4.1

Simplex Algorithm Using Canonical Tableaus

Consider LP in standard form:

$$
\begin{aligned}
\text{Min} \quad z = cx &+ \alpha \\
\text{subject to} \quad Ax &= b \\
x &\geq 0
\end{aligned}
$$

where $A_{m \times n}$ has rank m and α is a constant. In tableau form we record it as below.

Original Tableau

x	$-z$	
A	0	b
c	1	$-\alpha$

$x \geq 0$

Suppose a BFS $\bar{x}$ associated with a feasible basic vector x_B is available. Take $-z$ as the basic variable in the objective row, and we have $(x_B, -z)$ as a feasible basic vector for the original tableau.

Given below is the canonical tableau wrt it (we take $x_B = (x_1, \ldots, x_m)$).

Canonical Tableau

Basic	x_1	$\ldots$	x_m	x_{m+1}	$\ldots$	x_n	$-z$	
x_1	1	$\ldots$	0	$\bar{a}_{1,m+1}$	$\ldots$	$\bar{a}_{1n}$	0	$\bar{b}_1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
x_m	0	$\ldots$	1	$\bar{a}_{m,m+1}$	$\ldots$	$\bar{a}_{mn}$	0	$\bar{b}_m$
$-z$	0	$\ldots$	0	$\bar{c}_{m+1}$	$\ldots$	$\bar{c}_n$	1	$-z_0$

The Present BFS $\bar{x}$ is:

$$\begin{aligned} \text{nonbasics } x_{m+1} \text{ to } x_n &= 0 \\ \text{Basic} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} &= \begin{pmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_m \end{pmatrix} \\ z &= z_0 \end{aligned}$$

Columns in canonical tableau are called *updated columns* wrt present basic vector.

$\bar{c}_{m+1}, \ldots, \bar{c}_n$ are called *reduced* (or *relative*) cost coefficients of present nonbasics wrt present basic vector.

Theorem: Sufficient Opt. Criterion: If all nonbasic reduced cost coeffs. are ≥ 0, z_0 is the opt. obj. value, and present BFS is an opt. sol.

Theorem: Suppose $E = \{j : \bar{c}_j < 0\}$. Nonbasics x_j for $j \in E$ are said to be *eligible to enter the present basic vector*. Choosing any one of them as *entering variable* leads to:

- an extreme half-line along which $z \longrightarrow -\infty$ if its updated column has no positive entries (no pivot step carried out in this case).
- a strict improvement in objective value if pivot step is nondegenerate (a better adjacent BFS).
- another basic vector for the same BFS if pivot step is degenerate.

Change in obj. value in a pivot step = (Min ratio)×(Rel. cost coeff. of entering var.)

Simplex Algorithm

Inputs needed: Problem in standard form, a feasible basic vector.

1. If sufficient opt. criterion satisfied in present canonical tableau, present BFS optimal, terminate.

2. Select an entering variable among eligible vartiables.

  2.1 If updated col. of entering variable has no positive entries, z unbounded below, terminate.

  2.2 Otherwise perform pivot step, and go back to 1. with canonical tableau of new basic vector.

Example

$$\begin{aligned} \text{Min} \quad & z = -x_5 - 8x_6 \\ \text{subject to} \quad x_1 - x_5 + x_6 &= 2 \\ x_2 + x_5 + x_6 &= 1 \\ x_3 + 2x_5 + x_6 &= 5 \\ x_4 + x_6 &= 0 \\ x &\geq 0 \end{aligned}$$

Example

$$\begin{aligned} \text{Min} \quad & z = 2x_1 + x_2 - x_3 + 3x_4 - 6x_5 + 10x_6 \\ \text{subject to} \quad x_1 + x_3 - 2x_5 - x_6 &= 2 \\ x_2 + x_3 - 3x_5 + 2x_6 &= 4 \\ -x_3 + x_4 + x_5 + 3x_6 &= 1 \\ x &\geq 0 \end{aligned}$$

Example

$$
\begin{aligned}
\text{Min} \quad z = x_1 - 2x_5 - 2x_6 + 5x_7 + 100 & \\
\text{subject to} \quad x_1 - x_4 + x_7 &= 2 \\
x_1 + x_2 + x_5 + 2x_7 &= 3 \\
x_3 + x_6 + x_7 &= 5 \\
x_4 - x_7 &= 0 \\
x &\geq 0
\end{aligned}
$$

Simplex Method to solve General LP

1. First put problem in standard form.

(a) . Make all constraints into equations by introducing slacks.

(b) Eliminate any unrestricted variables by pivoting them out.

(c) Put objective in minimization form.

(d) Make RHS constants ≥ 0.

2. Look for a unit basis. Select basic variables corresponding to unit vectors, in as many rows as possible. If full unit basis found (it will be feasible because of (d)), starting with it go to the simplex algorithm (Phase II).

If full unit basis not found, introduce artificial variables corresponding to missing unit columns, and go to Phase I in which you apply simplex algorithm to minimize sum of artificial variables (Phase I objective function) starting with unit basis.

Phase I Once an artificial leaves basic vector, erase its column from tableau.

If $\bar{w}$ = min. of Phase I obj. value > 0, original model infeasible, go to *infeasibility analysis*.

if $\bar{w} = 0$, with basic vector at the end of Phase I, go to Phase II.

Transition From Phase I to Phase II:

- If all artificials left basic vector, delete last row ($-w$ row) , and the last col. ($-w$ column) in final Phase I tableau. What remains is canonical tableau wrt present basic vector for original LP. Begin Phase II with it.

- If some artificials still in basic vector, their values must be 0 (otherwise $w \neq 0$).

  Fod any j with $\bar{d}_j > 0$, fix $x_j = 0$ and delete it from further consideration from original tableau.

  Delete last row from final Phase I tableau, and go to Phase II with resulting tableau. In this process all artificials in basic vector will remain $= 0$, or leave basic vector.

Example

$$\begin{aligned}
\text{Min} \quad & z = x_1 + 2x_2 + 3x_3 \\
\text{subject to} \quad & 2x_1 + 3x_2 + x_3 - x_4 = 9 \\
& x_1 + 2x_2 - x_3 + x_5 = 5 \\
& x_1 + x_2 + 2x_3 = 4 \\
& x \geq 0
\end{aligned}$$

Examples

Original tableau

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	b
1	−1	0	0	2	0	0	0
−2	1	0	0	−2	0	0	0
1	0	1	0	1	−1	0	3
0	2	1	1	2	1	0	4
−40	−10	0	0	−7	−14	1	0

$x_j \geq 0$ for all j, minimize z

Theorem: Starting at an extreme point, the simplex algorithm traces an edge path along which the objective value is monotonic decreasing. Terminates after a finite number of pivot steps with either

- an extreme half-line along which $z \longrightarrow -\infty$
- or an extreme point optimum.