# Facets of an Assignment Problem with a $0-1$ Side Constraint

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November 4, 1999

**Abstract**

We show that the problem of finding a perfect matching satisfying a single equality constraint with 0-1 coefficients in an $n \times n$ incomplete bipartite graph, polynomially reduces to a special case of the same problem called the partitioned case. Finding a solution matching for the partitioned case in the incomplete bipartite graph, is equivalent to minimizing a partial sum of the variables over $Q^{n,r_1}_{n_1,n_2}$ = the convex hull of incidence vectors of solution matchings for the partitioned case in the complete bipartite graph. An important strategy to solve this minimization problem is to develop a polyhedral characterization of $Q^{n,r_1}_{n_1,n_2}$. Towards this effort, we present two large classes of valid inequalities for $Q^{n,r_1}_{n_1,n_2}$, which are proved to be facet inducing using a facet lifting scheme.

**Key Words:** Constrained assignment problem, integer hull, facet inducing inequalities, facet lifting scheme.

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# 1 Introduction

The well-known assignment problem of order $n$ deals with minimizing a linear objective function involving $n^2$ variables $x = (x_{ij} : i, j = 1, \ldots, n)$, usually written in the form of a square matrix of order $n$, subject to constraints (1)-(4). Associating the variable $x_{ij}$ with the edge $(i, j)$ in the complete bipartite graph $K_{n,n}$, $G = (I, J, I \times J)$, where $I = \{1, \ldots, n\}$, $J = \{1, \ldots, n\}$, each assignment $\bar{x} = (\bar{x}_{ij})$, i.e., feasible solution of (1)-(4), is associated with the perfect matching $\{(i, j) : \bar{x}_{ij} = 1\}$ in $G$. We will also find it convenient to associate the variable $x_{ij}$ and edge $(i, j)$ in $G$, with the $(i, j)$th cell in the two dimensional array $I \times J$. With the values of the variables entered in their associated cells in the array, each assignment becomes a permutation matrix.

However, in many applications, we need to find an assignment which has a specified value for a given objective function, rather than an assignment that minimizes it; i.e., we need to find a solution $x = (x_{ij})$ to the following system

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for all} \quad i = 1, \ldots, n \tag{1}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for all} \quad j = 1, \ldots, n-1 \tag{2}$$

$$x_{ij} \geq 0 \quad \text{for all} \quad i, j = 1, \ldots, n \tag{3}$$

$$x_{ij} \in \{0, 1\} \quad \text{for all} \quad i, j = 1, \ldots, n \tag{4}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} = r. \tag{5}$$

An example of such an application arises in the core management of pressurized water nuclear reactors [2, 4].

Solving (1)-(5) is NP-complete when $c_{i,j}$ are general integers [3]. The problem of solving (1)-(5) when all $c_{i,j}$ are $0-1$ has been described in [7] as a mysterious problem. In this special case necessary and sufficient conditions for the existence of a feasible solution to (1)-(5) have been derived in [5, 6], and an $\mathcal{O}(n^{2.5})$ algorithm for either finding a feasible solution to (1) -(5) or concluding that it is infeasible is also given in

[6].

In the sequel we assume that all $c_{ij}$ are 0 or 1, and $0 \leq r \leq n$, $r$ integer. In this paper we investigate some polyhedral aspects of this special case.

System (1)-(5) is defined on the complete bipartite graph $G$, i.e., all the $n^2$ variables $x_{ij}$ are allowed to assume values 0 or 1. This feature is used crucially in the algorithm discussed in [6] for solving (1)-(5). However, in applications, the problem is usually defined on an incomplete bipartite graph; i.e., we are given a subset of edges $F$ called the subset of *forbidden edges, or missing edges* of $G$ and all the variables $x_{ij}$ for $(i,j) \in F$ are deleted from system (1)-(5) and we need to solve the remaining system. This is equivalent to imposing a new constraint

$$x_{ij} = 0 \text{ for all } (i,j) \in F. \tag{6}$$

Whether an efficient algorithm exists for the problem in an incomplete graph, i.e., for solving (1)-(6) remains an open question.

Whether it is on the complete graph (this corresponds to $F = \emptyset$) or incomplete graph, our problem belongs to a special case called the *partitioned case* if there exist partitions $I = I_1 \cup I_2$, $J = J_1 \cup J_2$ such that

$$c_{ij} = \begin{cases} 1 & \text{for all } (i,j) \in (I_1 \times J_1) \cup (I_2 \times J_2) \backslash F \\ 0 & \text{for all } (i,j) \in (I_1 \times J_2) \cup (I_2 \times J_1) \backslash F. \end{cases}$$

In this partitioned case, the cells in the two dimensional array $I \times J$ are partitioned into 4 blocks: $B_1 = I_1 \times J_1$, $B_2 = I_1 \times J_2$, $B_3 = I_2 \times J_2$, and $B_4 = I_2 \times J_1$. Let $|I_1| = n_1, |J_1| = n_2$. The following facts have been proved in [6, 8] for this partitioned case, in the complete graph.

**(i)** In this case, for any $t = 1$ to 4, $|B_t \cap \{(p,q) : x_{pq} = 1\}|$ is the same, say $r_t$, for all solutions $x = (x_{pq})$ of (1) to (5), and if such a solution exists, then $r_1 = (-n + r + n_1 + n_2)/2$, $r_2 = (n - r + n_1 - n_2)/2$, $r_3 = (n + r - n_1 - n_2)/2$, $r_4 = (n - r - n_1 + n_2)/2$ since $r_2 = n_1 - r_1$, $r_4 = n_2 - r_1$, and $r_3 = n - r_1 - r_2 - r_4$.

**(ii)** In this case, system (1) to (5) has a solution iff $n + r + n_1 + n_2$ is an even number, and all the $r_1, r_2, r_3, r_4$ given in (i) are $\geq 0$. Hence all the $r$ for which system (1) to (5) has a solution in this case have the same odd-even parity, and the set of all such $r$ form an arithmetic progression in which consecutive elements differ by 2.

Furthermore, in this partitioned case, the following 6 constraints: $\sum_{(i,j)\in B_t} x_{ij} = r_t, \quad t = 1$ to 4; $\sum_{(i,j)\in B_1\cup B_3} x_{ij} = r$; $\sum_{(i,j)\in B_2\cup B_4} x_{ij} = n - r$; are all equivalent to each other in the sense that any one of them can replace (5) in system (1) to (5), leading to an equivalent system. In particular, consider

$$\sum_{(i,j)\in B_1} x_{ij} = r_1. \tag{7}$$

In this case, system (1) to (5); or the equivalent system (1) to (4) and (7), has a solution iff $r_1$ is a nonnegative integer and $\max\{0, n_1 + n_2 - n\} \leq r_1 \leq \min\{n_1, n_2\}$.

Color the edge $(i, j)$ in $G$ ( and the cell $(i, j)$ in the array $I \times J$) red if $c_{ij} = 1$, blue if $c_{ij} = 0$. Then any solution to (1)-(5) is the incidence vector of a perfect matching in $G$ with exactly $r$ red edges. Such a perfect matching will be called a *solution matching*.

We will assume that there is at least edge of each color, as otherwise the problem of finding a solution matching becomes the standard one of finding a perfect matching in a bipartite graph which is efficiently solvable.

With this coloring, the complete graph $G$, or the incomplete graph $H = (I, J, E = (I \times J)\backslash F)$ belongs to the partitioned case if there exists partitions $I = I_1 \cup I_2$, $J = J_1 \cup J_2$ such that

$$\begin{aligned} &\text{edge } (i,j) \text{ is red iff } (i,j) \in (I_1 \times J_1) \cup (I_2 \times J_2)\backslash F \\ &\text{edge } (i,j) \text{ is blue iff } (i,j) \in (I_1 \times J_2) \cup (I_1 \times J_2)\backslash F. \end{aligned} \tag{8}$$

Consider the incomplete graph case as defined earlier. The following lemma gives the necessary and sufficient conditions for the incomplete graph $H$ to belong to the partitioned case.

**Lemma 1** *Consider the incomplete colored bipartite graph $H = (I, J, E)$ where $E = (I \times J)\backslash F$. $H$ belongs to the partitioned case iff there exists no cycle in $H$ containing* *an odd number of red edges.*

**Proof.** Since $H$ is bipartite, if a cycle in $H$ contains an odd number of red edges, it must also contain an odd number of blue edges and vice versa. If partitions exist as defined earlier, clearly there can be no cycle containing an odd number of red edges in $H$.

Suppose there exist no cycle containing an odd number of red edges. Let $H_R = (I, J, E_R)$, $H_B = (I, J, E_B)$ denote the subgraphs of $H$ induced by the red and blue edges respectively but each of them containing all the nodes. Under these assumptions $H_R$ cannot be a connected graph, for suppose it is connected. Take any blue edge $(i, j)$. Since $H_R$ is connected, there exists a red simple path $\mathcal{P}$ say in $H_R$ from $i$ to $j$. Then $\mathcal{P} \cup \{(i, j)\}$ is a simple cycle containing an odd number, 1, of blue edges, contradicting our assumption. So $H_R$ must consist of two or more connected components, and no blue edge connects two nodes in the same component.

Construct an auxiliary graph $X = (\mathcal{N}, \mathcal{A})$ by the following rules:

1. Each node in $\mathcal{N}$ represents a connected component in $H_R$.

2. Nodes $p$ and $q$ in $\mathcal{N}$ are joined by an edge $(p, q) \in \mathcal{A}$ iff there is at least one blue edge in $H$ connecting one of the nodes in connected component $p$ of $H_R$ and another node from connected component $q$ of $H_R$.

By the hypothesis, the graph $X$ contains no odd cycles. Hence $X$ is bipartite. Suppose a bipartition for $X$ is $\mathcal{N}_1$, $\mathcal{N}_2$. Now place node $i \in I$ in $I_1$ if the component of $H_R$ containing node $i$ is in $\mathcal{N}_1$, or in $I_2$ if that component is in $\mathcal{N}_2$. Similarly place node $j \in J$ in $J_1$ if the component of $H_R$ containing node $j$ is in $\mathcal{N}_1$, or in $J_2$ if that component is in $\mathcal{N}_2$. Then the edges in $H$ in blocks $I_1 \times J_1$ and $I_2 \times J_2$ can not be blue, since the two nodes on any edge from these blocks come from the same connected component of $H_R$. On the other hand, the edges in $H$ in blocks $I_1 \times J_2$ and $I_2 \times J_1$ can not be red, since the two nodes on any edge from these blocks come from different components in $H_R$. Therefore, partitions $I = I_1 \cup I_2$, $J = J_1 \cup J_2$ satisfy the conditions given in (8). ∎

We will show now that the problem of solving (1)-(5) on the incomplete bipartite graph $H$ can be solved in polynomial time iff there exists a polynomial time algorithm for the same type of problem belonging to the partitioned case.

**Theorem 1** *The problem of solving (1)-(5) on the incomplete bipartite graph $H$ polynomially reduces to a problem of the same type belonging to the partitioned case*

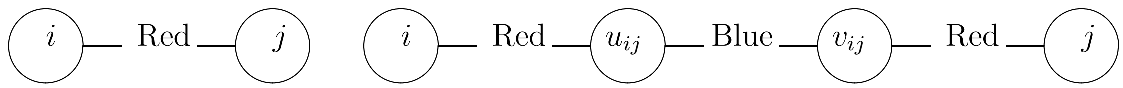
**Proof.** We consider two cases:

**Case 1:** Suppose that $H$ has no cycles containing an odd number of red edges. In this case by Lemma 1, our problem itself belongs to the partitioned case.

**Case 2:** $H$ has at least one cycle containing an odd number of red edges. Let $H_R = (I, J, E_R)$, $H_B = (I, J, E_B)$ denote the subgraphs of $H$ induced by the red and blue edges respectively. We will now enlarge $H$ into a new bipartite graph $H^*$ by adding $2|E_R|$ new nodes and $2|E_R|$ new edges by the following rule:

Replace each edge $(i, j) \in E_R$ by a path $i, (i, u_{ij}), u_{ij}$ ,$(u_{ij}, v_{ij}), v_{ij}, (v_{ij}, j), j$; (see Figure 1), where $u_{ij}, v_{ij}$ are two new nodes corresponding to the original red edge $(i, j)$ in $H$. On this path color the new edges $(i, u_{ij})$ and $(v_{ij}, j)$ red; and color the new edge $(u_{ij}, v_{ij})$ blue. Clearly the new graph $H^*$ has $n^* = 2n + 2|E_R|$ nodes and $|E_B| + 3|E_R| = |E| + 2|E_R|$ edges. Also notice that any cycle in $H^*$ that contains a new node of the type $u_{ij}$ say, must also include the nodes $v_{ij}, i, j$. Also each cycle in the original graph $H$ that contains $a$ red edges and $b$ blue edges becomes a cycle containing $2a$ red edges and $a + b$ blue edges. Hence all cycles in $H^*$ have an even number of red edges so by Lemma 1 the colored graph $H^*$ belongs to the partitioned case.

By replacing each red edge $(i, j)$ in a perfect matching with $r$ red edges in $H$ by the pair of edges $(i, u_{ij}), (v_{ij}, j)$, it becomes a perfect matching with $2r$ red edges in the new graph $H^*$. Also every perfect matching in $H^*$ that contains the red edge $(i, u_{ij})$ must also contain the red edge $(v_{ij}, j)$, as otherwise the node $v_{ij}$ will remain unmatched. Thus red edges in each perfect matching in $H^*$ occur in pairs,




Edge in original graph $H$.

Path with new nodes $u_{ij}$, $v_{ij}$ replacing the red edge $(i, j)$.

Figure 1: An edge, and the path that replaces it.

each pair belonging to a path of the form in Figure 1. Thus by replacing each pair of red edges in a path of the form in Figure 1 by the edge on the left of Figure 1 in the original graph $H$, every perfect matching with $2r$ red edges becomes a perfect matching in $H$ with $r$ red edges. Thus finding a perfect matching in $H$ containing $r$ red edges is equivalent to finding a perfect matching in the new graph $H^*$ containing $2r$ red edges, and this is a problem of the same type as the original problem, but belonging to the partitioned case. ∎

Because of Theorem 1, algorithmic studies of the problem of solving (1)-(6) can be restricted to the partitioned case without any loss of generality. So in the sequel we focus our attention on the partitioned case. Also, solving (1)-(6) is equivalent to the optimization problem

$$\begin{array}{ll} \min & \sum_{(i,j)\in F} x_{ij} \\ \text{subject to} & (1)-(5). \end{array} \tag{9}$$

(9) is a 0-1 integer program defined on the complete graph $G$ which we assume belongs to the partitioned case. An important strategy for solving a 0-1 integer program is to develop a polyhedral characterization of the convex hull of its set of feasible solutions, i.e., obtain a linear inequality representation for it. In this paper, we focus on a polyhedral characterization for (1)-(5) in the partitioned case. We present two large classes of facet-

inducing inequalities ( each containing an exponential number of inequalities) for this problem [1]. However, these classes do not completely characterize the convex hull of the set of feasible solutions of (1)-(5).

## 2 The Results

We consider the system (1) to (5) defined on the complete graph $G$ belonging to the partitioned case with partitions, $I = I_1 \cup I_2, J = J_1 \cup J_2$, blocks $B_1, B_2, B_3, B_4$, and $n_1, n_2$, $r_1$ to $r_4$ as defined earlier.

When one of the sets among $I_1, I_2$ is $\emptyset$, and one of the sets among $J_1, J_2$ is $\emptyset$, all the edges in $G$ have only one color, and all extreme points of the set of feasible solutions of (1), (2), (3), (5) satisfy (4) automatically. The same property holds when exactly one of the 4 sets among $I_1, I_2, J_1, J_2$ is $\emptyset$, and the other three are nonempty. So, we assume $0 < n_1 < n, 0 < n_2 < n$, and without loss of generality, we assume that the rows and columns of the array are rearranged so that $I_1 = \{1, 2, \ldots, n_1\}$, $I_2 = \{n_1 + 1, \ldots, n\}$, $J_1 = \{1, 2, \ldots, n_2\}$, $J_2 = \{n_2 + 1, \ldots, n\}$ (See Figure 2). Define

$P^{n,r_1}_{n_1,n_2}$ = Set of feasible solutions of (1), (2), (3), (7) [ or equivalently (1), (2), (3), (5) ]

$Q^{n,r_1}_{n_1,n_2}$ = *Integer hull* of $P^{n,r_1}_{n_1,n_2}$ defined as conv($\{x : x \in P^{n,r_1}_{n_1,n_2}$ and $x$ integer $\}$) = convex hull of set of feasible solutions of (1), (2), (4), (7).

It can be shown that $P^{n,r_1}_{n_1,n_2} \neq \emptyset$ iff $\max\{0, n_1 + n_2 - n\} \leq r_1 \leq \min\{n_1, n_2\}$, which we assume.

The polytope defined by (1),(2), and (3) is the well-known *assignment* , or *Birkoff polytope* $K_A$ with integral extreme points. However, with the side constraint (7), $P^{n,r_1}_{n_1,n_2}$ may have fractional extreme points. For example, when $n = 4$, $n_1 = n_2 = 2$, $r_1 = 1$,

$$x_{11} = x_{14} = x_{22} = x_{23} = x_{32} = x_{34} = x_{41} = x_{43} = \frac{1}{2} \quad , \quad x_{ij} = 0 \text{ otherwise}$$

is a fractional extreme point of $P^{4,1}_{2,2}$. Hence, $Q^{n,r_1}_{n_1,n_2}$ may not be equal to $P^{n,r_1}_{n_1,n_2}$.

In the sequel, an assignment $x = (x_{ij})$ of order $n$ is represented as a permutation $(\sigma_1, \sigma_2, \ldots, \sigma_s, \ldots, \sigma_n)$ such that $x_{s\sigma_s} = 1$ for $s = 1, 2, \ldots, n$, $x_{ij} = 0$ otherwise. For example, the diagonal assignment is represented by the permutation $(1, 2, \ldots, n)$.

## 2.1 Dimension and the Trivial Facets of $Q^{n,r_1}_{n_1,n_2}$

Here, we present one condition under which $Q^{n,r_1}_{n_1,n_2}$ coincides with $P^{n,r_1}_{n_1,n_2}$. For the general case when $Q^{n,r_1}_{n_1,n_2} \neq P^{n,r_1}_{n_1,n_2}$, we establish that $\dim(Q^{n,r_1}_{n_1,n_2}) = \dim(P^{n,r_1}_{n_1,n_2}) = n^2 - 2n$ when $Q^{n,r_1}_{n_1,n_2} \neq \emptyset$.

**Lemma 2** *Let $K_A$ be the assignment polytope, i.e., set of feasible solutions of (1), (2), (3). If one or more of $r_1$, $r_2$, $r_3$, $r_4$ are 0, $Q^{n,r_1}_{n_1,n_2} = P^{n,r_1}_{n_1,n_2}$= a face of $K_A$.*

**Proof.** From Theorem 1 we know that in system (1), (2), (3), (5), the constraint (5) can be replaced by

$$\sum_{(i,j)\in B_t} x_{ij} = r_t. \tag{10}$$

for any $t = 1$ to 4. Hence $P^{n,r_1}_{n_1,n_2}$ is the set of feasible solutions of (1), (2), (3), and (10). But if $r_t = 0$, under (3), constraint (10) is equivalent to

$$x_{ij} = 0 \quad \text{for each } (i,j) \in B_t. \tag{11}$$

Hence in this case $P^{n,r_1}_{n_1,n_2}$ is the set of feasible solutions of (1), (2), (3), (11), which by definition is a face of $K_A$, and hence all its extreme points are $0-1$ vectors. Hence $Q^{n,r_1}_{n_1,n_2} = P^{n,r_1}_{n_1,n_2}$ = a face of $K_A$ in this case. ∎

**Theorem 2** *Suppose that $r_t \geq 1$ for all $t$ =1 to 4, and $Q^{n,r_1}_{n_1,n_2} \neq \emptyset$. Then $Q^{n,r_1}_{n_1,n_2}$ and $P^{n,r_1}_{n_1,n_2}$ both have the same dimension $n^2 - 2n$. Also, each non-negativity restriction in (3) is a facet-inducing inequality for $Q^{n,r_1}_{n_1,n_2}$.*

**Proof.** Dim $P^{n,r_1}_{n_1,n_2} = n^2 - 2n$ can be shown rather easily. Hence, dim $Q^{n,r_1}_{n_1,n_2} \leq n^2 - 2n$. Now assume that dim $Q^{n,r_1}_{n_1,n_2} < n^2 - 2n$ then there exists a hyperplane $H = \{x \in \mathbb{R}^{n^2} : \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}x_{ij} = \beta\}$ containing $Q^{n,r_1}_{n_1,n_2}$, but not $P^{n,r_1}_{n_1,n_2}$. i.e., $H$ is not defined by a linear combination of the equality constraints (1), (2), and (7). We will show that no such hyperplane $H$ can exist thus establishing that dim $Q^{n,r_1}_{n_1,n_2} = n^2 - 2n$.

Let $Ax = b$ represent the system of equality constraints (1), (2), and (7). Then $A$ is a full row rank $2n \times n^2$ matrix. Let $x^0$ be a solution matching in $Q^{n,r_1}_{n_1,n_2}$ and $A = (B\ ,\ N)$ be a partition of $A$ into basic, nonbasic parts with $B$ being a $2n \times 2n$ basis for $A$, corresponding to basic vector $x_B$ containing the basic variables

$$x_{1,\,n_2+r_2-1}, x_{2,\,n_2+r_2-2}, \ldots, x_{n_1+r_4-1\,,1} \quad , \quad x_{n_1+r_4\,,n}, x_{n_1+r_4+1,\,n-1}, \ldots, x_{n\,,n_2+r_2}$$

$$x_{1,\,n_2+r_2}, x_{2,\,n_2+r_2-1}, \ldots, x_{n_1+r_4\,,1} \quad , \quad x_{n_1+r_4+1\,,n}, x_{n_1+r_4+2\,,n-1}, \ldots, x_{n\,,n_2+r_2+1}$$

with the basic variables in the top row having value 0 in $x^0$ ( the cells marked with ($\circ$) in Figure 2), and those in the bottom row having value 1 in $x^0$ ( the cells marked with a ($\star$) in Figure 2). Let $x_N$ denote the vector of nonbasic variables. From the results in [6] we know that in the partitioned case under discussion here, the rows and columns of the array can be rearranged so that the matched cells in any solution matching appear along one of the diagonals like the one marked with ($\star$)'s in Figure 2.

Let $(\alpha_B \quad \alpha_N)$ be the corresponding rearrangement of the row vector $(\alpha_{ij})$. Hence

$$H = \{x \in \mathbb{R}^{n^2} : \alpha_B x_B + \alpha_N x_N = \beta\}.$$

Let

$$\hat{H} = \{x \in \mathbb{R}^{n^2} : \hat{\alpha}_B x_B + \hat{\alpha}_N x_N = \hat{\beta}\}$$

where

$$(\hat{\alpha}_B,\ \hat{\alpha}_N,\ \hat{\beta}) = (\alpha_B,\ \alpha_N,\ \beta) - \lambda^T(B,\ N,\ b)$$

where $\lambda \in \mathbb{R}^{2n}$ will be chosen appropriately.

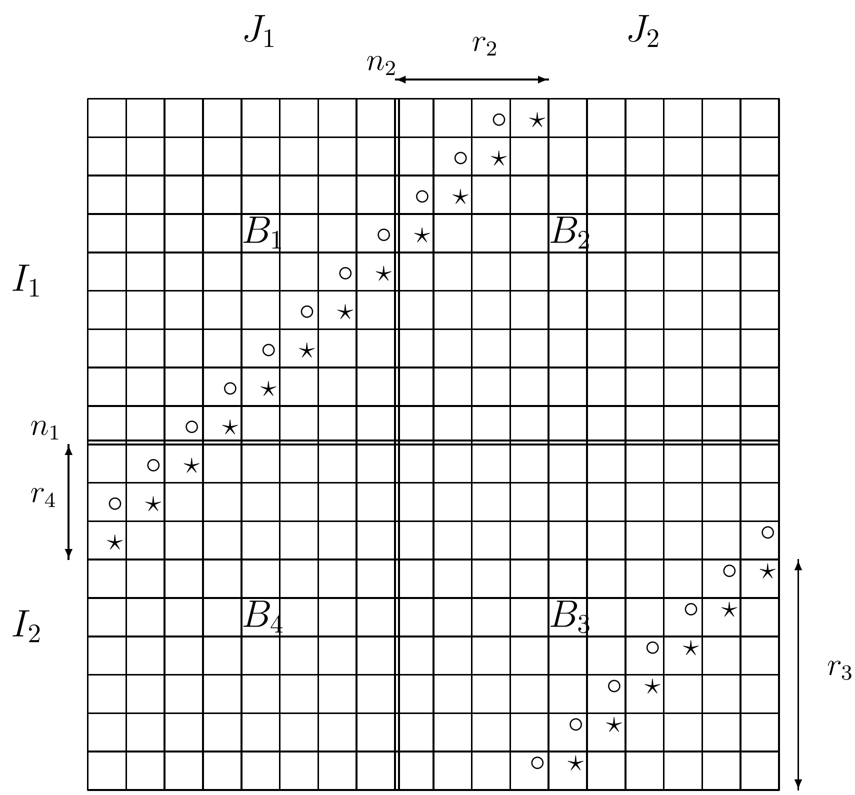



Figure 2: The double lines indicate the row and column partitions, and the four blocks $B_1$, $B_2$, $B_3$, and $B_4$ are shown. The $2n$ basic cells corresponding to basic vector $x_B$ are marked with ($\circ$) or ($\star$).

By construction $\hat{H}$ contains $Q^{n,r_1}_{n_1,n_2}$. Now if we can show that $\hat{\alpha}_B = 0$, $\hat{\alpha}_N = 0$, and $\hat{\beta} = 0$, for a proper choice of $\lambda$, it would follow that the equation defining $H$, is a linear combination of the equality constraints (1), (2), and (7), thus arriving at a contradiction.

To establish this, let $\lambda^T = \alpha_B B^{-1}$. Then $\hat{\alpha}_B = 0$. Represented as a permutation of (1,2, …,n), $x^0$ is

$$(n_2 + r_2,\ n_2 + r_2 - 1,\ n_2 + r_2 - 2 \ldots,\ 1, n, n-1, \ldots,\ n_2 + r_2 + 1).$$

Then $x^0_N = 0$. Since $Q^{n,r_1}_{n_1,n_2}$ lies in $\hat{H}$, it follows that $\hat{\alpha}_B x^0_B + \hat{\alpha}_N x^0_N = \hat{\beta}$. Since $\hat{\alpha}_B = 0$ and $x^0_N = 0$ it follows that $\hat{\beta} = 0$. Thus it remains to show that $\hat{\alpha}_N = 0$. Towards this effort, let $x^1$ be the assignment

$$x^1 = (n_2 + r_2 - 1,\ n_2 + r_2,\ n_2 + r_2 - 2 \ldots,\ 1, n, n-1, \ldots,\ n_2 + r_2 + 1)$$

whose representation as a permutation is obtained by interchanging the first two elements in the permutation corresponding to $x^0$ ( when represented as permutation matrices, $x^1$ is obtained by interchanging rows 1 and 2 in $x^0$). By the hypothesis in the theorem $n_1 = r_1 + r_2 \geq 2$, and hence the interchange does not alter the number of allocations within each of the four blocks, i.e., $x^1$ is also a solution matching, or $x^1 \in Q^{n,r_1}_{n_1,n_2}$. So $\sum_{i=1}^n \sum_{j=1}^n \hat{\alpha}_{ij} x^1_{ij} = \hat{\beta} = 0$, clearly this implies that the component $\hat{\alpha}_{2,n_2+r_2}$ in $\hat{\alpha}_N$ is zero.

In the same way we can generate a sequence of solution matchings $x^2, x^3, \ldots, x^k, \ldots,$ $x^{n^2-2n} \in Q^{n,r_1}_{n_1,n_2}$ written as permutation matrices, where $x^k$ is derived from some $x^i \in \{x^0, x^1, \ldots, x^{k-1}\}$, by interchanging either two rows ( both within $I_1$ or both within $I_2$) or two columns ( both within $J_1$ or both within $J_2$), and for each $k = 2$ to $n^2 - 2n$, using the equation $\sum_{i=1}^n \sum_{j=1}^n \hat{\alpha}_{ij} x^k_{ij} = 0$ we are able to establish that one more component of $\hat{\alpha}_N$ is zero. In the end we have $\hat{\alpha}_N = 0$. This establishes that dim $Q^{n,r_1}_{n_1,n_2} = n^2 - 2n$.

Now select any variable $x_{pq}$. From the above procedure it is clear that the dimension of the set of all solution matchings in each of which $x_{pq}$=0 has dimension $n^2 - 2n - 1$. This implies that the face $F = \{x \in Q^{n,r_1}_{n_1,n_2} : x_{pq} = 0\}$ is a facet of $Q^{n,r_1}_{n_1,n_2}$. ∎

## 2.2 Some Non Trivial Facets of $Q^{n,r_1}_{n_1,n_2}$

We assume that all of $r_1$, $r_2$, $r_3$, and $r_4 \geq 1$. This automatically implies $n \geq 4$.

**Proposition 1** *Let $x_{\tilde{I}\tilde{J}} = (x_{ij} : i \in \tilde{I}, j \in \tilde{J})$, where $\tilde{I}$, $\tilde{J}$ are arbitrary nonempty subsets of $I, J$ respectively, be the incidence matrix of a matching in $\tilde{I} \times \tilde{J}$. Let $\mathcal{K}_R$, $\mathcal{K}_C$ be subsets of $\tilde{I}$, $\tilde{J}$ respectively such that $|\mathcal{K}_R| \leq |\tilde{J} \backslash \mathcal{K}_C|$ and $|\mathcal{K}_C| \leq |\tilde{I} \backslash \mathcal{K}_R|$. Then*

$$\sum_{i \in \mathcal{K}_R \;\, j \in \mathcal{K}_C} x_{ij} + \sum_{i \in \mathcal{K}_R \;\, j \in \tilde{J} \backslash \mathcal{K}_C} x_{ij} + \sum_{i \in \tilde{I} \backslash \mathcal{K}_R \;\, j \in \mathcal{K}_C} x_{ij} \leq |\mathcal{K}_R| + |\mathcal{K}_C|.$$

*Equality holds for the matching $\bar{x}_{\tilde{I}\tilde{J}} = (\bar{x}_{ij} : i \in \tilde{I}, j \in \tilde{J})$ where*

$$\bar{x}_{ij} = \begin{cases} 1 & \text{for each } i \in \mathcal{K}_R, \text{ for some } j \in \tilde{J} \backslash \mathcal{K}_C \\ 1 & \text{for each } j \in \mathcal{K}_C, \text{ for some } i \in \tilde{I} \backslash \mathcal{K}_R \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** This follows directly from the definition of a matching.

### 2.2.1 The First Class of Facets

Facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$ of the first class are characterized by a cell $(p, q) \in I \times J$ called the *primary defining cell* or just the *defining cell*, and a nonempty set of row indices $\mathcal{K}_R$, and a nonempty set of column indices $\mathcal{K}_C$.

Look at the four blocks in our partition (Figure 2). Blocks $B_1$, $B_2$ lie in the same rows of the array, so we say that each of them is the *row adjacent block* of the other. Similarly, in blocks $B_3$, $B_4$, each is row adjacent block to the other. In the same way in the pairs $(B_1, B_4)$, $(B_2, B_3)$, each is the *column adjacent block* of the other. We say that two given blocks are *adjacent* if they are either row adjacent or column adjacent.

The defining cell $(p, q)$ for the first class of facets can be any cell in the array. Suppose it is contained in block $B_t$. Let $I_t$, $J_t$ denote the set of row and column indices of $B_t$ respectively. Let $B_u$ be the row adjacent block of $B_t$, and $B_v$ the column adjacent block of $B_t$. Let $B_w$ be the remaining block which is not adjacent to $B_t$. Let $\hat{I}$ denote the set of row indices of $B_v$, and $\hat{J}$ denote the set of column indices of $B_u$. (i.e., $\hat{I} = I \backslash I_t$ and $\hat{J} = J \backslash J_t$) Then the *defining subset of row indices* $\mathcal{K}_R$ must be a nonempty proper subset of $\hat{I}$, and the *defining subset of column indices* $\mathcal{K}_C$ must be a nonempty proper subset of $\hat{J}$, and together they have to satisfy $|\mathcal{K}_R| + |\mathcal{K}_C| = 1 + r_w$.

**Lemma 3** *Let* $(p, q)$ *be the defining cell and* $\mathcal{K}_R$*,* $\mathcal{K}_C$ *be the defining sets of row and column indices selected as discussed above. Then*

$$x_{pq} + \sum_{j \in \mathcal{K}_C} x_{pj} + \sum_{i \in \mathcal{K}_R} x_{iq} - \sum_{i \in \hat{I} \backslash \mathcal{K}_R \, , \, j \in \hat{J} \backslash \mathcal{K}_C} x_{ij} \ \leq 1 \tag{12}$$

*is a valid inequality for* $Q^{n,r_1}_{n_1,n_2}$.

**Proof.** First we observe that in any assignment $x = (x_{ij} : i \in I, j \in J)$

$$x_{pq} + \sum_{j \in \mathcal{K}_C} x_{pj} + \sum_{i \in \mathcal{K}_R} x_{iq} \tag{13}$$

is equal to 0, 1, or 2. This is easy to see since each of these terms is either 0 or 1 and since all of them can not be 1 at the same time.

For an assignment $x \in Q^{n,r_1}_{n_1,n_2}$, if the expression in (13) is equal to either 0 or 1, our lemma holds trivially. Therefore, assume that the expression in (13) is equal to 2 for an assignment $x \in Q^{n,r_1}_{n_1,n_2}$. This holds only when $x_{pq} = 0$, and $\sum_{j \in \mathcal{K}_C} \ x_{pj} = \sum_{i \in \mathcal{K}_R} \ x_{iq} = 1$. Suppose that $x_{pj_0} = x_{i_0q} = 1$ where $j_0 \in \mathcal{K}_C$ and $i_0 \in \mathcal{K}_R$. Thus

$$\sum_{j \in \hat{J}} x_{i_0 j} = \sum_{i \in \hat{I}} x_{i j_0} = 0. \tag{14}$$

Since $x \in Q^{n,r_1}_{n_1,n_2}$ we have $\sum_{(i,j) \in B_w} x_{ij} = r_w$, i.e.,

$$\sum_{i \in \mathcal{K}_R \, , \, j \in \mathcal{K}_C} x_{ij} + \sum_{i \in \mathcal{K}_R \, , \, j \in \hat{J} \backslash \mathcal{K}_C} x_{ij} + \sum_{i \in \hat{I} \backslash \mathcal{K}_R \, , \, j \in \mathcal{K}_C} x_{ij} + \sum_{i \in \hat{I} \backslash \mathcal{K}_R \, , \, j \in \hat{J} \backslash \mathcal{K}_C} x_{ij} = r_w.$$

Using Proposition 1 and (14) it follows that

$$\sum_{i \in \mathcal{K}_R \, , \, j \in \mathcal{K}_C} x_{ij} + \sum_{i \in \mathcal{K}_R \, , \, j \in \hat{J} \backslash \mathcal{K}_C} x_{ij} + \sum_{i \in \hat{I} \backslash \mathcal{K}_R \, , \, j \in \mathcal{K}_C} x_{ij} \leq |\mathcal{K}_R \backslash \{i_0\}| + |\mathcal{K}_C \backslash \{j_0\}| = r_w - 1$$

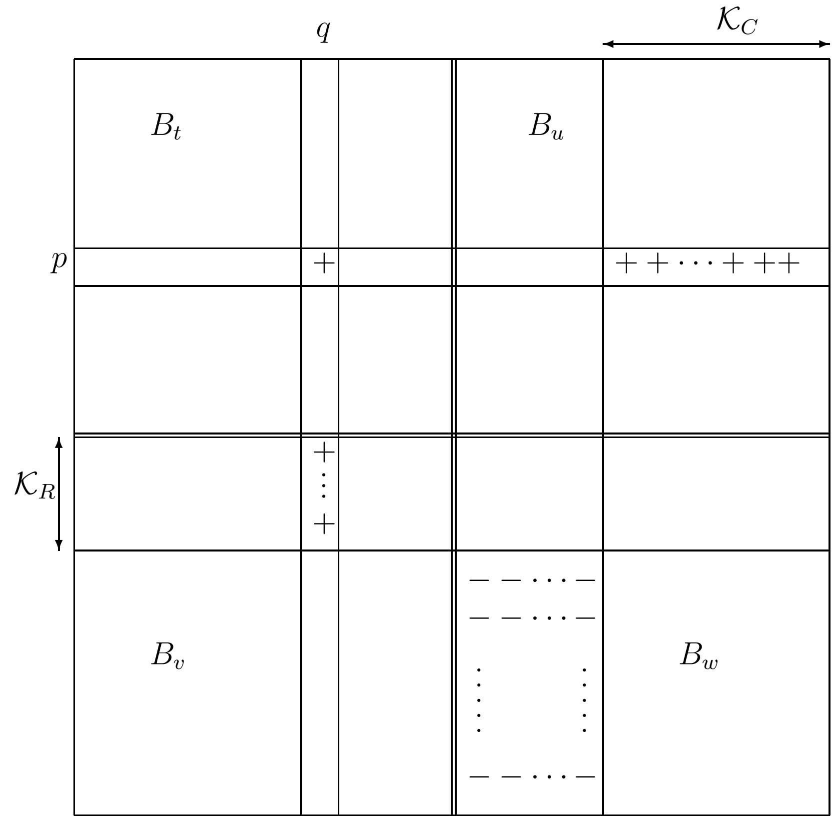



Figure 3: Pictorial representation of signs of nonzero coefficients in (12). The double lines indicate the row and column partitions.

hence $\sum_{i\in\hat{I}\setminus\mathcal{K}_R\,,\, j\in\hat{J}\setminus\mathcal{K}_C} x_{ij} \geq 1$ and hence (12) holds for $x$ and the lemma follows. ∎

As an example consider the case where $n = 5$, $n_1 = 2$, $n_2 = 3$ and $r_1 = 1$. Hence $r_2 = r_3 = 1$ and $r_4 = 2$. Let the defining cell be (1,1), and the defining sets be $\mathcal{K}_R = \{3\}$, $\mathcal{K}_C = \{4\}$. The valid inequality (12) corresponding to these choices is

$$x_{11} + x_{14} + x_{31} - x_{45} - x_{55} \leq 1$$

which is a valid inequality for $Q^{5,1}_{2,3}$. Note that all the nonzero coefficients in (12) are $+1$ or $-1$.

It is helpful to have a pictorial representation of inequality (12). In Figure 3, we show the array with the defining cell $(p, q)$ and the defining subsets $\mathcal{K}_R$, $\mathcal{K}_C$, and the cells in the array whose variables appear with a $+1$ coefficient (marked by $+$ symbol), and those with a $-1$ coefficient (marked by $-$ symbol) in this inequality.

**Theorem 3** *The valid inequality (12) in Lemma 3 is a facet-inducing inequality for* $Q^{n,r_1}_{n_1,n_2}$.

The proof of Theorem 3 is given in Section 2.3.

Inequalities (12) define the first class of facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$. For defining these inequalities, the defining cell $(p, q)$ can be selected as any cell in the array, so there are $n^2$ ways of choosing it. Once the defining cell $(p, q)$ is selected, the number of ways of selecting the defining subsets $\mathcal{K}_R$, $\mathcal{K}_C$ is

$$\sum_{N=1}^{r_w} \begin{pmatrix} \hat{I} \\ N \end{pmatrix} \begin{pmatrix} \hat{J} \\ r_w + 1 - N \end{pmatrix}$$

where $N = |\mathcal{K}_R|$ and $r_w + 1 - N = |\mathcal{K}_C|$, this number grows exponentially with $|\hat{I}|$ ,$|\hat{J}|$ and $r_w$. Hence the total number of these first class of facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$ grows exponentially with $n_1$, $n_2$, $r_1$.

### 2.2.2 The Second Class of Facets

Facet-inducing inequalities in this class are characterized by two defining cells called the *primary and secondary defining cells*, and by two defining subsets of row indices, and two defining subsets of column indices.

The primary defining cell, $(p, q)$ say, can be any cell in the array. Suppose it is contained in block $B_t$. The second class of facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$ only exist for the primary defining cell $(p, q) \in B_t$ if the numbers $r_u$, $r_v$ corresponding to the row adjacent block $B_u$, the column adjacent block $B_v$ of $B_t$, are both $\geq 2$. If this condition is satisfied, the secondary defining cell, $(m, l)$ say, can be any cell in the adjacent blocks $B_u$ or $B_v$ of $B_t$ satisfying $m \neq p$, $l \neq q$.

Let $B_w$ be the block not adjacent to $B_t$. If $(m, l) \in B_u$, the defining subsets of column indices $\mathcal{K}_C$, $\tilde{\mathcal{K}}_C$, say, can be any nonempty proper subsets of the column indices of the blocks $B_u$, $B_t$ respectively satisfying the condition that $l \notin \mathcal{K}_C$, $q \notin \tilde{\mathcal{K}}_C$; and the defining subsets of row indices, $\mathcal{K}_R$, $\tilde{\mathcal{K}}_R$, say, can be any nonempty mutually disjoint proper subsets of the row indices of $B_v$ which together satisfy $|\mathcal{K}_C| + |\mathcal{K}_R| = 1 + r_w$, and $|\tilde{\mathcal{K}}_C| + |\tilde{\mathcal{K}}_R| = r_v$.

If $(m,l) \in B_v$, the column adjacent block of $B_t$, the defining subsets of column indices, $\mathcal{K}_C$, $\tilde{\mathcal{K}}_C$, can be any nonempty mutually disjoint proper subsets of the column indices of $B_u$; and the defining subsets of row indices, $\mathcal{K}_R$, $\tilde{\mathcal{K}}_R$ can be any nonempty proper subsets of the row indices of $B_v$, $B_t$ respectively satisfying the condition that $m \notin \mathcal{K}_R$, $p \notin \tilde{\mathcal{K}}_R$; which together satisfy $|\mathcal{K}_C| + |\mathcal{K}_R| = 1 + r_w$, and $|\tilde{\mathcal{K}}_C| + |\tilde{\mathcal{K}}_R| = r_u$ (see Figure 4).

For this case where the secondary defining cell $(m,l) \in B_v$ (see Figure 4) we have the following lemma.

**Lemma 4** *Let the primary defining cell be $(p,q)$ from block $B_t$, and suppose its row, column adjacent blocks $B_u, B_v$ satisfy $r_u \geq 2$. Let $\hat{I}$ be the set of row indices of block $B_v$, and $\hat{J}$ be the set of column indices of block $B_u$. Let $I_t$, $J_t$ be the sets of row and column indices of $B_t$. Let $(m,l) \in B_v$ be the secondary defining cell, and let the defining subsets of row and column indices $\mathcal{K}_R$, $\tilde{\mathcal{K}}_R$, $\mathcal{K}_C$, and $\tilde{\mathcal{K}}_C$ be selected as discussed above. Let $B_w$ be the block not adjacent to $B_t$ (i.e., $B_w = \hat{I} \times \hat{J}$). Then*

$$\begin{aligned} x_{pq} + \sum_{j \in \mathcal{K}_C} x_{pj} + \sum_{i \in \mathcal{K}_R} x_{iq} \quad & - \sum_{i \in \hat{I} \setminus (\mathcal{K}_R \cup \{m\})} \; \sum_{j \in \hat{J} \setminus \mathcal{K}_C} x_{ij} \\ - \sum_{j \in \hat{J} \setminus (\mathcal{K}_C \cup \tilde{\mathcal{K}}_C)} x_{mj} \quad & - \sum_{i \in I_t \setminus (\tilde{\mathcal{K}}_R \cup \{p\})} \; \sum_{j \in \hat{J} \setminus (\mathcal{K}_C \cup \tilde{\mathcal{K}}_C)} x_{ij} \\ & - \sum_{i \in I \setminus (\mathcal{K}_R \cup \tilde{\mathcal{K}}_R \cup \{p,m\})} x_{il} \qquad \leq 1 \end{aligned} \tag{15}$$

*is a valid inequality of $Q^{n,r_1}_{n_1,n_2}$.*

**Proof.** For any assignment $x \in Q^{n,r_1}_{n_1,n_2}$ the sum

$$x_{pq} + \sum_{j \in \mathcal{K}_C} x_{pj} + \sum_{i \in \mathcal{K}_R} x_{iq} \tag{16}$$

is equal to 0, 1, or 2. If the expression in (16) is equal to either 0 or 1 the lemma follows trivially. Therefore, assume that the expression in (16) is equal to 2. This holds when

$x_{pj_0} = 1$ for some $j_0 \in \mathcal{K}_C$ and $x_{i_0q} = 1$ for some $i_0 \in \mathcal{K}_R$. Then by Proposition 1 we have

$$\sum_{i\in\mathcal{K}_R \; j\in\mathcal{K}_C} x_{ij} + \sum_{i\in\mathcal{K}_R \; j\in\hat{J}\backslash\mathcal{K}_C} x_{ij} + \sum_{i\in\hat{I}\backslash\mathcal{K}_R \; j\in\mathcal{K}_C} x_{ij} \le |\mathcal{K}_R\backslash\{i_0\}| + |\mathcal{K}_C\backslash\{j_0\}|. \tag{17}$$

Two cases will be considered

**Case 1:** $x_{mj} = 0$ for all $j \in \tilde{\mathcal{K}}_C$. Then since $\sum_{(i,j)\in B_w} x_{ij} = r_w$ and since $|\mathcal{K}_R\backslash\{i_0\}| + |\mathcal{K}_C\backslash\{j_0\}| = r_w - 1$ it follows that

$$\sum_{i\in\hat{I}\backslash\mathcal{K}_R \quad j\in\hat{J}\backslash\mathcal{K}_C} x_{ij} \ge 1$$

and since $\sum_{j\in\tilde{\mathcal{K}}_C} x_{mj} = 0$ by assumption, it follows that

$$\sum_{i\in\hat{I}\backslash(\mathcal{K}_R\cup\{m\}) \quad j\in\hat{J}\backslash\mathcal{K}_C} x_{ij} + \sum_{j\in\hat{J}\backslash(\mathcal{K}_C\cup\tilde{\mathcal{K}}_C)} x_{mj} \ge 1$$

and (15) holds for $x$.

**Case 2:** $x_{mj_1} = 1$ for some $j_1 \in \tilde{\mathcal{K}}_C$.

Then if (17) holds as a strict inequality, and by the same argument as in case 1, we have $\sum_{i\in\hat{I}\backslash(\mathcal{K}_R\cup\{m\}) \quad j\in\hat{J}\backslash\mathcal{K}_C} x_{ij} \ge 1$, and (15) holds for $x$. Therefore, assume that (17) holds as an equality. By Proposition 1, this corresponds to the case where for each $i \in \mathcal{K}_R\backslash\{i_0\}$, $x_{ij} = 1$ for some $j \in \hat{J}\backslash\mathcal{K}_C$; and for each $j \in \mathcal{K}_C\backslash\{j_0\}$, $x_{ij} = 1$ for some $i \in \hat{I}\backslash(\mathcal{K}_R \cup \{m\})$. This implies that

$$x_{il} = 0 \text{ for all } i \in \mathcal{K}_R \cup \{m\} \tag{18}$$

$$\sum_{i\in I_t\backslash\{p\} \; j\in\mathcal{K}_C} x_{ij} = 0. \tag{19}$$

Now applying Proposition 1 to block $B_u$ and using (19) we have

$$\sum_{i\in\tilde{\mathcal{K}}_R \; j\in\tilde{\mathcal{K}}_C} x_{ij} + \sum_{i\in\tilde{\mathcal{K}}_R \; j\in\hat{J}\backslash\tilde{\mathcal{K}}_C} x_{ij} + \sum_{i\in I_t\backslash\tilde{\mathcal{K}}_R \; j\in\tilde{\mathcal{K}}_C} x_{ij} \le |\tilde{\mathcal{K}}_R| + |\tilde{\mathcal{K}}_C\backslash\{j_1\}| \tag{20}$$

if (20) holds as a strict inequality and since $\sum_{(i,j)\in B_u} x_{ij} = r_u$ and $|\tilde{\mathcal{K}}_R| + |\tilde{\mathcal{K}}_C\backslash\{j_1\}| = r_u - 1$ it follows that

$$\sum_{i\in I_t\backslash(\tilde{\mathcal{K}}_R\cup\{p\}) \quad j\in\hat{J}\backslash(\mathcal{K}_C\cup\tilde{\mathcal{K}}_C)} x_{ml} \ge 1$$

and (15) holds for $x$.

Therefore assume that (20) holds as an equality. This corresponds to the case where for each $i \in \tilde{\mathcal{K}}_R$, $x_{ij} = 1$ for some $j \in \hat{J} \backslash \tilde{\mathcal{K}}_C \cup \{j_1\}$; and for each $j \in \tilde{\mathcal{K}}_C \cup \{j_1\}$, $x_{ij} = 1$ for some $i \in I_t \backslash (\tilde{\mathcal{K}}_R \cup \{p\})$ which implies that $x_{il} = 0$ for all $i \in \tilde{\mathcal{K}}_R \cup \{p\}$. Therefore, by (18) and the fact that $\sum_{i \in I} x_{il} = 1$ it follows that $\sum_{i \in I \backslash (\mathcal{K}_R \cup \tilde{\mathcal{K}}_R \cup \{p,m\})} x_{il} = 1$ Thus, (15) holds for $x$ and the lemma follows. ∎

A similar lemma for the case where the secondary defining cell $(m, l) \in B_u$ is given below.

**Lemma 5** *Let the primary defining cell be $(p, q)$ from block $B_t$, and suppose its row, column adjacent blocks $B_u, B_v$ satisfy $r_v \geq 2$. Let $\hat{I}$ be the set of row indices of block $B_v$, and $\hat{J}$ be the set of column indices of block $B_u$. Let $I_t$, $J_t$ be the sets of row and column indices of $B_t$. Let $(m, l) \in B_u$ be the secondary defining cell, and let the defining subsets of row and column indices $\mathcal{K}_R$, $\tilde{\mathcal{K}}_R$, $\mathcal{K}_C$, and $\tilde{\mathcal{K}}_C$ be selected as discussed above. Let $B_w$ be the block not adjacent to $B_t$ (i.e., $B_w = \hat{I} \times \hat{J}$). Then*

$$\begin{aligned} x_{pq} + \sum_{j \in \mathcal{K}_C} x_{pj} + \sum_{i \in \mathcal{K}_R} x_{iq} \quad & - \sum_{i \in \hat{I} \backslash \mathcal{K}_R \quad j \in \hat{J} \backslash (\mathcal{K}_C \cup \{l\})} x_{ij} \\ - \sum_{i \in \hat{I} \backslash (\mathcal{K}_R \cup \tilde{\mathcal{K}}_R)} x_{il} \quad & - \sum_{i \in \hat{I} \backslash (\tilde{\mathcal{K}}_R \cup \mathcal{K}_R) \quad j \in J_t \backslash (\tilde{\mathcal{K}}_C) \cup \{q\})} x_{ij} \\ & - \sum_{j \in J \backslash (\mathcal{K}_C \cup \tilde{\mathcal{K}}_C \cup \{q,l\})} x_{mj} \qquad \leq 1 \end{aligned} \tag{21}$$

*is a valid inequality of $Q^{n,r_1}_{n_1,n_2}$.*

The proof of Lemma 5 is similar to that of Lemma 4.

As an example consider the case where $n = 8, n_1 = 4, n_2 = 4$, and $r_1 = r_2 = r_3 = r_4 = 2$. Then, selecting $(p, q) = (1, 1) \in B_1$, $(m, l) = (5, 2) \in B_4$, $\mathcal{K}_R = \{6\}$, $\mathcal{K}_C = \{6, 7\}$, $\tilde{\mathcal{K}}_R = \{2\}$, $\tilde{\mathcal{K}}_C = \{5\}$ satisfying all the conditions for selection mentioned above, leads to the valid inequality for $Q^{8,2}_{4,4}$.

$$x_{11} + x_{16} + x_{17} + x_{61} - x_{32} - x_{38} - x_{42} - x_{48}$$

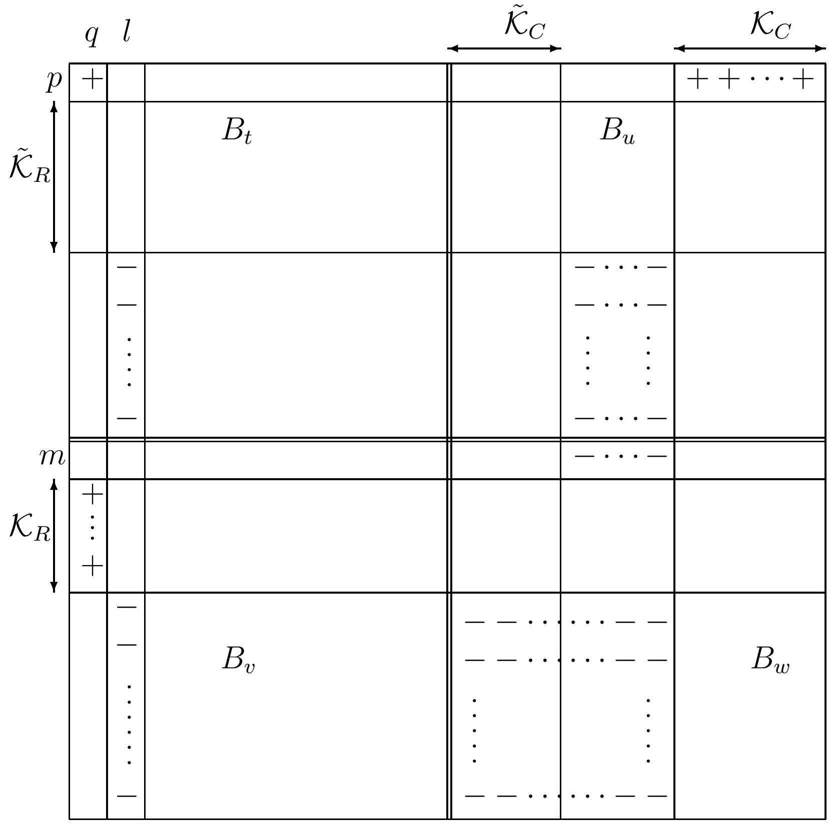



Figure 4: Pictorial representation of signs of nonzero coefficients in (15).

$$-x_{58} - x_{72} - x_{75} - x_{78} - x_{82} - x_{85} - x_{88} \leq 1.$$

In Figure 4, we give a pictorial representation of inequality (15). It shows the array with the defining cells $(p, q) \in B_t$, $(m, l) \in B_v$ and the defining subsets $\mathcal{K}_R$, $\mathcal{K}_C$, $\tilde{\mathcal{K}}_R$, $\tilde{\mathcal{K}}_C$ and the cells in the array whose variables appear with a $+1$ coefficient (marked by $+$ symbol), and those with a $-1$ coefficient ( marked by $-$ symbol) in the inequality.

**Theorem 4** *The valid inequalities (15 ) or (21) defined in Lemmas 4,5 are facet-inducing inequalities for* $Q^{n,r_1}_{n_1,n_2}$ *provided that both* $r_u, r_v \geq 2$.

Theorem 4 will be proved in section 2.3. Notice that in Lemma 4 we only require $r_u \geq 2$ for (15) to be a valid inequality for $Q^{n,r_1}_{n_1,n_2}$. Correspondingly in Lemma 5 we only require $r_v \geq 2$ for (21) to be a valid inequality for $Q^{n,r_1}_{n_1,n_2}$. But Theorem 4 establishes that these are facet-inducing when both $r_u$, $r_v \geq 2$.

Unfortunately, these two nontrivial classes of facets do not provide a complete description of the polytope $Q^{n,r_1}_{n_1,n_2}$ as demonstrated by the following fractional point $\hat{x} =$

$(\hat{x}_{ij})$ defined by

$$\hat{x}_{11} = \hat{x}_{15} = \hat{x}_{24} = \hat{x}_{27} = \hat{x}_{35} = \hat{x}_{38} = \hat{x}_{43} = \hat{x}_{44} = \hat{x}_{56} = \hat{x}_{57} =$$
$$\hat{x}_{62} = \hat{x}_{66} = \hat{x}_{73} = \hat{x}_{78} = \hat{x}_{81} = \hat{x}_{82} = \frac{1}{2} \;, \quad \hat{x}_{ij} = 0, \text{ otherwise.}$$

It can be verified that $\hat{x}$ is an extreme point of the polytope $P_{2,4}^{8,1}$ and that it satisfies all *first class* facet-inducing inequalities for $Q_{2,4}^{8,1}$. Since both $r_1$ and $r_2$ are $< 2$ (in fact equal to 1) for $Q_{2,4}^{8,1}$, we do not have a pair of nonadjacent blocks both of whose $r$-numbers are $\geq 2$. Hence the second class of inequalities of the form (15), (21) are not facet-inducing for this problem.

## 2.3 A Facet Lifting Procedure

In this section, a lifting procedure for facets of $Q_{n_1,n_2}^{n,r_1}$ is presented. Given a facet $F$ of $Q_{n_1,n_2}^{n,r_1}$, we show how to lift $F$ into a facet $F^*$ of $Q_{n_1,n_2}^{n+1,r_1}$, $Q_{n_1+1,n_2}^{n+1,r_1}$, $Q_{n_1+1,n_2+1}^{n+1,r_1+1}$, and $Q_{n_1,n_2+1}^{n+1,r_1}$. This procedure is used to prove Theorems 3 and 4 using mathematical induction. All symbols with a star (*) refer to assignments of order $n+1$. For any matrix $A$, we denote its $i$th row vector by $A_{i.}$, and its $j$th column vector by $A_{.j}$.

**Lemma 6** *Let $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_{ij} \leq a_0$ be a non trivial facet-inducing inequality for $Q_{n_1,n_2}^{n,r_1}$ and let $A^* = (a_{ij}^*)$ be the $(n+1)\times(n+1)$ matrix derived from $A = (a_{ij})$ such that*

$$A^* = \begin{pmatrix} A & A_{.j_0} \\ A_{i_0.} & 0 \end{pmatrix} \tag{22}$$

*for any $i_0 \in \{n_1+1,\ldots,n\}$ and any $j_0 \in \{n_2+1,\ldots,n\}$ satisfying $a_{i_0j_0} = 0$. Then $\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_{ij}^*x_{ij}^* \leq a_0$ is a facet-inducing inequality for $Q_{n_1,n_2}^{n+1,r_1}$ provided that it is a valid inequality for it.*

**Proof.** Let $F = \{x \in Q_{n_1,n_2}^{n,r_1} : \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_{ij} = a_0\}$ and $F^* = \{x^* \in Q_{n_1,n_2}^{n+1,r_1} : \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_{ij}^*x_{ij}^* = a_0\}$. Then there exist $n^2-2n$ affinely independent assignments $x^1, x^2, \ldots, x^{n^2-2n}$ in $F$, and for every $(i,j) \in \{1,\ldots,n\}\times\{1,\ldots,n\}$ there exists at least one $x^k \in \{x^1, x^2, \ldots, x^{n^2-2n}\}$ such that $x^k_{ij} = 1$. The last assertion follows since otherwise if $x^k_{rs} = 0$ for all $x^k \in \{x^1, x^2, \ldots, x^{n^2-2n}\}$ then $F$ would be contained in the intersection of two facetal hyperplanes $x_{rs} = 0$ and $\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_{ij} = a_0$ contradicting the assumption that $F$ is a facet of $Q^{n,r_1}_{n_1,n_2}$. Let $\{x^{i_1}, x^{i_2}, \ldots, x^{i_n}\} \subset \{x^1, x^2, \ldots, x^{n^2-2n}\}$ be such that $x^{i_1}_{1j_0} = x^{i_2}_{2j_0} = \cdots = x^{i_n}_{nj_0} = 1$. Likewise, let $\{x^{j_1}, x^{j_2}, \ldots, x^{j_n}\} \subset \{x^1, x^2, \ldots, x^{n^2-2n}\}$ be such that $x^{j_1}_{i_01} = x^{j_2}_{i_02} = \cdots = x^{j_n}_{i_0n} = 1$.

Let $x^{*k}$, for $k = 1, 2, \ldots, n^2 - 2n$, be the assignments of order $n+1$ defined as $x^{*k}_{n+1,n+1} = 1$, $x^{*k}_{ij} = x^k_{ij}$ for $i, j = 1, 2, \ldots, n$ then $x^{*1}, x^{*2}, \ldots, x^{*n^2-2n}$ belong to $F^*$ since by construction $a^*_{n+1,n+1} = 0$. Let $x^{*i_1}, x^{*i_2}, \ldots, x^{*i_n}$ be the assignments of order $n+1$ derived from $x^{i_1}, x^{i_2}, \ldots, x^{i_n}$ by switching columns $j_0$ and $n+1$ and by setting $x^{*k}_{n+1,j_0} = 1$ for all $k = i_1, i_2, \ldots, i_n$. Then $x^{*i_1}, x^{*i_2}, \ldots, x^{*i_n}$ belong to $F^*$ since $A^*_{\cdot n+1} = A^*_{\cdot j_0}$. Likewise, let $x^{*j_1}, x^{*j_2}, \ldots, x^{*j_n}$ be the assignments of order $n+1$ derived from $x^{j_1}, x^{j_2}, \ldots, x^{j_n}$ by switching rows $i_0$ and $n+1$ and by setting $x^{*k}_{i_0,n+1} = 1$ for all $k = j_1, j_2, \ldots, j_n$. Then $x^{*j_1}, x^{*j_2}, \ldots, x^{*j_n}$ belong to $F^*$ since $A^*_{n+1\cdot} = A^*_{i_0\cdot}$. Then, by construction, $x^{*1}, x^{*2}, \ldots, x^{*n^2-2n}$, $x^{*i_1}, x^{*i_2}, \ldots, x^{*i_n}$, $x^{*j_1}, x^{*j_2}, \ldots, x^{*j_n} \backslash \{x^{*j_{j_0}}\}$ is a set of affinely independent assignments. Thus $\dim F^* = n^2 - 2 = (n+1)^2 - 2(n+1) - 1$. ∎

Using a similar argument as in Lemma 6, it can be shown that if $\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_{ij} \le a_0$ is a facet-inducing inequality for $Q^{n,r_1}_{n_1,n_2}$ then

$$\sum_{i=0}^{n} \sum_{j=1}^{n+1} b^*_{ij} x^*_{ij} \le a_0 \ , \quad \sum_{i=0}^{n} \sum_{j=0}^{n} c^*_{ij} x^*_{ij} \le a_0 \ , \quad \sum_{i=1}^{n+1} \sum_{j=0}^{n} d^*_{ij} x^*_{ij} \le a_0$$

are facet-inducing inequalities for $Q^{n+1,r_1}_{n_1+1,n_2}$ , $Q^{n+1,r_1+1}_{n_1+1,n_2+1}$ , and $Q^{n+1,r_1}_{n_1,n_2+1}$ respectively provided that they are valid inequalities. $B^* = (b^*_{ij})$, $C^* = (c^*_{ij})$, and $D^* = (d^*_{ij})$ are defined by

$$B^* = \begin{pmatrix} A_{k_0\cdot} & 0 \\ A & A_{\cdot j_0} \end{pmatrix} \ , \quad C^* = \begin{pmatrix} 0 & A_{k_0\cdot} \\ A_{\cdot m_0} & A \end{pmatrix} \ , \quad D^* = \begin{pmatrix} A_{\cdot m_0} & A \\ 0 & A_{i_0\cdot} \end{pmatrix}$$

for any $k_0 \in \{1, \ldots, n_1\}$, any $j_0 \in \{n_2+1, \ldots, n\}$, any $m_0 \in \{1, \ldots, n_2\}$, and any $i_0 \in \{n_1+1, \ldots, n\}$ satisfying $a_{k_0j_0} = 0$, $a_{k_0m_0} = 0$, and $a_{i_0m_0} = 0$.

**Proof of theorem 3 .** For ease of notation, and without loss of generality assume that the *defining cell* $(p, q)$ belongs to Block $B_1$. Thus, $\hat{I} = I_2 = \{n_1 + 1, n_1 + 2, \dots, n\}$ and $\hat{J} = J_2 = \{n_2 + 1, n_2 + 2, \dots, n\}$ and $r_w = r_3$. The proof is by induction on $n$, the order of the assignment.

For $n = 4, n_1 = n_2 = 2$ and $r_1 = 1$. Let $(p, q) = (1, 1)$ and $\mathcal{K}_R = \mathcal{K}_C = \{3\}$. Then

$$x_{11} + x_{13} + x_{31} - x_{44} \leq 1 \tag{23}$$

is a facet-defining inequality of $Q^{4,1}_{2,2}$ since it is a valid inequality of $Q^{4,1}_{2,2}$ by Lemma 3 and since the following 8 feasible assignments, represented as permutations, are affinely independent and satisfy (23) as an equality. Recall that dim $Q^{4,1}_{2,2} = 8$.

$$\begin{array}{lll} x^1 = (1,3,4,2) & x^2 = (1,4,3,2) & x^3 = (1,4,2,3) \\ x^4 = (2,4,1,3) & x^5 = (3,1,4,2) & x^6 = (3,2,4,1) \\ x^7 = (3,2,1,4) & & x^8 = (4,2,1,3). \end{array}$$

Now assume $n \geq 4$ and that the assertion is true for assignments of order $n$, using the lifting procedure in Lemma 6, we will show that it is true for assignments of order $n+1$.

Let $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{ij} \leq 1$ be a facet-inducing inequality of form (12), shown in Figure 3, for the problem of order $n$ (i.e., for $Q^{n,r_1}_{n_1,n_2}$); and let $(p, q)$ be its defining cell, $\mathcal{K}_R$ ($\mathcal{K}_C$) be its defining subset of row( column) indices. We will refer to this valid inequality as VI($n$).

Consider the problem of order $(n + 1)$ and its corresponding array $I^* \times J^*$. Then $I^* \times J^*$ is obtained from $I \times J$, $I = J = \{1, 2, \dots, n\}$ by the addition of one new row and one new column. The new row can be added either at the top or at the bottom of the $n \times n$ array, and the new column can be added either to the left or to the right of the $n \times n$ array, leading to four separate cases:

**Case 1:** The added row and the added column are $n + 1$ and $n + 1$. This corresponds to the polytope $Q^{n+1,r_1}_{n_1,n_2}$ where $r^*_3 = r_3 + 1$. ( Recall that symbols with $(*)$ refer to the problem of order $n + 1$). Then VI($n$) can be lifted in two ways:

1. Select $i_0$ to be any row $\in \mathcal{K}_R$ and $j_0$ to be any column $\in J_2 \backslash \mathcal{K}_C$. Note that for this selection $a_{i_0 j_0} = 0$. Hence, $\sum_{i=1}^{n+1} \sum_{j=l}^{n+1} a^*_{ij} x^*_{ij} \leq 1$, where $A^* = (a^*_{ij})$ as defined in (22), is a valid inequality of $Q^{n+1,r_1}_{n_1,n_2}$ since it is of the form (12), with defining cell $(p,q)$, $\mathcal{K}^*_R = \mathcal{K}_R \cup \{n+1\}$, and $\mathcal{K}^*_C = \mathcal{K}_C$; and by Lemma 6 it is facet-inducing.
2. Select $j_0$ to be any column $\in \mathcal{K}_C$ and $i_0$ to be any row $\in I_2 \backslash \mathcal{K}_R$. Using the same argument as in 1, it follows that the valid inequality for $Q^{n+1,r_1}_{n_1,n_2}$ with defining cell $(p,q)$, and $\mathcal{K}^*_C = \mathcal{K}_C \cup \{n+1\}$ and $\mathcal{K}^*_R = \mathcal{K}_R$ is also facet-inducing.

**Case 2:** The added row and the added column are 0 and $n+1$ respectively. This corresponds to the polytope $Q^{n+1,r_1}_{n_1+1,n_2}$ where $r^*_2 = r_2 + 1$. Select $j_0$ to be any column $\in J_2 \backslash \mathcal{K}_C$ and $i_0$ to be any row $\in (\{1,2,\ldots,n_1\} \backslash \{p\})$. Notice that for this selection $A^*_{i_0 \cdot}$ is a row of all 0's. Using the same argument as in case 1, it follows that the valid inequality for $Q^{n+1,r_1}_{n_1+1,n_2}$ with defining cell $(p,q)$ and $\mathcal{K}^*_C = \mathcal{K}_C$, $\mathcal{K}^*_R = \mathcal{K}_R$ is facet-inducing.

**Case 3:** The added row and the added column are 0 and 0 respectively. This corresponds to the polytope $Q^{n+1,r_1+1}_{n_1+1,n_2+1}$ where $r^*_1 = r_1 + 1$. Select $j_0$ to be any column $\in \{1,2,\ldots,n_2\} \backslash \{q\}$ and and $i_0$ to be any row $\in (\{1,2,\ldots,n_1\} \backslash \{p\})$. For this selection $A^*_{i_0 \cdot} = A^*_{\cdot j_0} = 0$. Using the same argument as in case 1, it follows that the valid inequality for $Q^{n+1,r_1}_{n_1+1,n_2+1}$ with defining cell $(p,q)$ and $\mathcal{K}^*_C = \mathcal{K}_C$, $\mathcal{K}^*_R = \mathcal{K}_R$ is facet-inducing.

**Case 4:** The added row and the added column are $n+1$ and 0 respectively. This corresponds to the polytope $Q^{n+1,r_1}_{n_1,n_2+1}$ where $r^*_4 = r_4 + 1$. Select $i_0$ to be any row $\in I_2 \backslash \mathcal{K}_R$ and $j_0$ to be any column $\in (\{1,2,\ldots,n_2\} \backslash \{q\})$. Using the same argument as in case 1, it follows that the valid inequality for $Q^{n+1,r_1}_{n_1,n_2+1}$ with defining cell $(p,q)$ and $\mathcal{K}^*_C = \mathcal{K}_C$, $\mathcal{K}^*_R = \mathcal{K}_R$ is facet-inducing.

Now assume $n \geq 4$, we will show that every valid inequality of the form (12) for the problem of order $n+1$ can be established as being facet-inducing by lifting some facet-inducing inequality of the form (12) for the problem of order $n$. Since $n+1 \geq 5$, for the problem of order $n+1$ at least one of the $r_t^*$'s $\geq 2$ for t=1 to 4.

Assume that $r_3^* \geq 2$ and consider the valid inequality of form (12) for the problem of order $n+1$ with defining cell $(p,q)$ and defining subsets $\mathcal{K}_R^*$ and $\mathcal{K}_C^*$. We will refer to this inequality by VI$(n+1)$. Then $|\mathcal{K}_C^*| + |\mathcal{K}_R^*| \geq 3$. Thus either $|\mathcal{K}_C^*|$ or $|\mathcal{K}_R^*|$ must be $\geq 2$.

1. If $|\mathcal{K}_R^*| \geq 2$. Let $i_0$ be any row $\in \mathcal{K}_R^*$ and $j_0$ be any column $\in J_2^* \backslash \mathcal{K}_C^*$ and consider the problem P$(n)$ of order $n$ associated with array $(I^* \backslash \{i_0\}) \times (J^* \backslash \{j_0\})$. Then the inequality obtained from VI$(n+1)$ by deleting $i_0$ from $\mathcal{K}_R^*$ is of form (12) with defining cell $(p,q)$, $\mathcal{K}_R = \mathcal{K}_R^* \backslash \{i_0\}$, and $\mathcal{K}_C = \mathcal{K}_C^*$; and hence it is a facet-inducing inequality for problem P$(n)$. Furthermore, VI$(n+1)$ can be established as facet-inducing for the problem of order $n+1$ by lifting this inequality as in Case 1 above.

2. If $|\mathcal{K}_C^*| \geq 2$. Let $j_0$ be any column $\in \mathcal{K}_C^*$ and $i_0$ be any row $\in I_2^* \backslash \mathcal{K}_R^*$. Then the inequality of form (12) with defining cell $(p,q)$ and $\mathcal{K}_C = \mathcal{K}_C^* \backslash \{j_0\}$, and $\mathcal{K}_R = \mathcal{K}_R^*$ is a facet-inducing inequality for problem P$(n)$, and we can establish that VI$(n+1)$ is facet-inducing for the problem of order $n+1$ by lifting this inequality as in Case 1 above.

Similarly, if $r_2^* \geq 2$ let $i_0$ be any row $\in (I_1^* \backslash \{p\})$, and let $j_0$ be any column $\in J_2^* \backslash \mathcal{K}_C^*$. If $r_1^* \geq 2$ let $i_0$ be any row $\in (I_1^* \backslash \{p\})$, and let $j_0$ be any column $\in (J_1^* \backslash \{q\})$. If $r_4^* \geq 2$ let $i_0$ be any row $\in (I_2^* \backslash \mathcal{K}_R^*)$ and let $j_0$ be any column $\in (J_1^* \backslash \{q\})$. Then in all these cases, it is easy to show that the inequality with defining cell $(p,q)$ and $\mathcal{K}_C = \mathcal{K}_C^*$, and $\mathcal{K}_R = \mathcal{K}_R^*$ is of form (12) and hence it is a facet-inducing inequality for the problem of order $n$ associated with the array $(I^* \backslash \{i_0\}) \times (J^* \backslash \{j_0\})$ and that VI$(n+1)$ can be established to be facet inducing for the problem of order $n+1$ by lifting this inequality as in Cases 2,3, and 4 respectively. ∎

**Proof of Theorem 4.** We assume that the secondary defining cell $(m, l) \in B_v$. A proof similar to the following applies when $(m, l) \in B_u$. Also we use induction on $n$, the order of the assignment. For $n = 6, n_1 = n_2 = 3$ and $r_1 = 1$. Let $(p, q) = (1, 1)$, $(m, l) = (4, 2)$, $\mathcal{K}_C = \{5\}$, $\tilde{\mathcal{K}}_C = \{4\}$, $\mathcal{K}_R = \{5\}$, and $\tilde{\mathcal{K}}_R = \{2\}$. Then

$$x_{11} + x_{15} + x_{51} - x_{32} - x_{36} - x_{46} - x_{62} - x_{64} - x_{66} \leq 1 \tag{24}$$

is a facet-defining inequality of $Q^{6,1}_{3,3}$ since it is a valid inequality of $Q^{6,1}_{3,3}$ by Lemma 3 and since the following 24 feasible assignments, represented as permutations, are affinely independent and satisfy (24) as an equality. Recall that dim $Q^{6,1}_{3,3}$ =24.

$$\begin{array}{lll}
x^1 = (1, 4, 5, 2, 6, 3) & x^2 = (1, 5, 4, 2, 6, 3) & x^3 = (1, 6, 4, 5, 2, 3) \\
x^4 = (1, 6, 4, 2, 5, 3) & x^5 = (1, 6, 4, 2, 3, 5) & x^6 = (1, 6, 4, 3, 2, 5) \\
x^7 = (1, 6, 5, 4, 2, 3) & x^8 = (1, 6, 5, 2, 4, 3) & x^9 = (2, 6, 4, 5, 1, 3) \\
x^{10} = (3, 6, 4, 2, 1, 5) & x^{11} = (5, 1, 4, 2, 6, 3) & x^{12} = (5, 2, 4, 6, 1, 3) \\
x^{13} = (5, 2, 4, 1, 6, 3) & x^{14} = (5, 2, 4, 3, 6, 1) & x^{15} = (5, 2, 4, 3, 1, 6) \\
x^{16} = (5, 2, 6, 4, 1, 3) & x^{17} = (6, 2, 4, 5, 1, 3) & x^{18} = (5, 3, 4, 2, 6, 1) \\
x^{19} = (5, 4, 1, 2, 6, 3) & x^{20} = (5, 6, 2, 4, 1, 3) & x^{21} = (4, 6, 3, 2, 1, 5) \\
x^{22} = (5, 4, 3, 2, 6, 1) & x^{23} = (5, 6, 3, 4, 1, 2) & x^{24} = (5, 6, 3, 2, 1, 4).
\end{array}$$

Now assume $n \geq 6$ and that the assertion is true for assignments of order $n$. Using the lifting procedure in Lemma 6, we will show that it is true for assignments of order $n+1$.

Without loss of generality, we assume that the primary defining cell $(p, q) \in B_1$. Thus $\hat{I} = I_2 = \{n_1 + 1, \ldots, n\}$, $\hat{J} = J_2 = \{n_2 + 1, \ldots, n\}$, and $r_w = r_3$.

Let $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{ij} \leq 1$ be a facet-inducing inequality of form (15), shown in Figure 4, for the problem of order $n$ (i.e., for $Q^{n,r_1}_{n_1,n_2}$); and let $(p, q)$, $(m, l)$ be respectively its primary and secondary defining cells, $\mathcal{K}_R$, $\tilde{\mathcal{K}}_R$, $\mathcal{K}_C$, and $\tilde{\mathcal{K}}_C$ be its defining subset of row and column indices. We will refer to this valid inequality as VII($n$).

Consider the problem of order $(n + 1)$ and its corresponding array $I^* \times J^*$. Then $I^* \times J^*$ is obtained from $I \times J$, $I = J = \{1, 2, \ldots, n\}$ by the addition of one new row and one new column. As in the proof of Theorem 3, the new row can be added either at the top or at the bottom of the $n \times n$ array, and the new column can be added either to the left or to the right of the $n \times n$ array, leading to four separate cases:

**Case 1:** The added row and and the added column are $n+1$ and $n+1$. This corresponds to the polytope is $Q^{n+1,r_1}_{n_1,n_2}$ where $r_3^* = r_3 + 1$. Then VII($n$) can be lifted in two ways.

1. Select $i_0$ to be any row $\in \mathcal{K}_R$ and $j_0$ to be any column $\in J_2 \backslash (\mathcal{K}_C \cup \tilde{\mathcal{K}}_C)$. Note that for such selection $a_{i_0 j_0} = 0$. Hence, $\sum_{i=1}^{n+1} \sum_{j=l}^{n+1} a^*_{ij} x^*_{ij} \leq 1$, where $A^* = (a^*_{ij})$ as defined in (22), is a valid inequality of $Q^{n+1,r_1}_{n_1,n_2}$ since it is of the form (15) with defining cells $(p, q)$ and $(m, l)$, and defining subsets $\mathcal{K}^*_R = \mathcal{K}_R \cup \{n + 1\}$ , $\mathcal{K}^*_C = \mathcal{K}_C$ , $\tilde{\mathcal{K}}^*_C = \tilde{\mathcal{K}}_C$ , and $\tilde{\mathcal{K}}^*_R = \tilde{\mathcal{K}}_R$; and by Lemma 6 it is facet-inducing.
2. Select $j_0$ to be any column $\in \mathcal{K}_C$ and $i_0$ to be any row $\in I_2 \backslash (\mathcal{K}_R \cup \{m\})$. Using The same argument as in 1, it follows that the valid inequality for $Q^{n+1,r_1}_{n_1,n_2}$ with defining cells $(p, q)$ and $(m, l)$ and defining subsets $\mathcal{K}^*_C = \mathcal{K}_C \cup \{n + 1\}$ and $\mathcal{K}^*_R = \mathcal{K}_R$ , $\tilde{\mathcal{K}}^*_C = \tilde{\mathcal{K}}_C$ , and $\tilde{\mathcal{K}}^*_R = \tilde{\mathcal{K}}_R$ is also facet-inducing.

**Case 2:** The added row and the added column are 0 and $n + 1$ respectively. This corresponds to the polytope $Q^{n+1,r_1}_{n_1+1,n_2}$ where $r_2^* = r_2 + 1$. Then VII($n$) can also be lifted in two ways.

1. Select $j_0$ to be any column $\in \tilde{\mathcal{K}}_C$ and $i_0$ to be any $i \in \{1, 2, \ldots, n_1\} \backslash (\{p\} \cup \tilde{\mathcal{K}}_R)$. Using the same argument as in Case 1, it follows that the valid inequality for $Q^{n+1,r_1}_{n_1+1,n_2}$ with defining $(p, q)$ and $(m, l)$ and defining subsets $\mathcal{K}^*_R = \mathcal{K}_R$, $\mathcal{K}^*_C = \mathcal{K}_C$, $\tilde{\mathcal{K}}^*_C = \tilde{\mathcal{K}}_C \cup \{n + 1\}$, and $\tilde{\mathcal{K}}^*_R = \tilde{\mathcal{K}}_R$ is facet-inducing.
2. Select $i_0$ to be any row $\in \tilde{\mathcal{K}}_R$ and $j_0$ to be any column $\in J_2 \backslash (\tilde{\mathcal{K}}_C \cup \mathcal{K}_C)$. Using the same argument as in Case 1, it follows that the valid inequality for $Q^{n+1,r_1}_{n_1+1,n_2}$ with defining $(p, q)$ and $(m, l)$ and defining subsets $\mathcal{K}^*_R = \mathcal{K}_R$, $\mathcal{K}^*_C = \mathcal{K}_C$, $\tilde{\mathcal{K}}^*_C = \tilde{\mathcal{K}}_C$, and $\tilde{\mathcal{K}}^*_R = \tilde{\mathcal{K}}_R \cup \{n + 1\}$ is facet-inducing.

**Case 3:** The added row and the added column are 0 and 0 respectively. This corresponds to the polytope $Q^{n+1,r_1+1}_{n_1+1,n_2+1}$ where $r_1^* = r_1 + 1$. Select $j_0$ to be any column $\in \{1, 2, \ldots, n_2\} \backslash (\{q\} \cup \{l\})$ and and $i_0$ to be any row $\in \{1, 2, \ldots, n_1\} \backslash (\{p\} \cup \tilde{\mathcal{K}}_R)$. For this selection, $A^*_{\cdot j_0} = 0$. Using the same argument as in Case 1, it follows that the valid inequality for $Q^{n+1,r_1}_{n_1+1,n_2+1}$ with defining $(p, q)$ and $(m, l)$ and defining subsets $\mathcal{K}^*_R = \mathcal{K}_R$, $\mathcal{K}^*_C = \mathcal{K}_C$, $\tilde{\mathcal{K}}^*_C = \tilde{\mathcal{K}}_C$, and $\tilde{\mathcal{K}}^*_R = \tilde{\mathcal{K}}_R$, is facet-inducing.

**Case 4:** The added row and the added column are $n + 1$ and 0 respectively. This corresponds to the polytope $Q^{n+1,r_1}_{n_1,n_2+1}$ where $r_4^* = r_4 + 1$. Select $i_0$ to be any row $\in I_2 \backslash (\mathcal{K}_R \cup \{m\})$ and $j_0$ to be any column $\in \{1, 2, \ldots, n_2\} \backslash (\{q\} \cup \{l\})$. Using the same argument as in Case 1, it follows that the valid inequality for $Q^{n+1,r_1}_{n_1,n_2+1}$ with defining $(p, q)$ and $(m, l)$ and defining subsets $\mathcal{K}^*_R = \mathcal{K}_R$, $\mathcal{K}^*_C = \mathcal{K}_C$, $\tilde{\mathcal{K}}^*_C = \tilde{\mathcal{K}}_C$, and $\tilde{\mathcal{K}}^*_R = \tilde{\mathcal{K}}_R$ is facet-inducing.

We will now show that every valid inequality of form (15) for the problem of order $n + 1$ can be obtained by lifting some valid inequality of form (15) for the problem of order $n$.

Consider the valid inequality of form (15) for the problem of order $n+1$ with primary and secondary defining cells $(p, q)$, $(m, l)$, and defining subsets $\mathcal{K}^*_R$, $\tilde{\mathcal{K}}^*_R$, $\mathcal{K}^*_C$, and $\tilde{\mathcal{K}}^*_C$. Refer to this inequality as VII$(n + 1)$. Since $n + 1 \geq 7$, for the problem of order $n + 1$, one of the following must hold. $r_3^* \geq 2$, $r_2^* \geq 3$, $r_4^* \geq 3$, or $r_1^* \geq 2$.
If $r_3^* \geq 2$. In this case $|\mathcal{K}^*_R| + |\mathcal{K}^*_C| \geq 3$ which implies that either $|\mathcal{K}^*_R| \geq 2$ or $|\mathcal{K}^*_C| \geq 2$.

1. if $|\mathcal{K}^*_R| \geq 2$. Let $i_0$ be any row $\in \mathcal{K}^*_R$ and $j_0$ be any column $\in J_2^* \backslash (\mathcal{K}^*_C \cup \tilde{\mathcal{K}}^*_C)$ and consider the problem P2$(n)$ of order $n$ associated with array $I^* \backslash \{i_0\} \times J^* \backslash \{j_o\}$. Then the inequality obtained from VII$(n + 1)$ by deleting $i_0$ from $\mathcal{K}^*_R$ is of form (15) with defining cells $(p, q)$, $(m, l)$, and defining subsets $\mathcal{K}_R = \mathcal{K}^*_R \backslash \{i_0\}$, $\mathcal{K}_C = \mathcal{K}^*_C$, $\tilde{\mathcal{K}}_R = \tilde{\mathcal{K}}^*_R$, $\tilde{\mathcal{K}}_C = \tilde{\mathcal{K}}^*_C$; and hence it is a valid inequality for problem P2$(n)$. Furthermore, VII$(n + 1)$ can be lifted from this valid inequality as in Case 1.

2. if $|\mathcal{K}^*_C| \geq 2$. Let $j_0$ be any column $\in \mathcal{K}^*_C$ and $i_0$ be any row $\in I_2^* \backslash (\mathcal{K}^*_R \cup \{m\})$.

Then the inequality obtained from VII$(n+1)$ by deleting $j_0$ from $\mathcal{K}_C^*$ is of form (15) with defining cells $(p,q)$, $(m,l)$, and defining subsets $\mathcal{K}_C = \mathcal{K}_C^* \backslash \{j_0\}$, $\mathcal{K}_R = \mathcal{K}_R^*$, $\tilde{\mathcal{K}}_R = \tilde{\mathcal{K}}_R^*$, $\tilde{\mathcal{K}}_C = \tilde{\mathcal{K}}_C^*$; and hence it is a valid inequality for problem P2$(n)$, and VII$(n+1)$ can be lifted from it.

If $r_2^* \geq 3$. In this case $|\tilde{\mathcal{K}}_R^*| + |\tilde{\mathcal{K}}_C^*| \geq 3$ which implies that either $|\tilde{\mathcal{K}}_R^*| \geq 2$ or $|\tilde{\mathcal{K}}_C^*| \geq 2$.

1. if $|\tilde{\mathcal{K}}_R^*| \geq 2$. Let $i_0$ be any row $\in \tilde{\mathcal{K}}_R^*$ and $j_0$ be any column $\in J_2^* \backslash (\tilde{\mathcal{K}}_C^* \cup \mathcal{K}_C^*)$. Then the inequality obtained from VII$(n+1)$ by deleting $i_0$ from $\tilde{\mathcal{K}}_R^*$ is of form (15) with defining cells $(p,q)$, $(m,l)$, and defining subsets $\tilde{\mathcal{K}}_R = \tilde{\mathcal{K}}_R^* \backslash \{i_0\}$, $\mathcal{K}_C = \mathcal{K}_C^*$, $\mathcal{K}_R = \mathcal{K}_R^*$, $\tilde{\mathcal{K}}_C = \tilde{\mathcal{K}}_C^*$; and hence it is a valid inequality for problem P2$(n)$, and VII$(n+1)$ can be lifted from it.

2. if $|\tilde{\mathcal{K}}_C^*| \geq 2$. Let $j_0$ be any column $\in \tilde{\mathcal{K}}_C^*$ and $i_0$ be any row $\in I_1^* \backslash (\tilde{\mathcal{K}}_R^* \cup \{p\})$. Then the inequality obtained from VII$(n+1)$ by deleting $j_0$ from $\tilde{\mathcal{K}}_C^*$ is of form (15) with defining cells $(p,q)$, $(m,l)$, and defining subsets $\tilde{\mathcal{K}}_C = \tilde{\mathcal{K}}_C^* \backslash \{j_0\}$, $\mathcal{K}_C = \mathcal{K}_C^*$, $\mathcal{K}_R = \mathcal{K}_R^*$, $\tilde{\mathcal{K}}_R = \tilde{\mathcal{K}}_R^*$; and hence it is a valid inequality for problem P2$(n)$, and VII$(n+1)$ can be lifted from it.

If $r_4^* \geq 3$, let $i_0$ be any row $\in I_2^* \backslash (\mathcal{K}_R^* \cup \{m\})$ and $j_0$ be any column $\in I_1^* \backslash (\{q\} \cup \{l\})$. If $r_1^* \geq 2$, let $i_0$ be any row $\in I_1^* \backslash (\tilde{\mathcal{K}}_R^* \cup \{p\})$ and $j_0$ be any column $\in I_1^* \backslash (\{q\} \cup \{l\})$. Then in both these cases the inequality with defining cells $(p,q)$, $(m,l)$, and defining subsets $\tilde{\mathcal{K}}_C = \tilde{\mathcal{K}}_C^*$, $\mathcal{K}_C = \mathcal{K}_C^*$, $\mathcal{K}_R = \mathcal{K}_R^*$, $\tilde{\mathcal{K}}_R = \tilde{\mathcal{K}}_R^*$ is of form (15); and hence it is a valid inequality for problem P2$(n)$, and VII$(n+1)$ can be lifted from it. ∎

Since $Q_{n_1,n_2}^{n,r_1}$ in $R^{n^2}$ space of $(x_{ij} : i,j = 1 \text{ to } n)$ is not a full dimensional polytope (because of equality constraints (1), (2), (5) in the system of constraints defining it) it is possible that a pair of inequalities among (3), (12), (15), (21) may actually represent the same facet of $Q_{n_1,n_2}^{n,r_1}$. As an example, let $n = 5$, $n_1 = n_2 = 2$, $r_1 = 1$. Then the following two inequalities of the first class with their defining cells in blocks $B_1, B_3$ respectively; can be verified to represent the same facet using the equations $\sum_{j=1}^{5} x_{1j} = 1$ and $\sum_{i=1}^{5} x_{i5} = 1$.

$$Ineq1: \quad x_{11} + x_{13} + x_{14} + x_{31} - x_{45} - x_{55} \ \leq \ 1$$
$$Ineq2: \quad x_{35} + x_{25} + x_{31} - x_{12} \ \leq \ 1.$$

However, we have the following proposition.

**Proposition 2** *Let Ineq* $: \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_{ij} \leq 1$ *and Ineq2* $: \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 x_{ij} \leq 1$ *be two distinct facet-inducing inequalities of the first class whose defining cells lie in the same block. Then, Ineq and Ineq2 represent distinct facets.*

**Proof.** Without any loss of generality and for ease of presentation we assume the following:

1. The defining cells $(p, q)$ of $Ineq$, and $(p^2, q^2)$ of $Ineq2$ lie in Bock 1. In particular, let $p = 1$ and $q = n_2 - r_1 + 1$.

2. Let $\mathcal{K}_R$ and $\mathcal{K}_C$, respectively the defining subset of row and column indices of $Ineq$ be as follows:

$$\mathcal{K}_R = \{n_1 + 1, n_1 + 2, \ldots, n - 1 + |\mathcal{K}_R|\},$$
$$\mathcal{K}_C = \{n - |\mathcal{K}_C| + 1, n - |\mathcal{K}_C| + 2, \ldots, n\}.$$

Let $x^0$ be the assignment

$$x^0 = \{\ n_2 - r_1 + 1,\ n_2 - r_1 + 2, \ldots, n, 1, 2, \ldots, r_4\}, \tag{25}$$

represented in Figure 5 by cells marked with stars. Then clearly $x^0$ is a feasible assignment which satisfies $Ineq$ as an equality (since $a_{1,n_2-r_1+1} = 1$). Now we consider 3 cases depending on the location of $(p^2, q^2)$, the defining cell of $Ineq2$. Let $\mathcal{K}_R^2$ and $\mathcal{K}_C^2$ denote respectively the defining subset of row and column indices of $Ineq2$.

**Case 1:** $p^2 = p$ and $q^2 = q$, i.e., both $Ineq$ and $Ineq2$ have the same defining cell. Let $j_0 \in \mathcal{K}_C \backslash \mathcal{K}_C^2$ (such $j_0$ exists since if $|\mathcal{K}_C| = |\mathcal{K}_C^2|$, then $\mathcal{K}_C \neq \mathcal{K}_C^2$ since $Ineq$

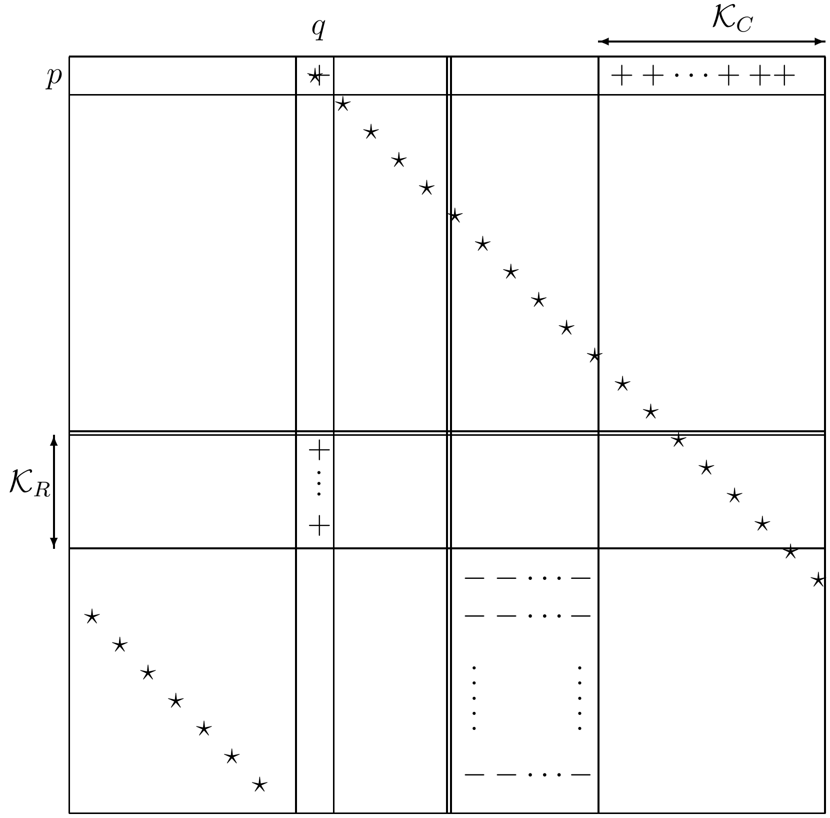



Figure 5: Pictorial representation of the facet-inducing inequality $Ineq$ and the assignment $x^0$.

and $Ineq2$ are distinct; and if $|\mathcal{K}_C| \neq |\mathcal{K}_C^2|$, then without loss of generality we assume that $|\mathcal{K}_C| > |\mathcal{K}_C^2|$). Let $i_0 \in I_2 \backslash \mathcal{K}_R^2$; and let $x^1$ be the assignment obtained from $x^0$ by switching columns $j_0$ and $n$ and rows $n_1 + r_3$ and $i_0$. Then clearly $x^1$ is a feasible assignment that satisfies $Ineq$ as an equality ( since $a_{i_0,j_0} = 0$) and $Ineq2$ as a strict inequality (since $a^2_{i_0,j_0} = -1$).

**Case 2:** $(p^2, q^2) \in \{(2, n_2 - r_1 + 2), (3, n_2 - r_1 + 3), \ldots, (r_1, n_2)\}$. Let $x^2$ be the assignment obtained from $x^0$ by switching columns $q^2$ and 1. Then clearly $x^2$ is a feasible assignment that satisfies $Ineq$ as an equality and $Ineq2$ as a strict inequality.

**Case 3:** Otherwise, i.e., $(p^2, q^2) \in B_1$ and $q^2 - p^2 \neq n_2 - r_1$. Then clearly $x^0$ is a feasible assignment that satisfies $Ineq$ as an equality and $Ineq2$ as a strict inequality. ∎

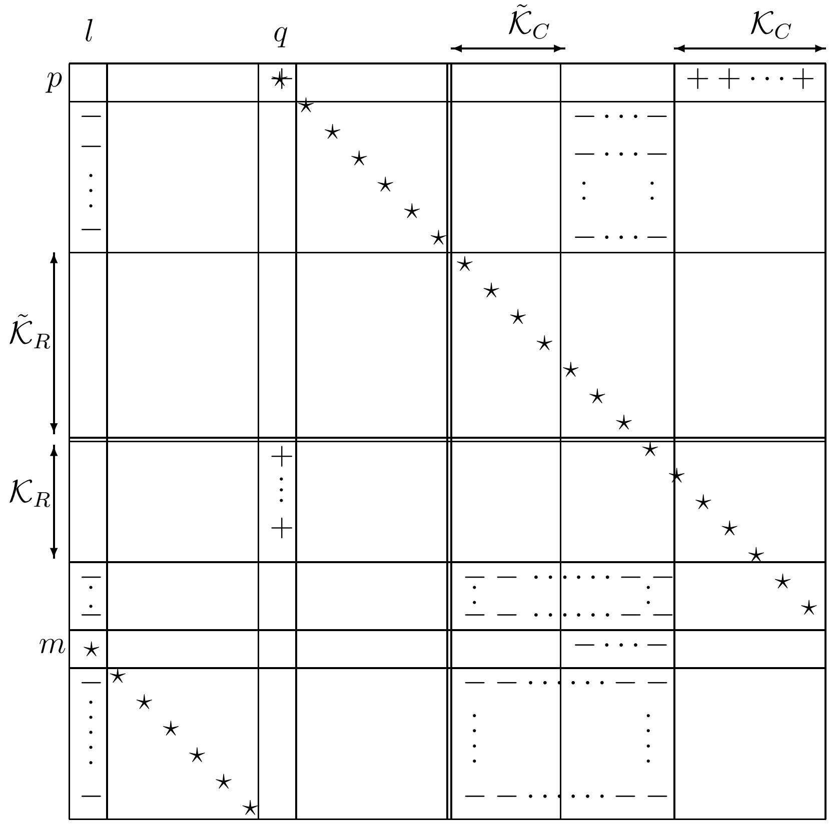



Figure 6: Pictorial representation of a facet-inducing inequality of form (15), and an assignment $x^1$ to be used in the proof.

Using the assignment $x^1$ in Figure 6 (cells with "1" entry marked with a star) in place of $x^0$, and arguments parallel to those in the above proposition, we can prove that two distinct facet-inducing inequalities of the second class whose primary defining cells lie in the same block represent distinct facets of $Q^{n,r_1}_{n_1,n_2}$.

# 3 Summary and Concluding Remarks

We have shown that the general 0-1 problem (9) polynomially reduces to the very special partitioned case. We have derived two large classes of facet inducing inequalities for the 0-1 integer program (9) in the partitioned case, the number in each class grows exponentially with the order of the problem. Whereas the first class of facet-inducing inequalities comes into play for $n \geq 4$, the second class plays a role only for $n \geq 6$. We are studying the separation problems for these classes with the aim of using these facet-inducing inequalities in a branch and cut scheme for solving (9).

These classes together with the non-negativity constraints on the variables do not completely characterize the convex hull of integer feasible solutions of the problem. Currently we are also investigating other facet-inducing inequalities for the problem that may lead to a complete characterization of its integer hull. We are also investigating whether all the facet-inducing inequalities for this problem can be shown to have coefficients 0, +1, or −1 only.

# 4 Acknowledgement

Our appreciation to the referee for his/her careful reading of the manuscript and suggestions for improving it.

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