3.1

Fundamental Concepts

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Active or Tight ; inactive constraints at a feasible solution:

Let $\bar{x}$ be feasible solution to system of constraints. An inequality constraint $g_p(x) \geq 0$ is:

Active or tight at $\bar{x}$	if $g_p(\bar{x}) = 0$, i.e., if satisfied as equation
Inactive at $\bar{x}$	if $g_p(\bar{x}) > 0$, i.e., if satisfied as strict inequality

An equality constraint in system, $h_i(x) = 0$, is always an active constraint at $\bar{x}$.

Direction of Movement, Step Length: Given $\bar{x} \in R^n$, and $0 \neq y \in R^n$, the point obtained by moving from $\bar{x}$ in the direction y a step length of $\alpha(> 0)$ is: $\bar{x} + \alpha y$.

Feasible Direction at a feasible point:

Given $K \subset R^n$, $\bar{x} \in K$, $0 \neq y \in R^n$ is said to be a **feasible direction at $\bar{x}$ for K** if there is a positive α s. th. $\bar{x} + \lambda y \in K$ for all $0 \leq \lambda \leq \alpha$.

Examples: A ball, a circle, a flat (set of feasible solutions of a ststem of linear equations), a convex polyhedron (defined by a system of linear eqs. and inequalities).

Descent Direction: $0 \neq y \in R^n$ is a descent direction for objective function $\theta(x)$ at $\bar{x} \in R^n$ if there is a positive α s. th., $\theta(\bar{x} + \lambda y) < \theta(\bar{x})$ for all $0 < \lambda < \alpha$.

THEOREM: If $\theta(x)$ is differentiable at $\bar{x}$, y is a descent direction for $\theta(x)$ at $\bar{x}$ iff $\nabla \theta(\bar{x}) y < 0$.

Descent Direction in an NLP: In minimization problem, descent direction (or descent feasible direction) at a feasible solution $\bar{x}$ is a feasible direction at $\bar{x}$ that is a descent direction for the objective function being minimized.

Steepest Descent direction at a point for an Unconstrained minimization problem: Consider:

$$\text{minimize} \quad \theta(x)$$

Given $\bar{x}$ and a $y \neq 0$, the directional derivative of $\theta(x)$ at $\bar{x}$ in the direction y is $\theta'(\bar{x}; y) = \lim_{\lambda \to 0^+} \frac{\theta(\bar{x}+\lambda y) - \theta(\bar{x})}{\lambda}$ if this limit exists.

If $\theta(x)$ differentiable at $\bar{x}$, $\theta'(\bar{x}; y) = \nabla\theta(\bar{x})y$.

$\theta'(\bar{x}; y)$ measures the *rate of change in* $\theta(x)$ *at* $\bar{x}$ *in the direction* y. y is a descent direction for $\theta(x)$ at $\bar{x}$ if $\theta'(\bar{x}; y) < 0$.

If descent directions exist for $\theta(x)$ at $\bar{x}$, the **steepest descent direction for** $\theta(x)$ **at** $\bar{x}$ is optimum solution of

$$\begin{aligned} \text{minimize} \quad & \theta'(\bar{x}; y) \\ \text{s. to} \quad & \text{Norm}(y) = 1 \end{aligned}$$

where $\text{Norm}(y)$ = distance between 0 and y. For Euclidean distance $\text{Norm}(y) = \sqrt{y^T y}$.

Commonly $\text{Norm}(y)$ defined to be $\sqrt{y^T D y}$ where D is some PD matrix.

If min in above is ≥ 0, no descent direction for $\theta(x)$ at $\bar{x}$, so no steepest descent direction.

Different norms may give different steepest descent directions.

Steepest Descent direction at a point for a constrained minimization problem: Consider: minimize $\theta(x)$ subject to some constraints.

If descent feasible directions exist for this problem at a feasible point $\bar{x}$, the steepest descent direction for this problem at $\bar{x}$ is optimum solution of

$$\text{minimize } \theta'(\bar{x}; y)$$

$$\text{s. to Norm}(y) = 1$$

and: y is a descent feasible direction at $\bar{x}$ to this problem.