Polyhedral Geometry

Let $K \subset R^n$ be a convex polyhedron.

Extreme or Corner Point, or Vertex: GEOMETRIC DEFINITION: $\bar{x} \in K$ is said to be an extreme or corner point or vertex of K if for any $0 < \alpha < 1$

$$x^1, x^2 \in K, \quad \bar{x} = \alpha x^1 + (1-\alpha)x^2 \quad \Longrightarrow x^1 = x^2 = \bar{x}$$

If K is specified by a system of linear constraints, checking efficiently whether a point in it is an extreme point, boils down to checking the linear independence of a set of vectors.

Active, Inactive Constraints at a feasible solution $\bar{x}$: Let K be the set of feasible solutions of the following system of constraints:

$$\begin{aligned} A_{i.}x &= b_i \qquad \text{for } i = 1 \text{ to } p \\ &\geq b_i \qquad \text{for } i = p+1 \text{ to } m \end{aligned}$$

Equality constraints are always active at every feasible solution $\bar{x} \in K$.

For $i = p+1$ to m, ith constraint (inequality)

active at $\bar{x} \in K$ if it holds as an eq. at $\bar{x}$, i.e., if $A_{i.}\bar{x} = b_i$

inactive or **slack** at $\bar{x} \in K$ if it holds as a strict ineq. at $\bar{x}$, i.e., if $A_{i.}\bar{x} > b_i$.

Basic Feasible Solution (BFS) for a System of Linear Constraints: ALGEBRAIC DEFINITION

Let (P) be a system of linear constraints, including possibly equations, inequalities, and bounds on variables.

A *feasible solution* for (P) is a vector that satisfies all constraints in (P).

BFS for (P): Let $\bar{x}$ be a feasible solution for (P). Let (S) be the system of linear equations obtained by treating all active constraints at $\bar{x}$ as equations. $\bar{x}$ is said to be a BFS for (P), iff it is the unique solution for the active system (S) treated as linear equations.

Nondegenerate, Degenerate BFSs: When $\bar{x}$ is a BFS for

(P), it is said to be a *nondegenerate BFS* if the system of equality constraints (S) defined above is a square system (i.e., the number of equations in it = the number of variables in it); *degenerate BFS* if the number of equations in (S) is more than the number of variables in it.

Theorem: Let K be the set of feasible solutions of the system of linear constraints (P). A feasible solution $\bar{x} \in K$ is an extreme point of K iff it is a BFS for (P).

Example:

x_1	x_2	x_3	x_4	x_5	
1	1	4	12	2	16
0	1	1	3	-4	4

$x_j \geq 0$ for all j

$x^1 = (12, 4, 0, 0, 0)^T, \quad x^2 = (0, 0, 1, 1, 0)^T.$

Nondegenerate, Degenerate systems, polyhedra: Let K be the set of feasible solutions of a system of linear constraints (P). $K, (P)$ are said to be *nondegenerate* if every BFS is nondegenerate; *degenerate* if at least one BFS is degenerate.

BFS for Standard Form: A feasible solution $\bar{x}$ for system

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

is a BFS iff $\{A_{.j} : j \text{ such that } \bar{x}_j > 0\}$ is linearly independent.

Nondegenerate, Degenerate BFSs for Standard Form: For system in standard form $Ax = b, x \geq 0$ where $A_{m \times n}$ has rank m; a BFS $\bar{x}$ is nondegenerate if number of positive variables in it is m, degenerate if this number is $< m$.

So, for a nondegenerate BFS the positive variables in it define the unique basic vector, basis matrix corresponding to it.

A basic vector corresponding to a degenerate BFS always contains some 0-valued basic variables, these can be choosen among the 0-valued variables in the BFS arbitrarily as long as the linear independence condition holds. So, usually, a degenerate BFS corresponds to many basic vectors.

Example:

x_1	x_2	x_3	x_4	x_5	
1	0	3	1	2	6
0	1	4	2	1	8

$$x_j \geq 0 \text{ for all } j$$

$\bar{x} = (6, 8, 0, 0, 0)^T$, $\tilde{x} = (0, 0, 2, 0, 0)^T$.

BFS of Standard Form for Bounded Variable System:

A feasible solution $\bar{x}$ for system

$$\begin{aligned} Ax &= b \\ \ell = (\ell_j) \leq x &\leq u = (u_j) \end{aligned}$$

is a BFS iff $\{A_{.j} : j \text{ such that } \ell_j < \bar{x}_j < u_j\}$ is linearly independent.

In the above system if $A_{m \times n}$ has rank m, a BFS $\bar{x}$ for it is nondegenerate if $J = \{j : \ell_j < \bar{x}_j < u_j\}$ has cardinality m; degenerate if $|J| < m$.

Purification Routine to Derive a BFS from a Feasible Solution

We describe for the system in standard form, $Ax = b, \quad x \geq 0$ where $A_{m \times n}$. Let $\bar{x}$ be a feasible solution with $J = \{j : \bar{x}_j > 0\}$.

If $\Gamma = \{A_{.j} : j \in J\}$ is linearly independent, $\bar{x}$ is a BFS, terminate.

If Γ linearly dependent, let the ℓ.d. relation be

$$\begin{aligned} \sum_{j \in J} \alpha_j A_{.j} &= 0 \\ \text{we also have} \quad \sum_{j \in J} \bar{x}_j A_{.j} &= b \\ \text{So} \quad \sum_{j \in J} (\bar{x}_j + \lambda \alpha_j) A_{.j} &= b \end{aligned}$$

where λ is a real valued parameter. Hence if we define $x(\lambda) = (x_j(\lambda))$ where

$$x_j(\lambda) = \begin{cases} \bar{x}_j + \lambda \alpha_j & j \in J \\ 0 & j \notin J \end{cases}$$

then $x(\lambda)$ satisfies $Ax = b$. Let

$$\theta_1 = \begin{cases} -\infty & \text{if } \alpha_j \leq 0 \text{ for all } j \in J \\ \max\{-\bar{x}_j/\alpha_j : j \in J, \alpha_j > 0\} & \text{otherwise} \end{cases}$$

$$\theta_2 = \begin{cases} +\infty & \text{if } \alpha_j \geq 0 \text{ for all } j \in J \\ \min\{-\bar{x}_j/\alpha_j : j \in J, \alpha_j < 0\} & \text{otherwise} \end{cases}$$

Then $\theta_2 > 0 > \theta_1$ and at least one of them is finite. For all $\theta_1 \leq \lambda \leq \theta_2$, $x(\lambda)$ is a feasible solution.

Let $\theta = \theta_1$ or θ_2 whichever is finite. Then $x(\theta)$ is a feasible solution in which number of positive variables is one less than that of $\bar{x}$. Repeat with $x(\theta)$.

Example

x_1	x_2	x_3	x_4	x_5	x_6	
1	0	1	-3	-6	12	2
-1	2	1	-1	17	18	10
0	1	1	-2	8	-5	6
0	1	1	-2	-9	-6	6

$x_j \geq 0$ for all j

$\bar{x} = (1, 2, 10, 3, 0, 0)^T$.

Example: Consider $\bar{x} = (1/3, 4/3)^T$ feasible to

$$
\begin{aligned}
x_1 + 2x_2 &\geq 2 \\
2x_1 + x_2 &\geq 2 \\
x_1 &\geq 0 \\
x_2 &\geq 0 \\
-x_1 + x_2 &\leq 2
\end{aligned}
$$

Theorem: If system in standard form $Ax = b, x \geq 0$ has a feasible solution, it has a BFS.

Theorem: If LP in standard form

$$
\begin{aligned}
\min \quad z &= cx \\
\text{subject to} \quad Ax &= b \\
x &\geq 0
\end{aligned}
$$

has an optimum solution, it has one which is a BFS.

Adjacency of Extreme Points, Edges

Adjacency: GEOMETRIC DEFINITION: Two extreme points x^1, x^2 of a convex polyhedron $K \subset R^n$ are said to be *adjacent* iff every point $\bar{x}$ on the line segment joining them satisfies:

$x^3, x^4 \in K$, $\bar{x} = \alpha x^3 + (1-\alpha)x^4$ for some $0 < \alpha < 1 \quad \Longrightarrow$ x^3, x^4 are also on the line segment joining x^1, x^2.

ALGEBRAIC DEFINITION: Suppose K is specified by system of linear constraints (P). Let (S) be the system of linear equations obtained as follows: (S) contains all the linear equations in (P), and all the inequality constraints and bound restrictions which hold as equations at some point in the interior of the line segment joining x^1, x^2, for example $(x^1 + x^2)/2$, treated as equations. x^1, x^2 are adjacent iff the set of solutions of (S) is the straight line joining x^1, x^2.

Theorem: Two BFSs x^1, x^2 of system in standard form $Ax = b, x \geq 0$ are adjacent iff rank of $\{A_{.j}$: j such that $\bar{x}_j > 0$, where

$\bar{x}$ is some point in the interior of the line segment joining x^1, x^2, for example $(x^1 + x^2)/2\}$ is one less than its cardinality.

Example: Put following system in standard form, & check whether x^1, x^2 are adjacent extreme points of it. What about x^1, x^3?

$$
\begin{aligned}
x_1 + 2x_2 &\geq 2 \\
2x_1 + x_2 &\geq 2 \\
x_1 &\geq 0 \\
x_2 &\geq 0 \\
-x_1 + x_2 &\leq 2
\end{aligned}
$$

$x^1 = (0,2)^T$, $x^2 = (2/3, 2/3)^T$, $x^3 = (2,0)^T$.

Bounded Edge: An *(Bounded) edge* of a convex polyhedron is the line segment joining two adjacent extreme points.

How to check whether $\bar{x}$ feasible to a system (P) is on an edge, and whether that edge is bounded or unbounded

If $\bar{x}$ is a BFS (Extreme point) it is of course on edges containing it. So assume $\bar{x}$ not a BFS.

Let (S) be the active system of constraints at $\bar{x}$ treated as a system of eqs. $\bar{x}$, which is not a BFS, is on an edge iff set of solutions of (S) is one dimensional, i.e., is a straight line.

If $\bar{x}$ on an edge, that edge bounded iff set of sols. of (S) & all other constraints in (P) not in (S) is a line segment; unbounded edge otherwise.

Example: Consider $\bar{x} = (5, 10, 3, 0, 0)^T$ feasible to

x_1	x_2	x_3	x_4	x_5	
1	−1	−1	1	−2	−8
0	1	2	3	8	16
−1	1	1	8	−9	8

$x_j \geq 0$ for all j

Example: Consider $\bar{x} = (2, 4, 8, 12, 0, 0)^T$ feasible to

x_1	x_2	x_3	x_4	x_5	x_6	
1	0	0	-1	1	17	-10
0	1	-1	0	2	18	-4
0	0	1	-1	3	19	-4

$x_j \geq 0$ for all j

Example: Consider $\bar{x} = (2, 4, 8, 12, 0, 0)^T$ feasible to

x_1	x_2	x_3	x_4	x_5	x_6	
1	0	0	1	1	17	14
0	1	-1	2	2	18	20
0	0	1	-1	3	19	-4

$x_j \geq 0$ for all j

How to Obtain Adjacent BFSs?, Pivot Steps

Let K be the set of feasible solutions of system in standard form, $Ax = b, x \geq 0$ where $A_{m\times n}$ and rank m.

A basic vector for this system is said to be a *feasible basic vector*, if corresponding basic solution is ≥ 0, i.e., it is a BFS.

Let x^1 be a BFS associated with basic vector x_B, basis B. Let x_D be vector of nonbasic variables. System can be rearranged into basic, nonbasic parts.

Original Tableau

x_B	x_D	
B	D	b

$x \geq 0$

The *canonical tableau* for syatem wrt x_B obtained by multiplying tableau on left by B^{-1}.

Canonical Tableau

Basic var.	x_B	x_D	
x_B	I	$B^{-1}D$	$B^{-1}b = \bar{b}$

$x \geq 0$

x^1, the BFS wrt x_B is given by:

$$\text{Nonbasic variables} \quad x_D = 0$$
$$\text{Basic vector} \quad x_B = B^{-1}b = \bar{b} \geq 0$$

Process of obtaining an adjacent BFS of x^1 starts with the following:

(i) Select one nonbasic variable, x_s say, as the *entering variable* into the basic vector x_B.

(ii) Fix all nonbasic variables other than x_s at 0.

(iii) Make x_s, the entering nonbasic $= \lambda$, a parameter.

(iv) Find unique values of basic variables in x_B as functions of λ to satisfy $\quad Ax = b$.

We first show these before describing the rest of the process. The whole process is called *pivot step for entering* x_s *into* x_B. Below we show remaining canonical tableau after doing (ii).

Remaining Canonical Tableau

Basic	Entering	
$x_1 \dots x_m$	x_s	
	$\bar{a}_{1s}$	$\bar{b}_1$
I	$\vdots$	$\vdots$
	$\bar{a}_{ms}$	$\bar{b}_m$

So $x(\lambda)$ is given by the following:

$$\begin{aligned} \text{Other nonbasics} &= 0 \\ \text{Entering } x_s &= \lambda \\ \text{Basic} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} &= \begin{pmatrix} \bar{b}_1 - \lambda \bar{a}_{1s} \\ \vdots \\ \bar{b}_m - \lambda \bar{a}_{ms} \end{pmatrix} \end{aligned}$$

This solution remains feasible if $\bar{b}_i - \lambda \bar{a}_{is} \geq 0$ for all i, i.e., if $0 \leq \lambda \leq \theta$ where

$$\theta = \begin{cases} +\infty \text{ if } \bar{a}_{is} \leq 0 \text{ for all } i \\ \min\{\bar{b}_i / \bar{a}_{is} : \text{over } \bar{a}_{is} > 0\}, \quad \text{otherwise} \end{cases}$$

This θ is called *minimum ratio* in this pivot step, process of computing it called *minimum ratio test*.

1. If $\theta = \infty$, $\{x(\lambda) : \lambda \geq 0\}$ is an *unbounded edge* or *extreme half-line* of K. Its direction given by:

$$\begin{array}{rcl} \text{Other nonbasics} & = & 0 \\ \text{Entering } x_s & = & 1 \\ \text{Basic} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} & = & \begin{pmatrix} -\bar{a}_{1s} \\ \vdots \\ -\bar{a}_{ms} \end{pmatrix} \end{array}$$

is an *extreme direction* of K. It is an extreme point of the normalized homogeneous system:

$$\begin{array}{rcl} Ax & = & 0 \\ \sum_{j=1}^{n} x_j & = & \beta \\ x & \geq & 0 \end{array}$$

for some strictly positive quantity β.

2. If $0 < \theta < +\infty$, $x(\theta)$ is an adjacent BFS of x^1. In $x(\theta)$ at least one present basic variable (one attaining min in the min ratio test) is 0, select one of them as the *dropping basic variable* to be replaced by entering variable, leading to a new feasible basic vector, $x_{\tilde{B}}$ say, for which $x(\theta)$ is the BFS.

If dropping basic variable is rth, performing pivot step with updated col of x_s as pivot col and rth row as pivot row in present canonical tableau, leads to canonical tableau wrt $x_{\tilde{B}}$.

Purpose of min ratio test: To determine dropping basic var. that entering var. should replace to make sure next basic vector is also feasible.

When min ratio $\theta > 0$ and finite, pivot step called *nondegenerate pivot step*. It always leads to an adjacent BFS. The line segment joining x^1 and $x(\theta)$ is an *edge (bounded edge)* of K.

3. If $\theta = 0$, $x(\theta) = x^1$. So, new basic vector $x_{\tilde{B}}$ is another basic vector corresponding to BFS x^1. In this case pivot step called *degenerate pivot step*. In a degenerate pivot step, BFS does not change, but basic vector changes.

Example: Consider basic vector (x_1, x_2, x_3) for

Original Tableau

x_1	x_2	x_3	x_4	x_5	x_6	
1	1	0	-1	1	1	7
0	1	0	-1	-1	1	0
0	0	1	0	0	1	5

$x_j \geq 0$ for all j

Example: Consider extreme point $x^1 = (1, 1)^T$ for the system:

$$
\begin{aligned}
x_1 + x_2 &\geq 2 \\
x_1 - x_2 &\geq 0 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

The Main Results

1. Consider the LP: to minimize $z = cx$ subject to some linear constraints.

Let K denote set of feasible sols.

Let $\bar{x}$ be an extreme point of K satisfying:

> Moving away from $\bar{x}$ along any edge incident at $\bar{x}$ either increases the value of $z = cx$, or keeps it unchanged.

Then $\bar{x}$ is an optimum solution of this LP.

2. Consider LP in standard form: min $z = cx$ subject to $Ax = b$, $x \geq 0$.

The minimum value of z in this LP is $-\infty$ (i.e., z unbounded below in this LP) iff there exists a BFS $\bar{x}$, and an unbounded edge incident at $\bar{x}$ along which z diverges to $-\infty$.

These results are the foundation for the **Simplex Algorithm** for LP.

Faces of a Convex Polyhedron

Let $K \subset R^n$ be a convex polyhedron.

Supporting Hyperplane for K: A hyperplane H is said to be a *supporting hyperplane for* K if $H \cap K \neq \emptyset$ and K is completely contained on one side of H.

Face of K : A *face* of K is either $\emptyset$, or K itself, or the intersection $H \cap K$ for some supporting hyperplane H of K.

Faces of a Polyhedron Defined by Linear Constraints: Let K be the polyhedron by system of linear constraints (P).

Take a subset of inequality constraints in (P) and make them into equations. Take a subset of variables with bound restrictions and make each of them = one of the bounds on it. Let (Q) be the resulting system.

The set of feasible solutions of (Q) is a face of K, and conversely every face of K is the set of feasible solutions of a system like (Q) obtained from (P).

Faces for System in Standard Form: Let K be set of feasible solutions of $Ax = b, x \geq 0$. Each face of K is the set of feasible solutions of remaining system after a subset of variables is fixed at 0.

Theorem: Let K be the set of feasible solutions of an LP. The set of optimum solutions of this LP is always a face of K.

The extreme points of a convex polyhedron are its 0-dimensional faces. The 1-dimensional faces are the edges. The vertices and edges put together define the *one dimensional skeleton* or *the graph* of the polyhedron.

Faces of the polyhedron whose dimension is one less than the dimension of the polyhedron are called its *facets*.

Boundedness of Convex Polyhedra

Homogeneous System: Let (P) be a system of linear constraints. The system (H) obtained by changing all the RHS constants in all constraints and bound restrictions in (P) to 0, is called the *homogeneous system* corresponding to (P).

Theorem: Every feasible solution $\bar{x}$ of the system in standard form: $Ax = b, x \geq 0$ can be expressed as:

(a convex comb. of BFSs) + (a homogeneous solution).

Extreme Homogeneous Solution: The homogeneous system corresponding to standard form above is: $Ay = 0, y \geq 0$.

An *extreme homogeneous solution* is an extreme point of the normalized homomgeneous system, i.e.,

$$\begin{aligned} Ay &= 0 \\ \sum y_j &= 1 \\ y &\geq 0 \end{aligned}$$

Theorem: Every homogeneous solution for system in standard form is a nonnegative comb. of extreme homogeneous solutions.

Theorem: If K = set of feasible solutions of system in standard form, and $K \neq \emptyset$, K bounded iff 0 is the unique homogeneous solution.

Theorem: Resolution Theorem for Systems in Standard Form: Every feasible solution can be expressed as

(a convex comb. of BFSs) + (a nonnegative comb. of extreme homogeneous solutions).

Theorem: The LP in standard form

$$\begin{aligned} \text{Min} \quad & z = cx \\ \text{subject to} \quad Ax &= b \\ x &\geq 0 \end{aligned}$$

has unbounded minimum (i.e., $z \longrightarrow -\infty$) iff it is feasible, and the following system is feasible in y

$$\begin{aligned} Ay &= 0 \\ cy &< 0 \\ y &\geq 0 \end{aligned}$$