6.1

Solving 2 Variable LP Models Geometrically

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Geometry Concepts I

Ray of a point $a \in R^n$: is $\{\alpha a : \alpha \geq 0\}$. It is the halfline joining 0 to a and continuing indefinitely in that direction.

Half-Line through b *parallel to the ray of* a: is $\{b + \alpha a : \alpha \geq 0\}$. b is its initial point, and a is said to be its direction.

Moving from b in the direction of a: means traveling from b along the half-line through b parallel to the ray of a.

Hyperplane in R^n: Set of solutions of a single nontrivial equation, i.e., $\{x : a_1x_1 + \ldots + a_nx_n = b_1$ where at least one of $a_1, \ldots, a_n \neq 0\}$.

In R^2 every hyperplane is a straight line and vice versa.

Half-Space in R^n: Set of solutions of a single nontrivial inequality, i.e., $\{x : a_1x_1 + \ldots + a_nx_n \geq b_1$ where at least one of $a_1, \ldots, a_n \neq 0\}$.

Each hyperplane divides the space into two half-spaces whose intersection is the hyperplane.

Geometric Method to Solve 2 Variable LPs

1. Set up the axes of coordinates. Identify the half-space corresponding to each constraint, and bound restriction. *Feasible region* K is the intersection of all these half-spaces.

If $K = \emptyset$, problem *infeasible*, terminate.

2. If $K \neq \emptyset$, select a point $\bar{x} \in K$ and draw the objective line $z(x) = \bar{z} = z(\bar{x})$.

Move objective line parallel to itself by increasing (*decreasing*) the RHS constant in its equation from $\bar{z}$ if $z(x)$ is to be maximized (*minimized*), always keeping its intersection with K nonempty. Two possibilities.

(a) The line can be moved indefinitely, keeping its intersection with K nonempty always. In this case $z(x)$ is unbounded above (*below*) on K.

(b) z_0 is the last value of RHS constant in above process. Any further change in it, makes the objective line loose its intersection with K.

In this case z_0 is the optimum value of $z(x)$ and all the points in $K \cap \{x : z(x) = z_0\}$ are optimum solutions of the problem.

Example: Fertilizer Problem

Points to Remember: This method not extended to higher dimensions. There objective line becomes objective hyperplane, and we cannot get visual information whether it can be moved while still keeping nonempty intersection with feasible region.

Path followed by simplex method very different in character.

Planning Information From LP Models

Optimum Solution: Solving model gives it, if one exists.

Infeasibility Analysis: If model infeasible, this helps in making necessary changes to make it feasible.

Identify Critical Items: Items corresponding to slacks $= 0$ in opt. sol. are at their bound, hence critical. Other items are not.

Marginal Values: The marginal value of an item (or the RHS constant in the corresponding constraint) is the rate of change in opt. obj. value per unit change in that RHS constant.

When model solved, we can determine

- Whether marginal values exist or not.
- Their values, if they exist.

Example: Fertilizer Problem.

Theorem: For LP in standard form, if opt. sol.

is the *nondegenerate basic solution* of system of equality constraints associated with basis B, then marginal value vector exists, and it is $\pi = c_B B^{-1}$ where c_B is the row vector of objective coefficients of variables associated with columns in B.

Marginal values have many planning applications.

Example: Evaluating Profitability of New Products

New fertilizer *Lushlawn*. Per ton made, needs inputs:

3 tons $RM1$

2 tons $RM2$

2 tons $RM3$

How much profit should lushlawn fetch per ton made, before company considers manufacturing it?

Note: The method discussed in class to compute MV geometrically, is a simplified method for computing MV in LPs involving 2 variables.

If LP has ≥ 3 variables, when it is solved by any algo. such as Simplex method, if algo. finds an opt. sol., it also outputs an opt. **dual sol.** (defined in Notes 8), which will be the MV vector for LP under some nondegeneracy conds. These conds. can be checked from final output of algo. If they do not hold, MV does not exist.

Homework problems: MV = Marginal values

6.1: Find the MV associated with all the RHS constants in problem 5.1 (a). Show all the work clearly.

A new product has been developed. Inputs needed to make this product are: (3, 2, 4 tons of RM1, RM2, RM3 respectively)/ton. What is the break even net profit/ton of this new product at which it becomes competitive with the existing two products?

(K) A new product P_4 needs as inputs (2, 5, 3 tons of RM1, RM2, RM3 respectively)/ton to make. This product is expected to yield a net profit of \$24/ton manufactured. Is it worth making this product?

(A) A new product P_5 needs as inputs (1, 4, 5 tons of RM1, RM2, RM3 respectively)/ton to make. This product is expected to yield a net profit of \$27/ton manufactured. Is it worth making this product?

6.2: Find the MV associated with all the RHS constants in problem 5.1 (b). Show all the work clearly.

A new product has been developed. Inputs needed to make this product are: (8, 2, tons of RM1, RM3 respectively)/ton; and this process also produces 0.5 ton of RM2 per ton of this product made, as a byproduct. What is the break even net profit/ton of this new product at which it becomes competitive with the existing two products?

(K) A new product P_4 needs as inputs (2, 3 tons of RM2, RM3 respectively)/ton to make, and this process also produces 2 tons of RM1 per ton of this product made, as a byproduct. This product is expected to yield a net profit of \$18/ton manufactured. Is it worth making this product?

(A) A new product P_5 needs as inputs (2, 1, 2 tons of RM1, RM2, RM3 respectively)/ton to make. This product is expected to yield a net profit of \$25/ton manufactured. Is it worth making this product?

6.3: Find the MV associated with all the RHS constants in problem 5.1 (c), including the upper bounds on the variables. Show all the work clearly.

A new product has been developed. Inputs needed to make this product are: (15, 10 tons of RM1, RM2 respectively)/ton. What is the break even net profit/ton of this new product at which it becomes competitive with the existing two products?

(K) A new product P_4 needs as inputs (30, 2 tons of RM1, RM2 respectively)/ton to make. This product is expected to yield a net profit of \$60/ton manufactured. Is it worth making this product?

(A) A new product P_5 needs as inputs (5, 10 tons of RM1, RM2 respectively)/ton to make. This product is expected to yield a net profit of \$15/ton manufactured. Is it worth making this product?

6.4: Find the MV associated with all the RHS constants in problem 5.1 (d), including the upper bounds on the variables. Show all the work clearly.

A new product has been developed. Inputs needed to make this product are: (2000, 1000 tons of RM1, RM2 respectively)/ton. What is the break even net profit/ton of this new product at which it becomes competitive with the existing two products?

(K) A new product P_4 needs as inputs (3000, 800 tons of RM1, RM2 respectively)/ton to make. This product is expected to yield a net profit of \$5400/ton manufactured. Is it worth making this product?

(A) A new product P_5 needs as inputs (2000, 200 tons of RM1, RM2 respectively)/ton to make. This product is expected to yield a net profit of \$2100/ton manufactured. Is it worth making this product?

6.5: Consider the problem problem 5.1 (e). If this problem has two distinct extreme point optimum solutions, write the system of equality constraints defining each of these solutions individually.

Working with each of these systems independently, compute the MV of all the RHS constants, showing your work clearly. Are the MV obtained from both the systems the same?

A new product has been developed. Inputs needed to make this product are: (10, 10, 5 tons of RM1, RM2, RM3 respectively)/ton. What is the break even net profit/ton of this new product at which it becomes competitive with the existing two products?

(K) A new product P_4 needs as inputs (20, 5, 10 tons of RM1, RM2, RM3 respectively)/ton to make. This product is expected to yield a net profit of \$490/ton manufactured. Is it worth making this product?

(A) A new product P_5 needs as inputs (25, 10, 10 tons of RM1, RM2, RM3 respectively)/ton to make. This product is expected to yield a net profit of \$505/ton manufactured. Is it worth making this product?

6.6: Find the MV associated with all the RHS constants in problem 5.1 (f). Show all the work clearly.

A new food has become available. Each unit of this food contains (10, 15, 10 units of C, P, F respectively). What is its breakeven price/unit at which it becomes competitive with existing foods?

(K) A new food has become available. Each unit of this food contains (20, 15, 5 units of C, P, F respectively), and costs \$5/unit. Is it worth including this food in the diet?

(A) A new food has become available. Each unit of this food contains (33, 22, 0 units of C, P, F respectively), and costs \$6/unit. Is it worth including this food in the diet?