Duality in LP, Optimality Conditions

Katta G. Murty Lecture slides

Every LP has another LP called its *dual*, which shares the same data, and is derived through *rational economic arguments*. In this context the original LP called the *primal LP*.

Variables in the dual problem are different from those in the primal; each dual variable is associated with a primal constraint, it is the *marginal value* or *Lagrange multiplier* corresponding to that constraint.

Example: Primal – Fertilizer Maker's Problem.

FERTILIZER MAKER has daily supply of:

1500 tons of RM 1

1200 tons of RM2

500 tons of RM 3

She wants to use these supplies to maximize net profit.

DETERGENT MAKER wants to buy *all of fertilizer maker's supplies* at cheapest price for his detergent process. Suppose he offers:

$$\$\ \pi_1/\text{ton for RM1}$$

$$\$\ \pi_2/\text{ton for RM2}$$

$$\$\ \pi_3/\text{ton for RM3}$$

These prices are the variables in his problem. Total payment comes to $1500\pi_1 + 1200\pi_2 + 500\pi_3$ which he wants to minimize.

FERTILIZER MAKER: won't sell supplies unless detergent maker's prices are competitive with each of hi-ph, lo-ph processes.

Hi-ph process converts a packet of {*2 tons RM1, 1 ton RM2, and 1 ton RM3*} into \$15 profit. In terms of detergent maker's prices, the same packet yields $\$(2\pi_1 + \pi_2 + \pi_3)$. So, she demands $2\pi_1 + \pi_2 + \pi_3 \geq 15$ for signing deal.

Similarly, by analyzing lo-ph process, she demands $\pi_1+\pi_2 \geq 10$ to sign deal.

From these economic arguments, we see that detergent maker's problem is:

Dual						Associated primal activity
Minimize	$1500\pi_1$	$+1200\pi_2$	$+500\pi_3$			
subject to	$2\pi_1$	$+\pi_2$	$+\pi_3$	$\geq$	15	Hi-ph
	π_1	$+\pi_2$		$\geq$	10	lo-ph
	$\pi_1,$	$\pi_2,$	$\pi_3,$	$\geq$	0	

From arguments, we see that if dual problem has unique opt. sol., it is the marginal value vector for primal.

Summary of Primal, Dual Relations:

1. There is a dual variable associated with each primal constraint, it is the marginal value of the RHS constant in that constraint.
2. There is a dual constraint corresponding to each primal variable.
3. RHS constants in primal (dual) are objective coefficients in dual (primal).

4. If primal is a maximization problem, dual is a minimization problem & vice versa.

5. The dual of the dual problem is the primal.

6. When optimum solutions exist, optimum objective values in the primal and dual are equal.

Dual of LP in Standard Form:

An LP is in STANDARD FORM if:

- All constraints are equations.
- All variables are required to be ≥ 0.
- Objective function in minimization form.

Here is the general LP in standard form in a detached coefficient tableau:

x_1	$\dots$	x_j	$\dots$	x_n	$-z$		Dual var.
a_{11}	$\dots$	a_{1j}	$\dots$	a_{1n}	0	b_1	π_1
$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{m1}	$\dots$	a_{mj}	$\dots$	a_{mn}	0	b_m	π_m
c_1	$\dots$	c_j	$\dots$	c_n	1	0	

$x_j \geq 0$ for all j, min. z

The Dual is:

$$\begin{array}{lcccccccccc} \text{Max. } v(\pi) = & b_1\pi_1 & + & \dots & + & b_m\pi_m & & & & & \text{Primal var.} \\ \text{S. to} & a_{11}\pi_1 & + & \dots & + & a_{m1}\pi_m & \leq & c_1 & & & x_1 \\ & \vdots & & & & \vdots & & \vdots & & & \vdots \\ & a_{1j}\pi_1 & + & \dots & + & a_{mj}\pi_m & \leq & c_j & & & x_j \\ & \vdots & & & & \vdots & & \vdots & & & \vdots \\ & a_{1n}\pi_1 & + & \dots & + & a_{mn}\pi_m & \leq & c_n & & & x_n \end{array}$$

The dual constraint corresponding to primal variable

x_j is $a_{1j}\pi_1 + \dots + a_{mj}\pi_m \leq c_j$

The dual slack variable denoted by $\bar{c}_j = c_j - (a_{1j}\pi_1 + \dots + a_{mj}\pi_m)$ is called *reduced* or *relative cost coefficient* of x_j WRT π.

Dual Feasibility: The row vector π satisfies all dual constraints, i.e., is dual feasible, iff $\bar{c}_j \geq 0$ for all j.

Example:

$$\text{Minimize } z = 3x_1 + 11x_2 - 15x_3 + 10x_4 + 4x_5 + 57x_6$$

$$\text{s. to} \quad x_1 + 2x_2 + 3x_3 - 2x_4 + x_5 + 16x_6 = 17$$

$$x_2 - 4x_3 + x_4 + x_5 + x_6 = 2$$

$$x_3 - 2x_4 + x_5 = 1$$

$$x_j \geq 0 \text{ for all } j.$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7		
1	1	1	1	1	−2	1	0	10
1	1	1	0	−1	−1	2	0	7
1	1	0	0	1	1	−1	0	3
1	0	0	0	−1	2	−2	0	1
4	2	2	−1	−2	5	2	1	0

$x_j \geq 0 \ \forall\, j$, min z

Opt. conds. for an LP in standard form for a given basic vector are:

1. Primal feasibility: Set nonbasics = 0.

Solve remaining system of eqs. for basic variables. They must be ≥ 0 in resulting primal basic sol.

Example: Take basic vector (x_1, x_2, x_3, x_4) in above problem.

2. Dual feasibility: Compute dual basic sol. & check dual feasibility.

To get dual basic sol. solve system of dual constraints corresponding to basic variables as a system of eqs.

Compute all $\bar{c}_j$ = slack in dual constraint of nonbasic var. x_j. If all these $\bar{c}_j \geq 0$, the basic vector is dual feasible.

Optimal basic vector: if it is both primal and dual feasible.

Primal sol. corresponding to optimal basic vector is Opt. BFS.

If optimal BFS nondegenerate, dual opt. sol. unique, it is

marginal value vector.

If optimal BFS degenerate, dual may have many opt. sols., & marginal values do not exist.

Example above.

Example: **Dual of the balanced transportation problem:**

Consider transporting iron ore from mines 1, 2 to plants 1, 2, 3 with following data (c_{ij} = cost (cents/ton) to ship ore from mine i to plant j

	Plant $j = 1$	Plant 2	Plant3	Availability a_i
Mine $i = 1$	$c_{11} = 11$	8	2	800
Mine 2	7	5	4	300
Requirement b_j	400	500	200	

Let $\quad x_{ij}$ = tons ore shipped from mine i to plant j. Model is:

x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	$-z$	RHS	Dual var.
1	1	1	0	0	0	0	800	u_1
0	0	0	1	1	1	0	300	u_2
1	0	0	1	0	0	0	400	v_1
0	1	0	0	1	0	0	500	v_2
0	0	1	0	0	1	0	200	v_3
11	8	2	7	5	4	1	0	

$$x_{ij} \geq 0 \; \forall \; i, j, \text{ min } z$$

The dual of this problem is:

In the same way for the general balanced transportation problem with m sources, n sinks, and

a_i = units available at ith source

b_j = units required at jth sink

c_{ij} = \$s/unit to ship from source i to sink j,

the dual problem with dual variables

u_i associated with source i

v_j associated with sink j

is: max $\Sigma_{i=1}^{m} a_i u_i + \Sigma_{j=1}^{n} b_j v_j$

s. to $u_i + v_j \leq c_{ij}$ (dual constraint associated with x_{ij})

$i = 1$ to m, $j = 1$ to n.

HW problems: 8.1: Consider LP;

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
0	1	1	−1	1	2	0	5
1	1	2	1	2	−1	0	7
2	1	−1	2	0	1	0	3
3	1	−1	6	−2	7	1	0

$x_j \geq 0 \ \forall\ j$, min z

Write the dual problem, showing the association of each dual constraint to its associated primal var.

The opt. canonical tableau for this problem is given below. Use info. in it to compute an opt. dual sol. & determine marginal value vector if it exists.

BV	x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
x_2	0	1	−1/3	−4/3	0	11/3	0	3
x_1	1	0	−1/3	5/3	0	−4/3	0	0
x_5	0	0	4/3	1/3	1	−5/3	0	2
$-z$	0	0	3	3	0	4	1	1

8.2 Consider the LP

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
1	0	0	−1	1	2	−1	0	4
−1	1	0	2	−2	2	−1	0	1
3	−1	1	1	9/2	3	3	0	9
2	−1	2	7	6	4	10	1	0

$x_j \geq 0 \; \forall \; j$, min z

Compute the primal and dual basic solutions corresponding to the basic vectors (x_1, x_2, x_3), (x_4, x_2, x_3), (x_5, x_2, x_3), (x_6, x_2, x_3) and determine which of these basic vectors are: primal feasible or infeasible; dual feasible or infeasible; nondegenerate or degenerate; optimal or nonoptimal.

8.3: For the following problem in standard form, write the dual, and compute the primal and dual basic solutions corresponding to the basic vector (x_1, x_2, x_3, x_4) by backsubstitution, & see if you can identify an opt. sol.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
1	2	1	1	1	−2	0	0	11
0	1	2	1	9	0	−3	0	11
0	0	1	2	−7	4	1	0	11
0	0	0	1	4	2	2	0	4
−1	−4	−4	2	−7	15	14	1	0

$x_j \geq 0 \; \forall \; j$, min z

8.4: For following LP write the dual & check whether (x_1, x_2, x_3) is an optimum basic vector.

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
1	1	1	1	−1	−3	0	12
1	2	−1	−3	1	−1	0	13
0	0	1	−1	−3	1	0	3
3	1	10	5	−16	10	1	0

$x_j \geq 0 \; \forall \; j$, min z

8.5: Write the dual of the following balanced transportation problem.

	Sink $j = 1$	Sink 2	Sink 3	Availability a_i
Source $i = 1$	$c_{11} = 10$	6	8	100
Source 2	15	14	15	100
Requirement b_j	33	150	17	

A basic vector for this problem is $(x_{12}, x_{13}, x_{21}, x_{22})$. Compute the primal basic solution corresponding to this basic vector by back substitution.

Compute the dual basic solution corresponding to this basic vector by assuming that v_3 = dual var.

associated with sink 3 is 0.