Bounded Variable Primal Simplex Algorithm

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The *standard form* is: min $z = cx$, subject to $Ax = b, 0 \leq x \leq u$ where $A_{m\times n}$ and rank m. Some of the u_j may be $+\infty$.

A BFS for this system is defined by a *partition* of the variables into 3 sets (B, L, U) where:

B = Vector of basic variables each of which can take any value within its bounds in the BFS, the columns of A corresponding to these variables must form a basis.

L = Vector of nonbasic variables which are fixed at lower bound 0 in the BFS.

U = Vector of nonbasic variables which are fixed at their upper bounds in the BFS.

If B is the basis matrix, then the basic solution corresponding to the partition (B, L, U) is given by:

$$
\begin{aligned}
x_L &= 0 \\
X_U &= u_U \\
x_B &= B^{-1}(b - A_{.U}u_U)
\end{aligned}
$$

The partition (B, L, U) is said to be a *feasible partition* if this basic sol. is feasible, i.e., if $0 \le B^{-1}(b - A_{.U}u_U) \le u_B$.

A feasible sol. $\bar{x} = (\bar{x}_j)$ is a BFS iff $\{A_{.j} : j$ such that $0 < \bar{x}_j < u_j\}$ is linearly independent.

BFS	:	Basic sol. corresponding to a feasible partition.
Nondegenerate BFS	:	One in which every basic variable is strictly within its bounds.
Degenerate BFS	:	One in which at least one basic variable is equal to its lower or upper bound.

Dual Basic Sol. Corresponding to Partition (B, L, U)

Let π be vector of dual variables associated with "$Ax = b$", and μ_j the dual variable associated with "$x_j \le u_j$" for $j \in J$.

Let $J = \{j : u_j < \infty\}$, $\bar{J} = \{j : u_j = +\infty\}$. The dual basic sol. associated with (B, L, U) is:

$$\begin{aligned} \bar{\pi} &= c_B B^{-1} \\ \text{for } j \in J,\ \bar{\mu}_j &= \text{Min } \{\bar{c}_j, 0\}, \text{ where } \bar{c}_j = c_j - \bar{\pi} A_{.j} \end{aligned}$$

Optimality Criterion

(B, L, U) be a feasible partition associated with BFS $\bar{x}$, and let $\bar{\pi} = c_B B^{-1}$, $\bar{c} = c - \bar{\pi} A$. This partition optimal if:

$\bar{c}_j \geq 0$ for all $x_j \in L$, and

$\bar{c}_j \leq 0$ for all $x_j \in U$

BOUNDED VARIABLE PRIMAL SIMPLEX ALGORITHM

Inputs Needed: Problem in bounded variable standard form, and a feasible partition for it.

1. Checking optimality: Let (B, L, U) be the present partition associated with the BFS $\bar{x} = (\bar{x}_B, \bar{x}_L, \bar{x}_U)$, where

$$\begin{aligned} \bar{x}_L &= 0 \\ \bar{x}_U &= u_U \\ \bar{x}_B &= \bar{g} = B^{-1}(b - A_{.U} u_U) \end{aligned}$$

Inverse tableau wrt x_B maintained as before, with only difference that nonbasics in U are at upper bound, and basic values vector is $\bar{g}$, and not $\bar{b} = B^{-1}b$.

Let $\bar{\pi} = c_B B^{-1}$ be present dual sol.

For all nonbasics x_j compute $\bar{c}_j$. If

$$\bar{c}_j \begin{cases} \geq 0 & \text{for all } x_j \in L \\ \leq 0 & \text{for all } x_j \in U \end{cases}$$

(B, L, U) is an optimum partition, $\bar{x}, \bar{\pi}$ are primal and dual opt. sols., terminate.

Otherwise let $E = \{j$: either $x_j \in L$ and $\bar{c}_j < 0$; or $x_j \in U$ and $\bar{c}_j > 0\}$, indices of eligible nonbasics to enter basic vector x_B.

2. Select Entering Variable: Select a nonbasic variable x_j for a $j \in E$ as the entering variable, say x_s.

3. Minimum ratio Test: Let $\bar{A}_{.s} = (\bar{a}_{is} : i = 1 \text{ to } m)$ be the updated column of x_s.

Case 1: If $x_s \in L$, we have $\bar{c}_s < 0$, and the objective value decreases if the value of x_s is increased from its present value of 0, to say λ. So, the new solution, $x(\lambda)$ as a function of λ is:

$$
\begin{aligned}
\text{Nonbasics other than } x_s &= \text{Present value (lower/upper bound)} \\
x_s &= \lambda \\
x_B &= \bar{g} - \lambda \bar{A}_{.s} \\
z &= \bar{z} + \lambda \bar{c}_s
\end{aligned}
$$

Need to make sure that x_s and all the variables in x_B remain within their lower and upper bounds in the new solution. The maximum value that λ can have subject to this condition is the *minimum ratio* θ in this step.

$$\theta = \min[u_s, \{\frac{u_i - \bar{g}_i}{-\bar{a}_{is}} : i \ni \bar{a}_{is} < 0\}, \{\frac{\bar{g}_i}{\bar{a}_{is}} : i \ni \bar{a}_{is} > 0\}]$$

where $\bar{g}_i, u_i$ represent the present value and upper bound for the ith basic variable, $i = 1$ to m.

- If $\theta = \infty$, z is unbounded below, terminate.
- If $\theta = u_s < \infty$, update the basic values vector by putting $\lambda = \theta$. Let $L' = L \backslash \{x_s\}, U' = U \cup \{x_s\}$. With (B, L', U') as the new partition and new BFS go back to 1.
- If $\theta < u_s$ make changes in the basic vector, and carry out the pivot step to update the inverse tableau, as given below in 4.

EXAMPLES:

Basic Var.	Basic Value	Upper Bound	Updated entering col.			
x_1	0	40	-1	-1	-1	-1
x_2	10	40	-1	-1	-1	-1
x_3	20	40	0	0	0	1
x_4	30	80	1	1	0	0
x_5	0	80	1	-1	-1	0
x_6	15	80	1	-1	-1	0
UB of entering var.			20	10	50	50

Case 2: If $x_s \in U$, we have $\bar{c}_s > 0$, and objective value decreases if the value of x_s is decreased from its present value of u_s. So, we make the new value of $x_s = u_s - \lambda$, leaving all other nonbasic variables at their present values. So, the new solution $x(\lambda)$ is:

$$\begin{aligned}
\text{Nonbasics other than } x_s &= \text{Present value (lower/upper bound)} \\
x_s &= u_s - \lambda \\
x_B &= \bar{g} + \lambda \bar{A}_{.s} \\
z &= \bar{z} - \lambda \bar{c}_s
\end{aligned}$$

Based on same arguments as before, in this case the minimum ratio θ in this step is

$$\theta = \min[u_s, \{\frac{u_i - \bar{g}_i}{\bar{a}_{is}} : i \ni \bar{a}_{is} > 0\}, \{\frac{\bar{g}_i}{-\bar{a}_{is}} : i \ni \bar{a}_{is} < 0\}]$$

where $\bar{g}_i, u_i$ represent the present value and upper bound for the ith basic variable, $i = 1$ to m.

All remaining work in this case is similar to that in Case 1.

4. **Pivot Step:** Select as the dropping basic variable, to be one whose ration ties for the minimum in the definition of θ.

   (a) Replace the dropping rth basic variable in the basic vector by x_s. Delete x_s from L or U where it is currently.

   (b) $x(\theta)$ is the new BFS. Replace the basic values column in the inverse tableau by the values of the new basic vector in

$x(\theta)$.

Include the dropping variable in L or U depending on whether it is at its lower bound 0, or upper bound, in $x(\theta)$.

Let (B', L', U') be the new partition after these changes.

(c) Update the inverse matrix in the inverse tableau by carrying out the pivot step with the updated column of x_s as the pivot column and row r as the pivot row.

Go back to Step 1 with the new partition, and inverse tableau for it.