assignment Problem.

Halmos and Vaughan (1950) used Assignment problem as a tool to analyze problem of marriage and divorce in society.

Consider club of 5 boys and 5 girls.

c_{ij} = (Mutual) happiness rating of boy i and girl j when they spend a unit time together.

Given $c = (c_{ij})$ determine what fraction of life each boy should spend with each girl, to max. club's happiness.

	c_{ij} for girl $j =$				
	1	2	3	4	5
boy $i = 1$	78	−16	19	25	83
2	99	98	87	16	92
3	86	19	39	88	17
4	−20	99	88	79	65
5	67	98	90	48	60

Multi-Period LP Models

Important applications in production planning for planning production, storage, marketing of product over a planning horizon.

Production capacity, demand, selling price, production cost, may all vary from period to period.

AIM: To determine how much to produce, store, sell in each period; to max. net profit over entire planning horizon.

YOU NEED Variables that represent how much material is in storage at end of each period, and a material balance constraint for each period.

EXAMPLE: Planning Horizon = 6 periods.

Storage warehouse capacity: 3000 tons.

Storage cost: \$2/ton from one period to next.

Initial stock: 500 tons. Desired final stock: 500 tons.

Period	Prod. cost	Prod. capacity	Demand (tons)	Sell price
1	20 \$/ton	1500	1100 tons	180 \$/ton
2	25	2000	1500	180
3	30	2200	1800	250
4	40	3000	1600	270
5	50	2700	2300	300
6	60	2500	2500	320

EXAMPLE:

Company sends 3 air pollutants (particulates (P), sulfur oxides (SO), and hydrocarbons (HC)) from blast furnaces, open hearth furnaces. Each measured in units of million lbs./year. New rules require reductions in these.

Pollutant	Required reductions
P	50
SO	140
HC	130

Emissions can be reduced by 3 techniques. They are:

TS – taller smoke stacks, i.e., increase height.

IF – use filter devices in smokestacks

HG – use more high grade fuels

Following tables give reductions that can be achieved using the method at maximum possible level.

Pollutant	Reduction in blastfurnaces*		
	TS	IF	HG
P	12	25	17
SO	35	20	55
HC	34	27	28
Yearly Cost (in 10^6 \$)+	8	7	11

* using at maximum level

+for maximum level

Pollutant	Reduction in open hearth furnaces*		
	TS	IF	HG
P	9	20	13
SO	40	30	47
HC	50	23	18
Yearly Cost (in 10^6 \$)+	10	6	9

* using at maximum level

+for maximum level

Reductions by 3 techniques are additive. Each can be used at any fraction of its maximum level in any furnace

to get corresponding fractional reduction ,

at corresponding fractional cost.

Determine combination of 3 techniques, as

fractions of the maximum possible levels, to use at each furnace,

to

achieve mandated reductions
at minimum cost.

Homework Problems:
5.1 (F, MV) Formulate following problems as linear programs.

(a) Company can make 2 products (P_1, P_2) using 3 raw materials (RM1, RM2, RM3) with follwing data. P_1 can be sold in unlimited quantities. Market for P_2 is limited to 10 tons. To maximize total net profit.

	Input (tons/ton) to make		
	P_1	P_2	Availability
RM1	2	5	60 tons
RM2	1	1	18
RM3	3	1	44
Net profit ($/ton sold)	8	14	

(b) Similar to (a). Amounts of RM1, 2, 3 available are 15, 12, 45 tons respectively. However, process of making P_1 takes as input RM2, RM3, but outputs additional quantities of RM1 as a byproduct. Info. & data on both processes given below. Both products have unlimited market. Net profit per ton of P_1, P_2 sold is $10, 20 respectively.

(Input 1 ton RM2 + 5 tons RM3)→ [P_1 process] → (1 ton P_1 + 1 ton RM1)

(2 tons RM1 + 1 ton RM2 + 3 tons RM3)→ [P_2 process] → (1 ton P_2)

(c) Similar to (a), but only RM1, 2. Maximum demand for P_1, P_2 is 20, 30 tons respectively. Here is other data.

	Input (tons/ton) to make		
	P_1	P_2	Availability
RM1	20	10	500 tons
RM2	5	5	165
Net profit ($/ton sold)	16	13	

(d) Same as (c) with following data

	Input (tons/ton) to make		
	P_1	P_2	Availability
RM1	5000	4000	6000 tons
RM2	400	500	600
Net profit ($/ton sold)	4500	4500	
Max. demand	1	1	

(e) Same as (a) but with following data, and unlimited demand for both products.

	Input (tons/ton) to make		
	P_1	P_2	Availability
RM1	15	5	300 tons
RM2	10	6	240
RM3	8	12	450
Net profit ($/ton sold)	500	300	

(f) Diet problem with 3 nutrients (C = carbohydrates, P = protein, F = fat), 2 foods. Need to find min cost diet.

	Units/unit of		
Nutrient	Food 1	Food 2	Requirement
C	5	15	at least 50
P	20	5	at least 40
F	15	2	at most 60
Cost ($/unit)	4	2	

5.2 (F) Formulate following problems as LPs.

(a) Transportation problem. To min total shipping cost.

	Shipping cost ($/unit) to			
	Sink 1	2	3	Available
Source 1	7	6	8	6
Source 2	6	4	9	5
Exact requirement	3	2	4	

(b) Nonferrous metals company makes 4 alloys from 2 metals, according to following requirements. To maximize gross revenue.

Metal	proportion of metal in alloy 1	2	3	4	Availability per day
1	0.5	0.6	0.3	0.1	25 tons
2	0.5	0.4	0.7	0.9	40 tons
Alloy price ($/ton)	750	650	1200	2200	

(c) The grains that can be included in a multigrain flour have the following composition and price.

	% of Nutrient in Grain 1	2	3	4
Starch	30	20	40	25
Fiber	40	65	35	40
Protein	20	15	5	30
Gluten	10	0	20	5
Cost (cents/kg.)	70	40	60	80

For taste, % of grain 2 in mix cannot exceed 20, of grain 3 has to be at least 30, and of grain 1 has to be between 10 to 25.

The % protein in must be at least 18, of gluten has to be between 8 to 13, and of fiber at most 50.

Cost has to be minimized.

(d) To make 3 juice mixes TM, Pa.D, HN, using high-sugar pineapple juice (HSP), normal pineapple juice (NP), orange juice (OJ), tangerine juice (TJ), and white grape juice (GJ). To maximize net profit.

Juice	Specs. on % of juice in mix TM	Pa.D	HN	Price/unit	Available (units)
HSP	≥ 5	≥ 10		65	1000
NP	$10 - 33$	≤ 20	$30 - 50$	30	10,000
OJ	$50 - 80$	≤ 70	≥ 50	20	60,000
TJ		≥ 15		75	2000
GJ	$10 - 20$	$10 - 20$		25	40,000
Selling price/unit	100	150	80		

(e) To make 3 grades of gasoline by blending 3 different crude oil distillates A, B, C. To maximize net profit.

Distillate	Oc.R	Availability (brls./day)	Cost ($/brl)
A	83	20,000	26
B	88	25,000	30
C	93	15,000	34

Gasoline	Min. Oc.R	Max. % A	Min. % C	Selling price ($/brl)
Regular	87		20	33
Mid-grade	89	15	30	41
Premium	90	60	40	48

(f) To max total net profit by selling currant jelly (CJ) & orange marmalade (OM).

Cost is $0.80 and $1.50 to produce a jar of OM and CJ. With no advertising, sales are 3500 jars for OM, 5500 jars for CJ at prices of $2.20 and $4 per jar, per week.

By advertizing, weekly sales of OM can be increased at rate 2 jars per $ spent on it until total sales reach 4200 jars; and of CJ can be increased at rate 1.2 jars per $ spent on it until total sales reach 6000 jars. Can spend $\leq$ $1000 on advertising per week, total. Determine opt. production and advertisement plan, ignore integer requirements on jars produced and sold.

(g) To make 2 types of desks. Available are: 150 man hours of welders time, 280 man hours of assembler time per day. Below is data on the man hours needed to make a unit of 5 desks. Max total revenue.

	Type 1	Type 2
Assembler man hours/unit	1.8	2.7
Welder man hours/unit	1.9	1.2
Revenue produced/desk	$575	450

(h) To make 5 products. 3 types of machines used. Labor cost is $6/hour on machine types 1, 2, and $4/hour on machine type 3. Max net profit.

	Mts. mc. time/unit of product					Mc. time available/week
	1	2	3	4	5	
Mc. type 1	12	7	8	10	7	149 hrs.
2	9	7	5	0	14	129 hrs.
3	7	8	5	4	3	1118 hrs.
Price/unit	$17	15	16	13	14	
Raw material cost/unit	$3	2	0.9	1.2	1	

(i) Available are 1000 acres of land, 4500 man hours of labor per season, to grow corn, soybeans. To find best allocation of available acreage to crops.

	Corn	Soybean
Man hrs. labor required/acre	6	4
Cost of seed, fertilizer, insecticide/acre	$85	$35
Yield Bushels/acre	120	35
Selling price/bushel	$3.15	$6.25