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Network Flow & Routing Models

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Introductory Lecture

Network optimization & routing problems occur in many diverse fields

Network	Application
Phone network	Route calls, messages, data.
Electrical power network	Transmit power from power plants to consumers (industry, business, offices, households).
Highway & street networks	Transport people, vehicles, goods from where they are, to where they need to be.

Rail, airline, shipping networks	”
Computer networks	Transmit information, data, messages, programs etc. from one place to another.
Pipeline networks	Transport crude oil, natural gas, or other fluids.
Satellite networks	Worldwide personal communication system based on low earth orbit satellites being developed by IRIDIUM using 66 satellites. Calls sent up to satellite network by origin ground station, routed through satellite network, & finally to destination on ground.

All these applications have an underlying **graph** or **network**.

Graph = $(\mathcal{N}, \mathcal{A})$ where:

$\mathcal{N}$ = (Finite) Set of points (or nodes, or vertices)

$\mathcal{A}$ = Set of lines (or links), each joining a pair of points.

Graph called a **network** if there is data associated with points &/or lines.

DEFINITIONS:

Line	Has exactly two nodes, it joins them.
Adj. Nodes	Those on a line.
Edge	$(i; j)$ Can be used in both directions.
Arc	(i, j) Directed or oriented line (like a one-way street). Can only be used from i to j. Node i called its **Tail**, node j its **Head**.

Arc incident into i	
Arc incident out of i	
Parallel edges	Join same nodes.
Parallel arcs	Tail same, head same.
Not parallel arcs	
Self-loops	Used only in some models (e.g., generalized network flows). Arc joining a node to itself.

Directed network	All lines are arcs.
Undirected networks	All lines are edges.
Mixed network	Has some arcs & some edges.
Node Degree	No. of lines incident at that node.
In-degree, out-degree	Defined only in directed networks. No. of arcs incident into, out of, a node.

Subnetwork	Of network $G = (\mathcal{N}, \mathcal{A})$ is $(\mathcal{N}, \bar{\mathcal{A}})$ where $\bar{\mathcal{A}} \subset \mathcal{A}$. Same set of nodes, but subset of lines.
Partial network	Of network $G = (\mathcal{N}, \mathcal{A})$ is $(\hat{\mathcal{N}}, \hat{\mathcal{A}})$ where $\hat{\mathcal{N}} \subset \mathcal{N}$ and $\hat{\mathcal{A}}$ is set of all lines in $\mathcal{A}$ with both nodes on them from $\hat{\mathcal{N}}$. Also called *partial network* or *subnetwork induced by subset of nodes* $\hat{\mathcal{N}}$.
Partial subnetwork	Partial network of a subnetwork.

Forward (*Reverse*) star of a node	In directed networks only, the set of all arcs incident out of (*incident into*) that node. Network can be stored by storing the forward star (or reverse star, or both) of each node. Algorithms have need for searching arcs satisfying certain property in every step. This representation very convenient for such searches.

After, before , sets of a node	In directed networks only. $A(i) = \{j : (i,j) \in \mathcal{A}\}$ $B(i) = \{j : (j,i) \in \mathcal{A}\}$.
Graph	Undirected network with no self-loops & no parallel edges.
Multigraph	Undirected network with no self-loops.

Digraph	Directed network with no self-lines & no parallel lines.
Multidigraph	Directed network with no self-loops.
Network	$G = (\mathcal{N}, \mathcal{A})$ with some data on arcs/nodes.

TYPES OF NODES

Source or Supply node	Node i with say $V(i)$ of material available to ship out. Here $V(i) > 0$, it is called **exogenous** or **external flow** at node i
Sink or demand node	Node i with some quantity of material required to be shipped to it from some source nodes. Exogenous flow here is $V(i) < 0$ where $\vert V(i)\vert$ is the requirement.
Transshipment or transit node	Node i with exogenous flow $V(i) = 0$.

DATA ON THE LINES

Bounds for arc flows	For arc (i, j), k_{ij} = **capacity of arc** (i, j) = max. amount that can flow through this arc per unit time. ℓ_{ij} = **lower bound of arc** (i, j) = min. amount that must flow through this arc per unit time. Often, this is 0.
Bounds for edge flows	We assume that lower bounds for edge flows are 0. For edge $(i; j)$, capacity k_{ij} applies to both directions In single commodity flow problems, edge $(i; j)$ can be represented by pair of arcs $(i, j), (j, i)$.

And only one of these two arcs need to be used in any solution because

This type of cancellation not possible if the two flows are of different commodities.

So for studying single commodity flow problems, can assume network directed (each edge replaced by corresponding pair of arcs as above).

Another way to convert edges into arcs

In single commodity flow problems, consider edge $(i; j)$ with capacity k_{ij}. Orient this edge in any direction as an arc, say (i, j), & make lower bound, capacity for this arc $-k_{ij}, +k_{ij}$.

In any solution, if flow on this arc is $f_{ij} > 0$, then in original network edge $(i; j)$ used in direction i to j with flow f_{ij}.
On other hand if $f_{ij} < 0$, edge $(i; j)$ used in direction j to i with flow $|f_{ij}|$.
This technique does not increase no. of lines.

Bounds in multicommodity flows	In multicommodity flow models, all lines are assumed to be arcs, & all commodities measured in common units (e.g., truckloads). On all arcs lower bounds are assumed 0, & capacity applies to sum of flows of all commodities thro' that arc. Traffic flow problems are all multi-commodity flow problems, because even though all flows are of vehicles, each may have a separate origin and destination.
Cost coeffs.	c_{ij} = cost incurred for unit flow on line (i, j). For an edge, cost coeff. assumed to be same for flows in both directions.

PURE & GENERALIZED NETWORKS

Pure networks	For every line (i, j), if w units sent from i to j through arc (i, j), exactly w units reach node j.
Generalized networks	If w units enter arc (i, j) at node i, amount that reaches j is $p_{ij}w$, where p_{ij} called **multiplier of arc** (i, j), i.e. flow quantity gets modified in transit.

If $0 \leq p_{ij} < 1$ arc (i, j) called **lossy arc**.

If $p_{ij} > 1$ arc (i, j) called **gainy arc**.

Lossy arcs occur in models of power transmission through HV power lines.

Gainy arcs occur in financial models over a multiperiod horizon. Money invested earns dividend or interest, hence grows larger.

However, generalized network algorithms not too popular because practitioners unwilling to get multiplier data.

TYPES OF PROBLEMS

Max flow	Find maximum quantity (per unit time) that can be shipped from a source node to a sink node in a capacitated network. Single commodity, multicommodity versions.
Shortest route	Find min cost route from an origin to a destination in a network with cost coefficients on arcs. A fundamental network problem.
Min cost flow	Find most economical way to ship material from sources to sinks in a network with arc costs and capacities.

Assignment, transportation	Very important special min cost flow problems in bipartite networks.
Generalized network flow	Above problems on generalized rather than pure networks.
Project planning	Special network flow problems that arise in scheduling and managing jobs in projects.
Capacity planning, network design	To design an optimal network for a job.
Matchings	Problems of finding optimal subnetworks satisfying certain properties.
Routing problems	Problems of finding optimal routes for a variety of jobs.