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Designing Earth Dams Optimally

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Abstract : Engineering design of an earth dam is a crucial issue from the view point of safety and economy of construction cost. Following a scientific approach, it aims at formulating the problem of designing earth dams as an optimization problem. The problem is formulated as a nonlinear program with dam cross sectional area - which represents the major portion of cost of construction - as the objective function and safety factor of the design as the main constraint. The paper provides mathematical modeling for optimizing earth dam designs and for computing the factor of safety. It also discusses issues in obtaining optimal solutions to the formulations, presents a heuristic approach and the results of the application of the methodology to a live earth dam design

Key words and Phrases: Safety constraint, Berms and slopes, Nonlinear optimization, Heuristic approach

1. Introduction

A dam is a barrier that impounds water or underground streams. Dams generally serve the primary purpose of retaining water, while other structures such as floodgates or levees (also known as dikes) are used to manage or prevent water flow into specific land regions. Hydropower and pumped - storage hydroelectricity are often used in conjunction with dams to generate electricity. A dam can also be used to collect water or for storage of water which can be evenly distributed between locations. See the sites http://en.wikipedia.org/wiki/Dam and http://osp.mans.edu.eg/tahany/dams1.htm for a detailed discussion on dams.

An earth dam is a dam built with highly compacted soils. It is classified as a type of embankment dam, being built in the shape of an embankment or wedge which blocks a waterway. In addition to soil, other materials such as rock and clay are also used in earth dams. Earth dams have been built by various human societies for centuries as they are most cost effective and are built from materials available in the nearby locations. Safety of earth dams is a crucial issue (http://

www.wisegeek.com/what-is-an-earth-dam.htm). Failure of moderate or large size earth dams can cause severe damage to life and property of the people living in the nearby areas. Overtopping, seepage and structural failures are the problems encountered due to poor designing and maintenance.

Engineering design of an earth dam is a crucial issue from the view point of safety and economy of construction cost. Evaluation of safety of a given design involves intensive computation. Despite the advances in soft computing and development of software for analyzing design safety, their utilization in practical applications appears to be limited. Perhaps accessibility, cost considerations, awareness and user-friendliness could be some of the reasons for this. Under the present practices, the emphasis is laid more on ensuring safety and the economic aspects are often given little or no consideration while designing the dams. The concept of looking for an optimal design itself appears to be a novel idea for many designing engineers in practice.

This work deals with producing optimal designs for earth dams. It has evolved from the efforts of a senior engineer (Raghupathy, one of the co- authors of this article) who was working as Superintending Engineer, Department of Irrigation, Government of Andhra Pradesh, India, and was responsible for designing earth dams. Following a scientific approach, it aims at formulating the problem of designing earth dams as an optimization problem. The problem is formulated as a nonlinear program with dam cross sectional area - which represents the major portion of cost of construction - as the objective function and safety factor of the design as the main constraint. In order to study the problem, it was necessary to develop a computer program to compute the factor of safety for a given design. The engineers find this program extremely useful in the preparation and analysis of earth dam designs and are using it as a decision support system.

The outline of this paper is as follows. Section 2 presents a brief interesting overview of history of dams. Section 3 presents a short discussion on different types of dams and safety aspects. Section 4 introduces the description of earth dams and basic terminology to set up the stage for the optimization problem in question. Section 5 presents concept and definition of factor of safety of an earth dam. Section 6 deals with the computational aspects of safety and some innovative ideas of computing factor of safety. Section 7 provides mathematical modeling for optimizing earth dam designs and for computing the factor of safety. It also discusses issues in obtaining optimal solutions to the formulations, presents a heuristic approach and the results of the application of the methodology to a live earth dam design. Section 8 deals with development of computer programs and the concluding Section 9 gives a summary of the paper.

2. Brief History of Dam Building

Dams are barriers constructed across streams (above ground or underground) to impound water or the underground streams. Any discussion on the history of dams is incomplete without a mention of beavers (industrious furry animals of the rodent family who live under water) that build astonishing water impounding structures across streams (see Fig 1) and rivulets using tree branches which they cut themselves, chopped wood, twigs, and mud; to provide themselves with comfortable ponds to live in.

The earliest human dam builders may have received their inspiration from beavers, human dam construction dates back to earlier than 5000 BCE and was practiced by several civilizations across the globe; the earliest perhaps in Mesopotamia, Middle East, and India. The Jawa dam in Jordan 100 kilometers (km) northeast of Amman (a gravity dam, 9 meter (m) high, 1 m wide stone wall supported by 50 m earth rampart) is dated to 3000 BCE.

In ancient India, the Indus valley civilization in Mohenjodaro and Harappa going back even earlier than 3000 BCE built a dam on the Saraswati (also referred to as the Ghaggar-Hakra) river and flourished to a high state of development, but that civilization scattered and vanished without any trace, for reasons not fully understood today; one of the contributing factors is perhaps the drying up and eventual disappearance of the Saraswati river.

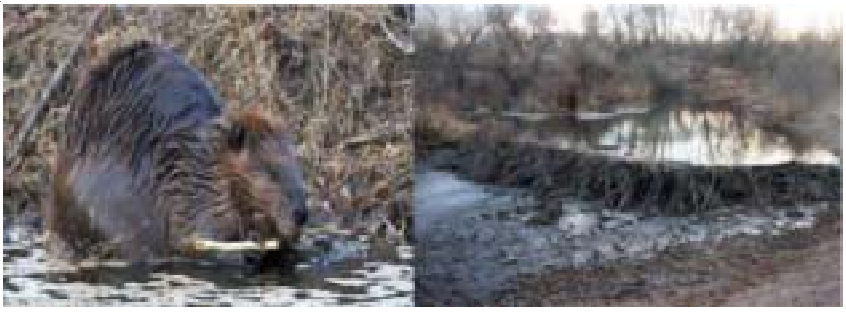

Fig. 1: A hard working beaver building a dam

In 3rd century BCE an intricate water storage and management system consisting of 16 reservoirs, dams, and various channels for collecting, storing, and distributing water was built in modern day India at Dholovira in Gujarat State in western India. Romans introduced the then novel concept of large reservoir dams which could secure water supply for urban settlements year round including the dry season, by pioneering the use of water-proof hydraulic mortar and Roman concrete for much larger dam structures than previously built, such as the Lake Homs dam in Roman Syria in 284 CE, with a capacity of 90 million m3 and in use even today.

The Kallanai Dam (Fig 2) across the main stream of the Kaveri river in Tamilnadu, South India, constructed of unhewn stone and 300 m long, 4.5 m high, and 20 m wide, dated to the 2nd century

CE, considered the oldest water diversion structure in the world, is still in use for diverting the waters of the Kaveri across the fertile delta region for irrigation via canals. Dujiangyan finished in 251 BCE is a large earthen dam in modern-day northern Anhi province in China created an enormous irrigation reservoir 100 km in circumference, a reservoir still present today.

Fig. 2: The Kallanai Dam in India

Also, in China is the Three Gorges Dam, a hydroelectric dam made of concrete and steel, spanning the Yangtze river by the side of the town of Sandouping in Hubei province, the world's largest power station in terms of installed capacity (22,500 MW). It is also intended to increase the Yangtze River's shipping capacity and reduce the potential for flood downstream by providing flood water storage space. Before the construction of this dam the Yangtze river floods used to be an annual phenomenon ravaging that area in China, and after witnessing these floods in 1954, then Chairman Mao Zedong authored a poem titled "Swimming" and initiated plans to construct this dam. The dam body was completed in 2006, and the dam project was completed and fully functional as of 4 July 2012. This project is a historic engineering, social and economic success. Project plans also included a unique method for moving ships, the ships will move into locks located at the lower and upper ends of the dam, and then cranes with cables would move the ships from one lock to the next. The ship lift is expected to be completed in 2014. The dam has raised the water level in the reservoir to 1725 m above sea level by October 2010. The expected annual electricity output will be over 100 TWh.

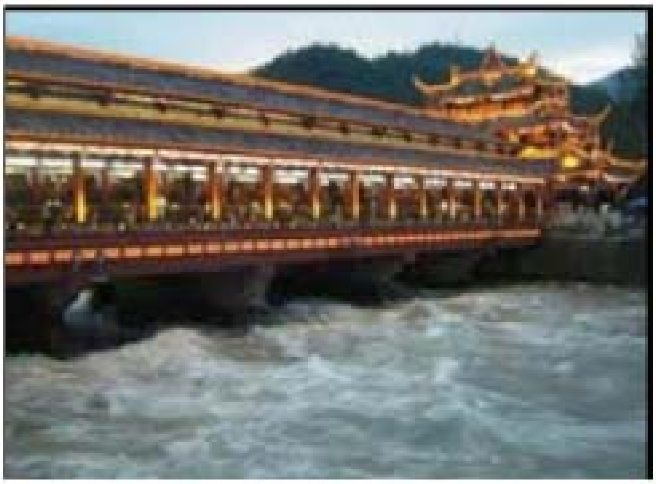

Fig.3 : Dujiangyan Dam in China

The Itaipu is a hydroelectric dam on the Parana river located on the border between Brazil and Paraguay, its name comes from that of an isle that existed near the construction site (in the local Guarani language, the word “Itaipu” means “the sound of a stone”). This dam is the largest operating hydroelectric facility in terms of annual energy generation, generating 94.7 TWh in 2008; its installed capacity of 14,000 MW is second to that of the Three Gorges dam. This dam is considered to be one of the seven modern wonders of the world; it is a concrete and steel dam. The era of large dams was initiated after Hoover Dam which was completed on the Colorado river near Los Vegas in 1936. By the end of the 20th century there were an estimated 800,000 dams worldwide, of which about 40,000 are over 15 m high.

Ancient people who lived in the present day Egypt at the time of the Pharoas constructed the Sadd-el-Kafara dam at Wadi-al-Garawi (25 km South of Cairo), a 102 m long at the base and 87 m wide structure, around 2800 BCE, as a diversion dam for flood control; unfortunately this structure had a very short life as it was destroyed by very heavy rains a few years after its construction. When a dam gets washed away its effects could be devastating resulting in great damage to life and property of the people living in the downstream areas of the dam. For this reason, it is utmost important to enszure that the dams are safe. This chapter deals with one of the important aspects of safety of a particular type of dams - the earth dams.

3. Types of Dams

There are several types of dams. The types may be classified by type of materials used for construction, the height of the dam, the purpose for which the dam is built, and so on. Broadly, dams

Fig. 4: Itaipu dam in South America, considered as one of the 7 modern wonders of the world

can be divided into two types: (i) embankment dams and concrete dams. Embankment dams are earth or rock-filled while gravity, arch and buttress dams are made of concrete. The type of dam for a particular site is selected on the basis of technical and economical data and environmental considerations. Factors that determine the type of a dam are topography, geology, foundation conditions, hydrology, earthquakes, and availability of construction materials. The foundation of the dam should be as sound and free of faults as possible. Narrow valleys with shallow sound rock favour concrete dams. Wide valleys with varying rock depths and conditions favour embankment dams. Earth dams are the most common type (http://encyclopedia2.thefreedictionary.com/dam).

Fig. 5: The Hoover Dam

3.1 Earth Dams

In this chapter we are mainly concerned with earth dams. Earth dams are constructed of well compacted earth, whose cross-section shows a shape like a bank, or hill. A uniformly rolled-earth dam is entirely constructed of one type of material but may contain a drain layer to collect seep water. A zone dearth dam has distinct parts or zones of dissimilar material, typically a locally available shell with a watertight clay core composed of an impermeable material to stop water from seeping through the dam. Since they exert little pressure on their foundations, they can be built on hard rock or softer soils. Compared to concrete dams, earth dams are most economical. To give a rough idea, the cost of a concrete dam is about 20 times that of an earth dam.

3.2 Safety of Dams

Throughout history there have been several instances of dam failure and discharge of stored water, usually causing considerable loss of life and great damage to property. The main causes of dam failure include spillway design error (South Fork Dam), geological instability caused by changes to water levels during filling or poor surveying (Vajont Dam, Malpasset), poor maintenance, especially of outlet pipes (Lawn Lake Dam, Val di Stava Dam collapse), extreme rainfall (Shakidor Dam), and human, computer or design error (Buffalo Creek Flood, Dale Dike Reservoir, Taum Sauk pumped storage plant, (http://en.wikipedia.org/wiki/List of dam failures)). As the dam failures caused by the structure being breached or significantly damaged are generally catastrophic, it is extremely important to analyse the safety of the dam properly. During a recent flood, the reservoir level of the famous Nagarjun Sagar dam in Andhra Pradesh, India, reached alarming heights threatening dam failure. The efficient management and timely actions by the concerned engineers averted a possible catastrophe of unimaginable proportions.

4. Earth Dam Description and Terminology

One of the most important steps while building a dam is to ensure that it is safe well beyond doubt, as any failure of the dam results in calamities. Safety depends on the materials used, the method and quality of construction, and above all the engineering design of the dam. In fact, the design parameters can be chosen to offset, to a reasonable extent, the lacunae in the qualities of materials, method and construction. The measure for safety is called the factor of safety and has a precise mathematical definition. It depends upon the design parameters, the material qualities and implicitly on the method of construction. The optimization problem being considered in this chapter aims at deriving the design parameters in the most economical way without compromising the factor of safety. To facilitate a good understanding of the problem and its solution, it is essential to have a detailed description of an earth dam, its structure and design parameters, the mathematical definition of factor of safety and its evaluation and the underlying assumptions. This section describes the earth dam and its design parameters and the underlying assumptions made for determining the factor of safety. The next two sections are devoted for the precise mathematical definition of factor of safety and the methods for evaluating the same.

4.1 Earth Dam Description

Throughout this chapter the units of measurement of various quantities will be as follows. All the linear measurements (length, width, depth, height) will be in meters (m); areas in square meters (m^2); volume in cubic meters (m^3); time units in seconds; force units in KgF (one KgF is 9.8 Newtons). Weight being gravitational force of an object, its units is KgF. In this chapter we use densities of material for computation of factor of safety. These densities are the weight densities. Weight density is defined as the weight per unit volume and the units will be KgF/m^3.

Fig. 6: Picture of an earth dam

As described earlier, an earth dam is constructed of compacted earth in a structured way, usually between two elevated places (see Fig 6). Fig 7 is drawn to show the picture of cross-sectional view of an earth dam. In this picture, the top arrow is along the length of the dam. The left hand side of the dam in the picture is that side of the dam where the water is impounded by the dam. The horizontal arrows indicate the natural flow of water. This side of the dam is called the upstream of the dam. The right side of the dam in the picture, horizontally opposite to the upstream, is called the downstream of the dam. The front portion of the picture shown by the vertically downward arrows is called a cross section of the dam at a point along the length of the dam. Broadly speaking, earth dams comprise of two components - the core and the shell. The interior portion of the dam is called the core and the portion of the dam surrounding it is called the shell (see Fig 7). The purpose of the core is to arrest the seepage of water from upstream side to downstream side. For this purpose, the core is constructed using materials of low permeability such as clay. In certain earth dams the core may not be present.

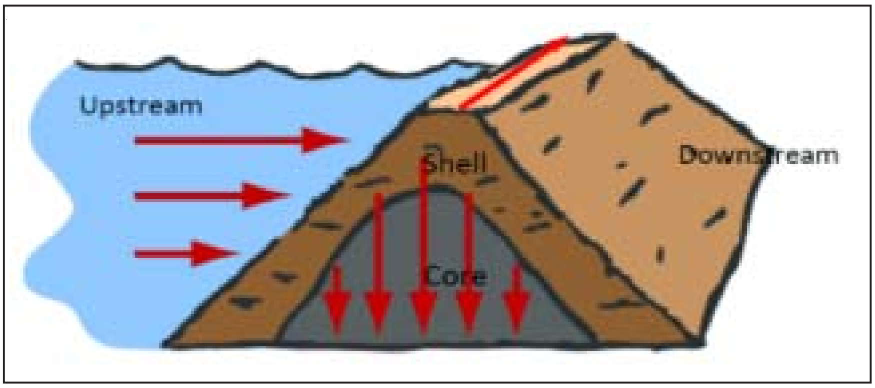

Fig. 7 : Cross Section of an earth dam

4.2 Cross Section of Earth Dam

In Fig 7, the top portion of the dam is shown as path. Technically speaking, the elevation of the upper most surface of a dam, usually a road or walkway, excluding any parapet wall, railings, etc, is called the top of the dam. The vertical height from the ground level to the top of the dam is called the dam height. The road or walkway at the top of the dam is used for passages, inspections of the dam and so on. When the dam height is large, one needs to have accessibility to various points on the dam. For this purpose, roads or walkways are also built at different altitudes on the two slanting surfaces - the upstream surface and the downstream surface - of the dam. One such road/walkway is visible on the upstream side in Fig 6. As per the conventional norms, a dam is supposed to have a road/walkway at every 15 meters altitude. However, this is not a hard and fast rule. For instance the Khandaleru dam in Andhra Pradesh, India, is 50 meters high, yet it has only one road at the top of the dam and no road/walkways anywhere else on the dam. If we look at the cross sectional view of the dam (that is, a person viewing from the ground at a point orthogonal to the water flow direction so that the core and shell are visible), then the picture of the cross section would look like the one in Fig 8. Any road/walkway on the slanting surfaces of the dam is called a berm. Note that the road/walkway at the top of the dam is not called a berm. For the dam in Fig

8, there are four berms - two on the upstream side and two on the downstream side. These correspond to the edges CD, EF, IJ and KL in the figure.

Imagine a dam of one kilometer long. We can slice this dam into 1000 vertical pieces along the length of the dam, each of one meter top length. The volume of the material in each slice is equal to a cubic meters, where *a* is the area of cross-section in m^2 of the dam. Thus, the material cost of the dam is a function of the cross section of the dam. Therefore, it is the cross sectional area that determines the volume of the material used for building the dam. In fact, the engineers use the cross section of the dam to design the dam. It is the cross sectional view that provides all the design parameters and these are described below.

4.3 Basic Terminology of an Earth Dam

When an engineer designs an earth dam she/he has to specify certain dimensions. These dimensions are described below. We shall call them the design parameters. In order to understand these parameters, we must first understand the basic structure and terminology of an earth dam. These are described with the help of Fig 8. Firstly, the cross section is viewed as a two-dimensional object in the xy-plane. Here, the x-axis is parallel to the direction of the water flow and is assumed to be aligned with the ground. Origin of the two-dimensional plane is taken as the starting point of the dam on the upstream side (point O in the picture). Some of the terms defined below are not the engineering terms but are coined to facilitate the description

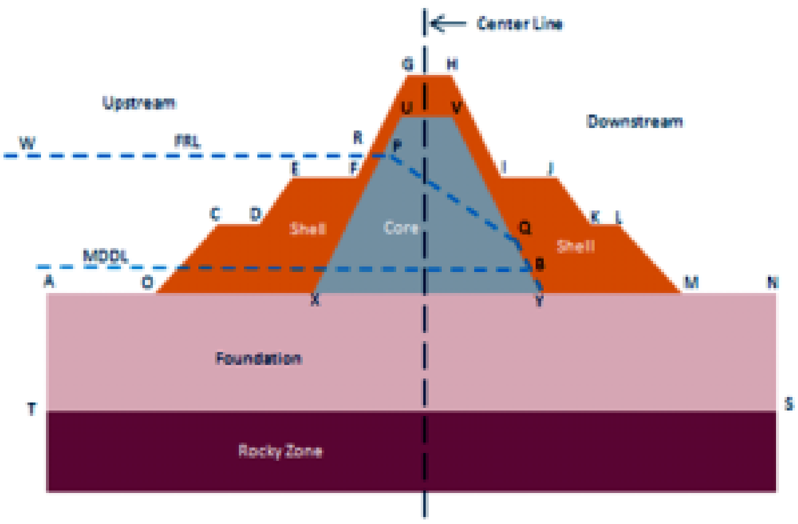

Fig. 8: Basic structure of earth dam, cross-sectional view.
The terms in this figure are explained in Section 4.3

Base Line of the Dam: It is the x-axis assumed to be aligned with the ground level. It is the horizontal line AN in the picture.

Base of the Dam: It the line segment OM in the picture. It is on this line segment that the dam is erected. The width of the line segment OM is called the base width of the dam.

Top of the Dam: The horizontal line segment GH is called the top of the dam. The width of the line segment GH is called the top width of the dam.

Center Line of the Dam: The perpendicular bisector of the line segment GH is called the central line of the dam. It divides the plane into two halves. The half that impounds water is called the upstream of the dam and the other half is called downstream of the dam.

Berms: The horizontal line segments CD, EF, IJ and KL are called the berms, and their widths are called respective berm widths.

Core: The trapezium XUVY is called the core of the dam. It comprises of material that has low permeability. The line segments XU, UV, VY and XY are called the core edges, and their union is called the core border. The width of the core top, UV, is called the core top width, and the width of the core base, XY, is called the core base width. The distance between the core top and the core base is called the core height.

Shell: The area of the polygon BCDEFIJKLM excluding the core is called the shell. The line segments OC, CD, DE, EF, FI, IJ, JK, KL, LM and OM are called the shell edges.

Foundation of the Dam: The rectangle ATSN is called the foundation of the dam, and its depth, length of AT, is called depth of the foundation of the dam.

Full Reservoir Level (FRL): Dams are designed to hold water safely up to a certain level, called the full reservoir level, which normally takes into account the flood situations. When the water level is getting closed to FRL or rises above FRL, the water is released through an outlet channel called the spill way. In the figure, FRL is the height of the horizontal line WRP from the ground level (equals to the y-coordinate of W or R or P).

Phreatic Line: By definition, it is the upper flow line, or the free surface of the seepage, along which the pressure in the soil water is equal to the atmospheric pressure. It is assumed to be the union of line segments WP, PQ and QY in the figure. In reality, the line segment PQ in the phreatic line is a curve rather than a straight line. For the purpose of computation of factor of safety, PQ is also assumed to be a line segment. In many of the designs, the slope of PQ is assumed to be -0.25. However, other criteria are also in practice. For instance, the IS Code 7894 - 1975 (1975) defines the point Q as the point of intersection of the downstream core edge and the horizontal line at $y = FRL/2$.

Minimum Draw Down Level (MDDL): It is the level at which water is always maintained. This is also known as sill level.

Rocky Zone: The portion below the foundation is called the rocky zone or impervious zone.

Moist or Dry Zone: Moist zone is the portion of shell and core that is above the phreatic line. Moist zone is also referred to as Dry zone in technical terms.

Buoyant Zone: This is the area of the dam bounded by MDDL, the line segment BY and the x-axis.

Saturated Zone: This is the area of the dam which is neither moist nor buoyant. It is the area of the dam bounded by theMDDL line and the phreatic line. Saturated zone is also referred to as wet zone.

Homogeneous Dams: We shall call a dam homogeneous if the structure of shell and core is symmetric around the central line of the dam. Otherwise, it will be called a heterogeneous dam. While Fig 8 depicts a homogeneous dam, Fig 9 is an example of a heterogeneous dam.

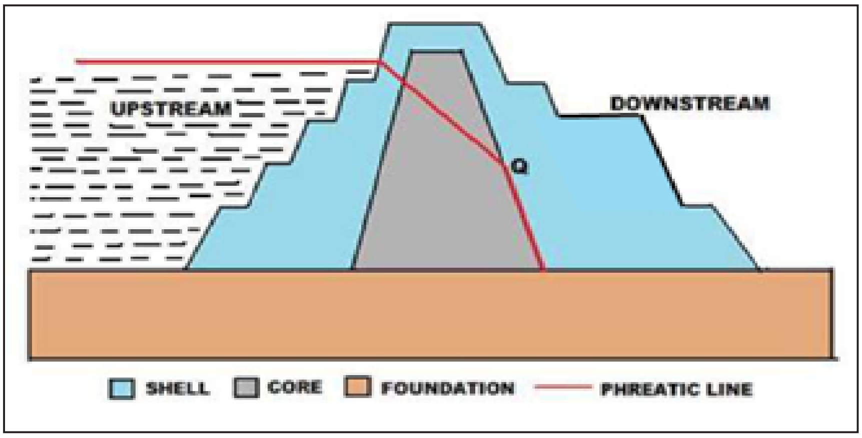

Fig. 9 : Picture of a heterogeneous dam

4.4 Design Parameters of an Earth Dam

As mentioned earlier, the safety and economics of an earth dam depend upon the cross section of the dam. Therefore, designing a dam essentially means the determination of cross sectional parameters of the dam. Of course, the determination of materials and methods to be used in the construction of the dam is equally important for designing the dam. But for the purpose of the optimization problem considered in this chapter, it is assumed that these factors are identified and fixed so that their determination is kept out of the scope of this project. Their role in our problem is only to use their strength characteristics in terms of friction and cohesiveness. We shall now identify the design parameters of the cross-section of an earth dam. Since we are dealing with only the cross-section of the dam, it will be convenient to use the two-dimensional coordinate system to represent various points and objects in the design. Before proceeding further, we need the following definitions.

4.5 Number of Berms

In a heterogeneous dam, the number of berms on the upstream side need not be same as the number of berms on the downstream side. Let n_U denote the number berms on the upstream side and let n_D denote the number of berms on the downstream side. For the dam in Fig 8, n_U and n_D are both equal to 2. And for the dam in Fig 9, $n_U = 4$ and $n_D = 3$.

4.6 Berms and Slopes

Berms and their widths have already been defined earlier. We shall number the berms as Berm 1, Berm 2,... starting from the upstream side. Thus, for the dam in Fig 8, Berm 1 is CD, Berm 2 is EF, Berm 3 is IJ, and Berm 4 is KL. Notice that each berm has a slanting edge attached to it and below the berm. For example, Berm 1 in Fig 8, CD, has the slanting edge CO below it. Similarly, the berms EF, IJ and KL are adjacent to the slanting edges ED, JK and LM respectively. In addition, the top two slanting edges are always adjacent to the top of the dam. These are GF and HI in Fig 8. Notice that each slanting edge corresponds to a right angled triangle in which the slanting edge is the hypotenuse. One edge of this triangle is in the vertical direction and the other in the horizontal direction. The horizontal edge will be called the base of slanting edge. The widths of horizontal and vertical edges of the triangle will be called the width and height of the slanting edge respectively. For the purpose of this paper, designing an earth dam means specifying the berm widths, widths and heights of the slanting edges and the core bottom width (XY in Fig 8). In other words, the decision variables of the earth dam design optimization problem are the widths and heights mentioned in the previous sentence. These decision variables shall be denoted by the vector $u = (u_1, u_2, ..., u_k)$. Normally, the dam height, dam top width, the core height and core top width are fixed and the designer has no choice over these parameters. In view of this, the coordinates of u are taken in the following order. Take u_1 as the left most width, u_2 is the width next to it, and so on. If u_k is the width of the right most slanting edge of the dam, take u_{k+1} as the width of the core, take u_{k+2} as the height of the left most slanting edge of the dam, u_{k+3} as the height of the next slanting edge and so on. While considering the slanting edge heights, the heights of the slanting edges adjacent to dam top are ignored. The reason for this is that the dam height is fixed.

The decision variables u for a 3-berm dam (total number of berms is 3) shown in Fig 10 has 12 variables, that is, $u = (u_1, u_2,...,u_{12})$. Notice that u_8 is the width of the right most slanting edge of the dam. Next, u_9 is the width of the core bottom. Having exhausted all the widths, u_{10} represents the height of left most slanting edge of the dam, and u_{11} is the height of the immediate right slanting edge. As the dam height is fixed, say h, the height of the upstream slanting edge adjacent to dam top is equal to $h - u_{10} - u_{11}$. Therefore, this height is not a decision variable. Similarly, the height of the downstream slanting edge adjacent to the dam top is also fixed as the slant height of the right most slanting edge (u_{12}) is taken as a decision variable. Thus, we have identified and defined all the decision variables of the earth dam design optimization problem. The variables are: n_U, n_D and u.

5. Factor of Safety with Respect to a Slip Circle - Theory, Definitions and Computation

The material or the mass of an earth dam has a tendency to slide or slip downward due to its weight (the gravitational force on the mass). This may lead to a partial or complete failure of the dam. The force that actuates the downward movement of the mass is a fraction of the weight of the dam determined in a particular direction. The weight of the mass depends upon the condition of the dam. For instance, when the water is up to FRL, then the weight of the dam is heavy compared to the situation where the water is only up to MDDL. Similarly, if there is an earthquake,

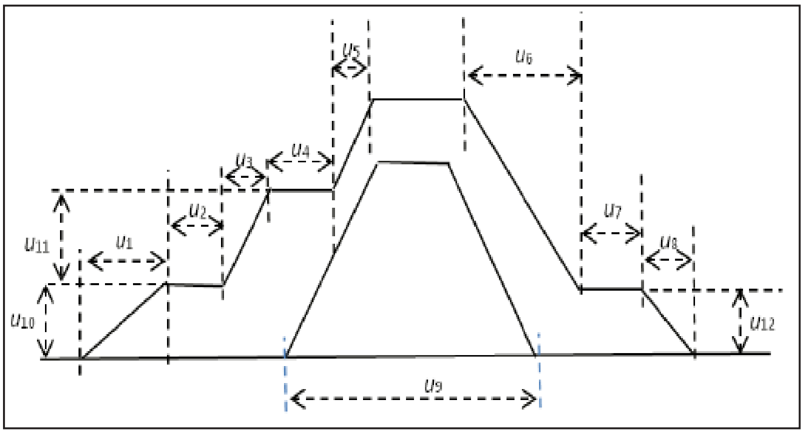

Fig. 10 : The design parameters of earth dam

there is an additional force shaking the dam structure adding to the downward movement. In view of this, the safety of the dam is evaluated under different conditions. For the purpose of this chapter, we shall confine ourselves to only one of those conditions. The forces causing downward movement are typically termed as driving or actuating forces. As has been pointed out just now, the nature and quantum of these forces depend on various conditions and the dam properties such as the material characteristics, the quality of construction and so on. A dam has a natural tendency to offer resistance to the driving forces. The resistance arises from the characteristics of the materials used, the method of construction, the external environment (which changes the characteristics of the soil etc.), and the design parameters which were identified in the previous section. The resistance arises from the frictional characteristics and the cohesiveness of the materials used and the design characteristics. The safety of a dam is a function of the driving forces and resistances. The factor of safety is computed using the driving forces and resistance and their resulting moments. If the moment of resistance is greater than the moment of driving forces, then the dam is expected to be stable and safe. A factor of safety is calculated by dividing the resisting moment by the driving moment.

Several methods have been proposed to determine the factor of safety. These include Culman Wedge Block Method, Swedish Circle or Fellenius Method, Janbu Method, Bishop Method, Morgenstern-Price Method, Spencer Method, Bishop-Morgenstern Method, Taylor Method, Barnes Method, Lowe-Karafiath Method, Sarma Method, etc. Each method has its own assumptions, Hammouri, 2008;http...://www.vulcanhammer.net/ utc/ence361/f2001/361-sl14.pdf). Of these, Fellenius or Swedish Circle or ordinary method (Fellenius, 1936, Sarma, 1979] is widely accepted and used in practice. This method is developed by Wolmar Fellenius as a result of slope failures in sensitive clays in Sweden. It is the simplest method of slices. It reduces the force resolution of the slope to a statically determinate structure (http://en.wikipedia.org/wiki/Embankment Dam). Despite the advances in evaluating the stability and the availability of analysis software (Sinha 2008, http://tailingsandminewaste.org/courses/2D 3D% 20Description.pdf), it is observed that the slip circle method is the method that is by and large used to evaluate the safety of an earth dam. The method is described in this section.

5.1 Slip Circle

In the slip circle method it is assumed that a part of the dam slides away along the arc of a circle. The circle is called a slip circle. Its centre lies in the two dimensional plane above the cross section of the dam. It cuts the shell outer line at two points to form a chord. The chord divides the circle into two arcs - the major (bigger) arc and the minor (smaller) arc. By definition, the slip circle must be such that the major arc is above the chord and the minor arc must have a nonempty intersection with the cross section of the dam. A picture of a slip circle is shown in Fig 11.

According to the theoretical assumptions, if the moment of the weight of the mass of the dam above the minor arc exceeds the resistance offered by the dam, then the mass above the minor arc slides away along the minor arc causing a failure of the dam. If a slide occurs, the mass slips into the upstream side of the dam provided the slope of the chord of the slip circle is positive; otherwise, the mass slips into the downstream side. The dam is expected to be safe with every possible slip circle. From engineering and physics knowledge, it is known that failure cannot occur with respect to certain slip circles. Such a slip circle is called an invalid slip circle and it is assumed that factor of safety with respect to an invalid slip circle is infinity. There are certain rules to verify whether a given circle is valid or not. For a circle to be a valid slip circle, the conditions to be satisfied by it, besides the ones mentioned above, are: it is not a very small circle (i.e., its radius is larger than a specified parameter ‘), the circle does not extend beneath the foundation bottom, and it does not intersect any vertical line in the cross-section [i.e., a line parallel to the straight line joining the midpoints of the core base and the core top (line segments XY, UV in Figure 8)] at two distinct points both lying inside the cross-section of the dam.

If the center of the slip circle lies in the upstream side, then it is called upstream slip circle; and if the center lies in the downstream side, then the circle is called downstream slip circle. The slope of any upstream slip circle chord is positive and that of a downstream slip circle chord is negative. In what follows, reference to a slip circle means that it is a valid slip circle unless stated otherwise. We are concerned with only the minor arcs of the slip circles and whenever we refer to arc, unless stated otherwise, we mean that it is the minor arc that is being referred.

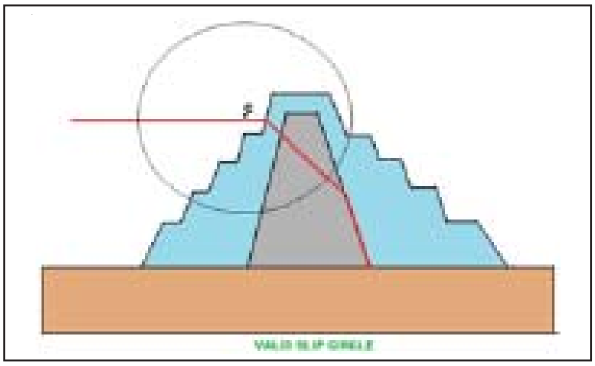

Fig. 11: A slip circle

5.2 Weight of Mass of the Dam at a Point on the Arc

Consider a slip circle arc and a slice of the dam of small width δx and depth 1 m above the arc at the point (x, y) on the arc. Here width is measured along the cross section and depth along the length of the dam. The slice can be decomposed into seven different components, namely, shell slice in the dry (also referred to as moist) zone ($h_{ds}(x)$), shell slice in the wet (also known as saturated) zone ($h_{ws}(x)$), shell slice in the buoyant zone ($h_{bs}(x)$), core slice in the dry zone ($h_{dc}(x)$), core slice in the wet zone ($h_{wc}(x)$), core slice in the buoyant zone ($h_{bc}(x)$), and the slice of foundation

(h_f (x)). The notation in the parentheses against each of these components is the approximate height (actually height at x which is taken as the approximate height of the slice) of the respective slice component. For the instance in Fig 12, only three of these components have positive heights, namely, $h_{ds}(x)$, $h_{dc}(x)$ and $h_{wc}(x)$. The heights of other components are zero. These heights depend on the arc and the position (x, y) on the arc. As the shell, core and foundation materials are different, they have different densities and their densities also depend on the zone (recall that densities are weight densities as mentioned earlier). If d_{ds}, d_{ws}, d_{bs}, d_{dc}, d_{wc}, d_{bc} and d_f are the densities of seven components mentioned above (in the same order), then the expression $d_f h_f(x)+d_{wc}h_{wc}(x)+d_{dc}h_{dc}(x) + d_{ws}h_{ws}(x) + d_{ds}h_{ds}(x)$ multiplied by the width δx of the slice gives the approximate weight of the slice above the arc. The densities depend on the material and the zone in which the material is present.

5.3 Driving Forces and Resistance at a Point on the Arc

Consider the slip circle with center at C in Fig 13. The chord XY is formed by the points of intersection of the circle with the shell outer line. Since the sliding takes place only on the minor arc, our interest and discussion is confined only to the minor arc which is below the chord. Hence, any reference to the arc means it is with respect to the minor arc only as stated earlier. Let (x, y) be an arbitrary point on the arc. Functionally, let w(x, y) denote the weight of the material vertically above at the point (x, y). For brevity, we write simply w for w(x, y). From the previous discussion, we have:

$$w(x, y) = d_f h_f(x) + d_{bs}h_{bs}(x) + d_{ws}h_{ws}(x) + d_{ds}h_{ds}(x)+ d_{bc}h_{bc}(x) + d_{wc}h_{wc}(x) + d_{dc}h_{dc}(x), \qquad (1)$$

where d_f , d_{bc}, d_{wc}, d_{dc}, d_{bs}, d_{ws} and d_{ds} are the material densities of the foundation, buoyant core, wet core, dry core, buoyant shell, wet shell and dry shell respectively; and h_f (x), $h_{bc}(x)$, $h_{wc}(x)$, $h_{dc}(x)$, $h_{bs}(x)$, $h_{ws}(x)$ and $h_{ds}(x)$ are the vertical height components of foundation, buoyant core, wet core, dry core, buoyant shell, wet shell and dry shell above the point (x, y) respectively.

The weight w is the force always acting vertically downward. This force is resolved into two components: (i) the driving force resulting from the weight w and (ii) the force resulting from friction. We shall denote the driving force due to weight at (x, y) by DF(x, y), and the resistance resulting from friction by FF(x, y). DF(x, y) is the component of w(x, y) in the tangential direction and FF(x, y) the component of w(x, y) in the normal direction with respect to the slip circle. Note that these components are determined at each point (x, y) on the arc. If $\alpha(x, y)$ is the angle (acute) between the weight (force vector acting vertically downward) and the normal to the slip circle at the point (x, y), then $DF(x, y) = w(x, y) \sin \alpha(x, y)$ and $FF(x, y) = w(x, y) \cos \alpha(x, y) \tan \varphi(x, y)$, where $\tan \varphi(x, y)$ is the coefficient of friction. Here, $\varphi(x, y)$ is called the angle of friction and it varies with material and the zone. For the purpose of this work, the coefficients of friction are taken as inputs.

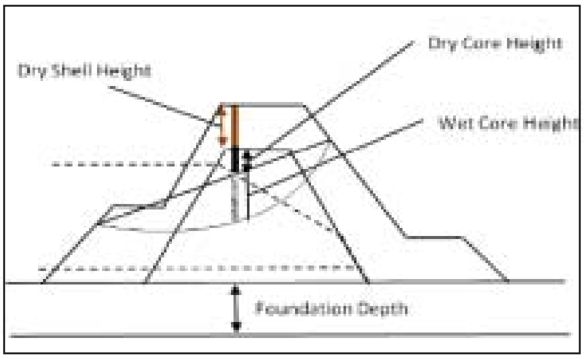

Fig. 12: Weight composition of material at a point on the arc

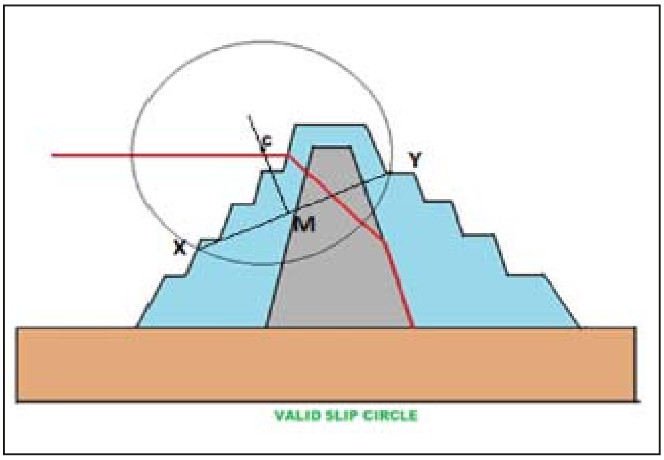

Fig. 13: Weight composition of material at a point on the arc

5.4 Cohesion

In addition to friction, there is the cohesive component which adds to the resistance. The cohesion acts along the arc. The amount of resistance offered by cohesion at a point (x, y) on the arc is measured by the cohesion coefficient. The cohesion coefficient, defined as the force per unit area, depends on the material at the point (x, y) and the zone to which the point belongs. Consider the arc of a small length, say μ m, so that the entire arc is in the same zone comprising of same material. The strip of the arc along the unit length of the dam has an area of μ m^2. Therefore, the resistance offered by cohesion of this strip is equal to $c \times \mu$, where c is the coefficient of cohesion. The unit of c is KgF/m^2. The cohesion coefficients are denoted by c_f for material in the foundation, c_{ds} for dry shell material, c_{ws} for wet shell material, c_{bs} for buoyant shell material, c_{dc} for dry core material, c_{wc} for wet core material, and c_{bc} for buoyant core material. We shall denote the cohesion at the point (x, y) by CF(x, y). Thus, if (x, y) is in the foundation, then CF(x, y) = $c_f\mu$; if (x, y) is in dry shell zone, then CF(x, y) = $c_{ds}\mu$; and so on.

5.5 Effect of Earthquakes

When earthquake is taken into account for computing the factor of safety, its effects are added to the actuating force and resistance. Technically, it is assumed that earthquake acts in the horizontal direction and its magnitude is taken as a fraction of the weight of the object. This fraction is called the earthquake coefficient which is unit-less number. The components of earthquake at (x, y) are: $w(x, y) \cos\alpha(x, y)C_{EQ}$ to the driving force and $w(x, y)C_{EQ} \sin \alpha(x, y) \tan \varphi(x, y)$ to the resistance, where C_{EQ} is the earthquake coefficient. The total driving force (TDF) and the total resistance (TRF) at (x, y) are given by $TDF(x, y) = w(x, y)\{\sin \alpha(x, y)+C_{EQ} \cos \alpha(x, y)\}$ and $TRF(x, y) = CF(x, y) + w(x, y)\{\cos \alpha(x, y) + C_{EQ} \sin \alpha(x, y)\}\tan \varphi(x, y)$.

5.6 Definition of Factor of Safety

The driving moment at point (x, y) on the arc is defined as r.TDF(x, y) and the resisting moment at (x, y) is defined as r.TRF(x, y), where r is the radius of the slip circle. The total driving moment on the arc, TDM(Arc), is defined as the integral of r.TDF(x, y) over the arc, that is the line integral of r.TDF(x, y) over the arc of the slip circle. Similarly, the total resisting moment on the arc, TRM(Arc), is defined as the integral of r.TRF(x, y) over the arc, that is the line integral of r.TRF(x, y) over the arc of the slip circle. Thus, the cumulative driving and resisting moments on the arc are given by

$$TDM(Arc) = \int_{Arc} r.TDF\ ds \text{ and } TRM(Arc) = \int_{Arc} r.TRF\ ds \tag{2}$$

where the integrals represent the line integrals over the arc (http://en.wikipedia.org/wiki/Line integral#Definition). The factor of safety of the dam with respect to the slip circle with center C and radius r is defined by

$$FS(C, r) = (TRM(Arc)) / (TDM(Arc)) \tag{3}$$

It may be noted that the factor of safety is a unitless number. Higher the factor of safety, the better it is.

5.7 Computation of Factor of Safety

From the previous section, it is clear that computation of factor of safety involves evaluation of line integrals of functions that have no simple closed form expressions. In practice, these integrals are determined using numerical approximation. The usual approach is to divide the structure of the dam above the arc into vertical slices of small widths, determine the weight components of each slice, compute the driving and resisting moments for each slice, and then sum them up to get the total driving and total resisting moments. The accuracy of this procedure depends upon the number of slices considered for the computation. The Indian standard code IS 7894-1975 recommends about 15 to 20 slices to be considered for evaluation of the factor of safety. Computation of weight of the material, w(x, y), requires determination of vertical height components $h_f(x)$, $h_{ds}(x)$, $h_{ws}(x)$, $h_{bs}(x)$, $h_{dc}(x)$, $h_{wc}(x)$, and $h_{bc}(x)$. This in turn requires finding points of intersection of the phreatic line with various line segments of the cross section and the arc of the slip circle. Once these heights are determined, they need to be multiplied by their densities. As per the recommended procedure, the densities used for computing driving forces may be different from those used for computing resistance. The reason for this is when the densities cannot be determined precisely, lower values are taken for the resistances to give benefit of doubt for the factor of safety computation. For this reason, the densities are specified separately for the driving forces and resistance computations. As an illustration, we shall now explore the computational aspects of factor of safety with respect to a specific slip circle for the example below.

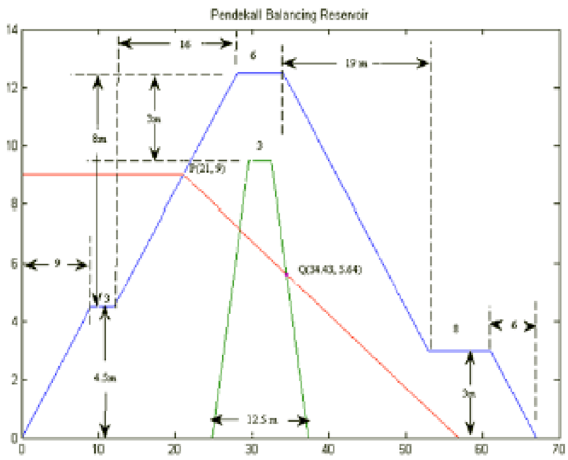

Fig. 14 : Cross-sectional design of the Pendekal dam

5.8 Example

The Pendekal dam, a heterogeneous dam with two berms - one in the upstream and the other in the downstream, is located near Ananthapur in Andhra Pradesh, India. Its cross-sectional design is shown in Fig 14. For this dam, the dam height is 12.5 m, dam top width is 6 m, core height is 9.5 m, core top width is 3 m, the FRL is 9 m, the foundation depth = 10 m, and MDDL = 3 m. The design vector has 9 variables. The design vector for the constructed dam is given by

$$u = (9, 3, 16, 19, 8, 6, 12.5, 4.5, 3). \tag{4}$$

These dimensions are shown in Fig 14. Table 1 gives list of required inputs for computing the factor of safety. Note that the densities are different for driving force and resistance.

Let us compute the factor of safety for this example with respect to the slip circle with center at (11.66, 13.36) and radius 21.35. Also, we shall examine the effect of number of slices on the accuracy of factor of safety. A computer program is developed to carry out these computations. Table 2 presents the factor of safety with different number of slices used.

Table 1 : Densities and Coefficients of Friction and Cohesion

Zone	Densities (KgF/m^3)		Cohesion (KgF/m^2)	Coefficient of Friction
	Driving	Resistance		
Dry Shell (ds)	2006	2006	2900	0.325
Dry Core (dc)	2000	2000	2000	0.141
Wet Shell (ws)	2054	1054	2900	0.325
Wet Core (wc)	2010	1010	2000	0.141
Foundation (f)	1073	1073	3200	0.231
Buoyant Shell (bs)	1054	1054	2900	0.325
Buoyant Core (bc)	1010	1010	2000	0.141

Coefficient of Earthquake (C_{EQ}): 0.06
Read parameters in the following fashion: $d_{ws} = 2.25$ for resistance, $d_{dc} = 1.76$ for Driving, $c_f = 1.90$, etc., last column gives $\tan\phi$ values.

Table 2 : Effect of number of slices in computation of factor of safety

SNo	No. of Slices	Driving Force (KgF)	Cohesive Force (KgF)	Resistance (KgF)	Factor of Safety
1	15	225.117	153.292	203.015	1.583
2	20	215.197	150.721	197.719	1.619
3	50	218.063	152.037	198.239	1.606
4	100	219.563	152.123	198.513	1.597
5	200	220.138	153.333	198.641	1.599
6	500	220.860	153.552	198.931	1.596
10	5000	220.318	153.644	198.745	1.599

Table 3 gives factor of safety for different slip circles. This information is useful in throwing some light on the extent of error if 50 slices are used. From these observations, it sounds reasonable to use 50 slices for computing the factor of safety. The above procedure is useful for computing the factor of safety with respect to a given slip circle. But the problem is to find the minimum factor of safety for upstream and downstream separately over all possible valid slip circles. As the number of valid slip circles is infinite, the usual practice is to select, based on experience, a few slip circles which are likely to be critical and evaluate the factor of safety with respect to only those slip circles. If the factor of safety is satisfactory with respect to these selected slip circles, then the design of the dam is accepted to be safe. The IS code suggests that about 12 circles be analyzed for examining the stability of a dam.

Table 3: Extent of error if 50 slices are used

Slip Circle	Number of Slices	Circle Center x_0	Circle Center y_0	Radius (m)	Factor of Safety	Error
1	50	48.179	8.646	28.180	1.961	0.002
	5000	48.179	8.646	28.180	1.959	
2	50	37.975	8.770	17.977	2.443	0.046
	5000	37.975	8.770	17.977	2.490	
3	50	40.170	4.519	30.170	2.325	0.004
	5000	40.170	4.519	30.170	2.321	
4	50	26.421	3.006	28.579	2.314	0.022
	5000	26.421	3.006	28.579	2.292	

With the power of computing available today, a large number of slip circles can be analyzed in no time. In fact, we developed an approach that analyzes infinitely many circles. This is plausible based on a phenomenon that we have observed while working on this project. The approach and the phenomenon are elaborated below.

6 Factor of Safety of an Earth Dam

In the preceding section we have looked at the factor of safety with respect to a specific slip circle. However, the dam failure may occur due to any slip circle with low factor of safety. For this reason, there is a need to find the minimum factor of safety over all possible slip circles. In practice, the factor of safety is split into two parts - the upstream factor of safety and the downstream factor of safety. These are defined below.

6.1 Upstream and Downstream Factors of Safety

Two factors of safety are defined - one for the upstream and the other for the downstream. The upstream factor of safety of a dam, FS_U, is defined as the minimum of FS(C, r) over all upstream slip circles (C, r). Similarly, the downstream factor of safety, FS_D, is defined as the minimum of FS(C, r) over all downstream slip circles (C, r). In practice, more importance is given to the downstream factor of safety compared to the upstream factor of safety. The conventional lower specifications are: 1.3 for the upstream factor of safety, and 1.5 for the downstream factor of safety.

6.2 The Chord Based Approach for Evaluating Factors of Safety

The procedure described earlier is useful in computing the factor of safety with respect to a given slip circle. But to compute the factors of safety, FS_U and FS_D, one must explore all valid slip circles. Notice that the arc of any slip circle has a chord associated with it. But this chord is common to infinitely many slip circles. Fig 15 exhibits two slip circles with a common chord XY. Every slip circle associated with a chord will have its center lying on the perpendicular bisector of the chord. Therefore, exploring all slip circles is equivalent to exploring all valid chords. By exploring a valid chord, we mean the following: take a chord of a valid slip circle and explore all slip circles that are associated with the chord. There is a substantial reduction in the effort required if this approach is used. The reduction is due to a special feature observed while working on this project. The special feature is that the factor of safety of slip circles associate with a chord is a *near* convex function. The details are explained below.

Notation for a Chord: Throughout this chapter, we shall use the following notation for a chord. Let X = (x_1, y_1) and Y = (x_2, y_2) be the end points of the chord. Note that for a given design, y_i is a function of x_i, i = 1, 2. Therefore, for a given design, a chord is uniquely determined by the x-coordinates of its end points. For this reason, we use the representation *Chord*(x_1, x_2) for the chord XY with $x_1 < x_2$.

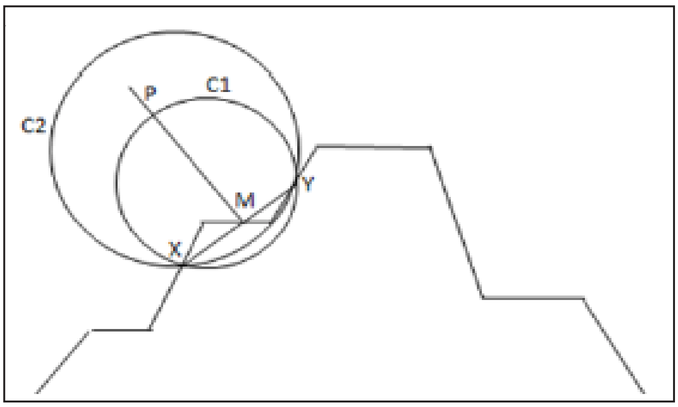

Fig. 15: Common slips circles of a chord

6.3 Convex Nature of FS(C, r) on a Chord

All slip circles associated with a chord will have their centers lying on the perpendicular bisector of the chord and they are uniquely determined by the distance between their centers and the chord. That is, if C_1 and C_2 are two slip circles associated with a chord (see Fig 15) with the distances from their centers to the chord as d_1 and d_2 respectively, then C_1 and C_2 are same if, and only if, $d_1 = d_2$. Now fix a valid chord of length 2L and consider the slip circle of radius r associated with the chord whose center is at a distance d from the chord. Fig 16 shows the chord, arc, radius and the distance for this slip circle. Notice the lengths of the edges of the right-angled triangle ABC in the figure. Since the length of the chord is fixed, r is a function of d given by $r(d) = \sqrt{L^2 + d^2}$. Let θ(d) denote the angle ACB and define the parameter λ(d) = d/r(d). Note that λ(d) = cos θ(d). As d tends to infinity, the angle θ(d) goes to zero and cos θ(d) goes to 1. This means as d approaches infinity λ(d) approaches 1. Also, note that for any given λ ε (0, 1), there is a unique d such that θ(d) = λ. Now consider the factor of safety of slip circles associated with the chord as a function of the parameter λ and denote it by f(λ). The behavior of the function f(λ) is observed empirically by plotting the function for some selected chords of the Pendekal dam example (see Fig 17). The interesting and useful observation is that the function is approximately convex. We can exploit this observation in computing the minimum factor of safety with respect to the family of circles associated with the chord. This is done as follows:

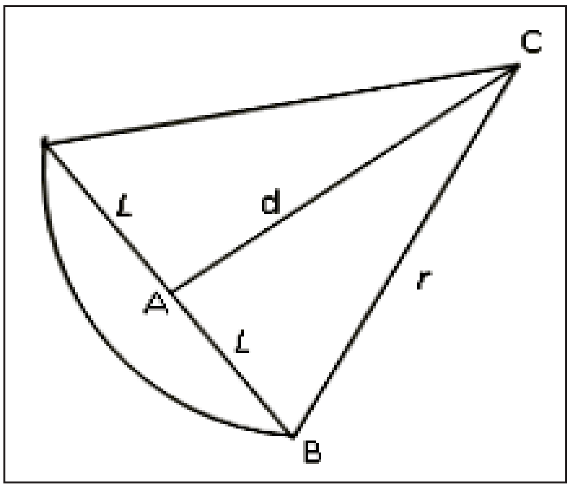

Fig. 16: Relationship between radius and distance to chord

Determine the smallest λ, say λ_0, and compute the minimum of $f(\lambda)$ in the interval $[\lambda_0, 1)$. The question is: What is λ_0? This is determined as follows. One of the rules for a valid slip circle is that it cannot have a vertical line intersecting the arc at two distinct points inside the dam. This means that the y-coordinate of the center of the slip circle must be greater than or equal to the y-coordinates of the two end points of the chord. Let the chord be PQ, and let Q be the end point of the chord with the largest y-coordinate (whose value we will denote by y_0). Therefore, the center of the slip circle with smallest d is the intersection point of the perpendicular bisector of PQ and the parallel line $y = y_0$. Let d_0 be this smallest distance. Let r_0 be the radius of the corresponding slip circle. Then $\lambda_0 = d_0/r_0$ is the required smallest λ_0.

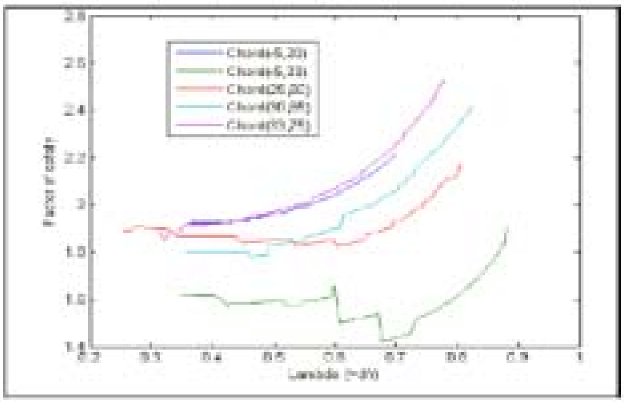

Fig. 17: Factor of safety as a function of lambda

Next consider $\lambda_2 = d_2/r_2$ and $\lambda_3 = d_3/r_3$, where $r_2 < r_3$. Since $\lambda = d/r = \cos\theta$ for $0 < \theta < \pi/2$, there is a unique d for every λ. Choose λ_1 sufficiently close to 1 and let d_1 be the distance corresponding to λ_1. To find out the minimum of $f(\lambda)$ in the interval $[\lambda_0, 1)$ we use the following approach. Let $d_2 = (d_1+2d_0)/3$ and let $d_3 = (2d_1 + d_0)/3$. If $f(\lambda_2) < f(\lambda_3)$, then set $d_1 = d_3$; otherwise set $d_0 = d_2$. Repeat this process until d_0 and d_1 are sufficiently close. For a strict convex function, this procedure leads to the minimum. Since the function $f(\lambda)$can have local minima, we compute the function at a number of points in the interval $[\lambda_0, \lambda_1)$ where λ_1 is a number close to 1. Our algorithm for computing minimum of $f(\lambda)$ in the interval $[\lambda_0, 1)$ is given below.

Algorithm for Computing Minimum of f (λ):

Recall that the x-axis is assumed to be aligned with the ground level. It is the horizontal line AN in Fig 8.

Step 0: Fix λ_1 close to 1 and less than 1. Fix a small positive number ε to be used for incrementing d.

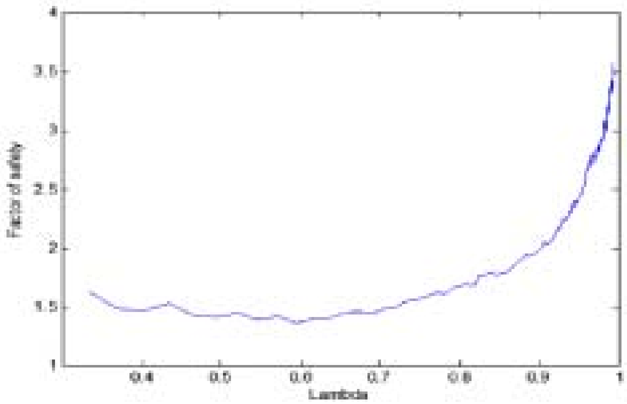

Fig. 18: Factor of safety as a function of lambda (λ)

Step 1: Determine d_0 corresponding to λ_0 and d_1 corresponding to λ_1.

Step 2: Let $d_2 = (d_1 + 2d_0)/3$ and $d_3 = (2d_1 + d_0)/3$. If $f(\lambda_2) < f(\lambda_3)$, then set $d_1 = d_3$; otherwise set $d_0 = d_2$.

Step 3: If d_0 and d_1 are sufficiently close, take $f(\lambda_0)$ as the minimum of $f(\lambda)$, and stop. Otherwise go to Step 2.

This algorithm is applied to the Pendekal dam design and it is found to be very efficient. The graph of factor of safety for the Chord(28, 59) is shown in Fig 18. It took 35 iterations (examination of 35 slip circles) to arrive at the minimum. The time taken is about 4 seconds. The factor of safety for this chord is 1.334 and the critical slip circle of the chord has its center at (12.69, 21.56) with a radius of 24.08.

6.4 Evaluation of FS_U and FS_D

So far, we have a procedure for computing the minimum factor of safety on a family of slip circles associated with any given chord. From here we should have a procedure to compute the upstream and downstream factors of safety. This calls for exploring all valid chords. This can be formulated as a mathematical programming problem involving two decision variables (this will be shown in the next section). But for simplicity, we shall explore the minimum factor of safety on a large number of chords and then take the minimum of all those minimums. It should be noted that the procedure considers a subset of all possible slip circles and hence the factors of safety evaluated are near approximations but not exact theoretically. Our procedure for doing this is given below.

6.5 Procedure for Computing FS_U and FS_D

We shall describe the procedure for computing FS_U. A similar procedure can be used for computing FS_D. Consider the chord(a_1, a_2) with end points X = (a_1, b_1) and Y = (a_2, b_2). Recall that $a_1 < a_2$. If $b_1 < b_2$, then the chord corresponds to an upstream slip circle; and if $b_1 > b_2$, then the chord corresponds to a downstream slip circle. For ease of notation, let us define the nonnegative function *shell*(x) as the height of the shell at x. Note that *shell*(x) = 0 for all x ≤ 0 and for all x greater than the base width of the dam. Note that b_1 = *shell*(a_1) and b_2 = *shell*(a_2). Fix a lower bound B_L for $a_2 - a_1$. It may be noted that $a_2 - a_1$ is the length of the projection of the chord PQ where P = (a_1, *shell*(a_1)) and Q = (a_2, *shell*(a_2)). Since small slip circles can be ignored, chords of small width can also be ignored. For this reason, we may set a minimum width, say L_0, and consider only chords whose width is at least L_0. Instead of setting a lower limit on chord width (L_0), we may set a lower limit on the width of the projection of the chord. Thus, we can choose B_L to be a practically meaningful figure. Once B_L is set as the lower limit, all chords of length less than B_L will be ignored. The starting point of a chord may have its x-coordinate (a_1) well before the origin. A practically meaningful value for this is taken as L where L is -0.25 times the base width of the dam. If a chord is an invalid chord, its minimum factor of safety is taken as infinity.

FS_{ij}

According to this procedure, we shall take $FS_U = \min_{i,j} FS_{ij}$, where is the minimum factor of safety over the $Chord(x_1^i, x_2^j), x_1^i = L + i\delta x_1, x_2^j = x_1^i + B_L + j\delta x_2,$ and i, j are nonnegative integers. Here δx_1 and δx_2 are used as increments for the chord first and second parameters respectively. Let s_{Left} and s_{Right} denote the x-coordinates of the dam top left and right points respectively (points G and H in Fig 8). It may be noted that x_1^i being the initial x-coordinate of an upstream chord, we must have $x_1^i \leq s_{Left}$. If $x_2^j > s_{Right}$, then $Chord(x_1^i, x_2^j)$ is flatter than $Chord(x_1^i, s_{Right})$ and has higher factor of safety. For this reason it suffices to consider only j such that $x_2^j < s_{Right}$. The steps for determining FS_U are given below:

Step 0: Choose and fix L, δx_1 and δx_2. Set i = 0, j = 0 and FS_{Min} = 1000.

Step 1: Set $x_1 = L + i\delta x_1$ and $x_2 = \min(x_1 + B_L + j\delta x_2, s_{Right})$.

Step 2: If $x_2 \leq s_{Right}$, , then find out the minimum factor of safety on the Chord(x_1, x_2). If this minimum is less than FS_{Min}, then reset FS_{Min} equal to this new minimum. Set j = j + 1 and go to

Step 1.

Step 3: If $x_1 \leq s_{Left}$, then set i = i + 1, j = 0 and go to Step 1.

Step 4: Take $FS_U = FS_{Min}$ as the upstream factor of safety of the dam. Stop.

Applying the above algorithm to the Pendekal dam design, it is found that the factors of safety of the design, evaluated over 105 chords spread across the dam are: 1.270 for the upstream and 1.506 on the downstream. It should be noted that the implemented design does not meet the minimum requirement of 1.3 on the upstream. The failure occurs on the Chord(-1.288, 34). The failure slip circle on this chord has its center at (11.6, 19.67) with radius of 23.52 m. The chord length is 34.18 m and the distance between the chord and the center is 17.42 m. Also, one of the engineers pointed out, based on an independent analysis, that the design does not meet the factor of safety requirements.

7. Mathematical Model for the Design Optimization

It is understood that at present the problem of designing an earth dam is merely viewed from the angle of safety and once a design is reached with required factors of safety the book is closed. In this work we have formulated the problem as a constrained optimization problem. The main objective of the problem is to minimize the area of cross section of the dam which is related to minimizing the cost of material for building the dam. The main constraint is that of ensuring the factors of safety. The existing standards are: the upstream factor of safety should be at least 1.3 and the downstream factor of safety should be at least 1.5. As mentioned in the previous section, the problem of finding the factor of safety of a given design can itself be formulated as a mathematical programming problem. We however determine the factor of safety approximately by partial enumeration. We first present this model and then proceed to consider the main problem of optimizing the design.

7.1 Model for Obtaining Factor of Safety of a Given Design

We shall describe the model for obtaining the upstream factor of safety, FS_U. Similar model can be used for FS_D as well. For this model, we shall assume that the factor of safety over any Chord (a_1, a_2) can be determined using the algorithm described earlier. In light of this, the problem of determining FS_U boils down to determining the factor of safety over all upstream chords. Let $g(a_1, a_2)$ denote the minimum factor of safety over the Chord(a_1, a_2). Then the problem of determining the upstream factor of safety is given by

$$\begin{aligned} &\textit{Minimize} \quad g(a_1, a_2) \quad \textit{subject to} \\ &a_1 < a_2 \\ &shell(a_1) < shell(a_2) \\ &a_2 > 0 \end{aligned}$$

The first constraint in the above problem means that we are looking for a chord of positive length. The second constraint means that the Chord (a_1, a_2) is an upstream chord, and the third constraint, $a_2 > 0$, together with the other constraints ensures that chord intersects the dam structure. For practical purposes, we can convert the strict inequalities in the above problem by introducing a small ε positive as follows:

$$\begin{aligned} &\textit{Minimize} \quad g\,(a_1, a_2) \quad \textit{subject to} \\ &a_1 \le a_2 + \varepsilon \\ &shell(a_1) \le shell(a_2) + \varepsilon \\ &a_2 \le \varepsilon \end{aligned}$$

It may be observed that the constraints are piecewise linear (as s is a piecewise linear function) and the objective function is nonlinear.

7.2 Model for Optimal Design

We shall now look at the main problem of this work. The problem is to determine the design vector u. The number of berms in the upstream side, n_U, and the number of berms on the down-stream side, n_D, are also decision variables. The two main constraints are that the two factors of safety should be above their minimum requirements. Clearly, these are nonlinear constraints. Besides, there will be boundary constraints on the variables and the slopes. For instance it may be required that a slope u_i/u_j should at most ½, where u_i and u_j are the height and width of a slanting edge. Such a constraint can be treated as linear constraint $2u_i - u_j \le 0$.

The core material is generally costlier compared to the cost of shell material. Therefore, in the formulation, it is justified to assume different costs to these two materials. Let C_{Shell} and C_{Core} be the unit costs (say, in rupees per m^3) of shell and core materials respectively. Let $f_{Shell}(n_U, n_D, u)$ and $f_{Core}(n_U, n_D, u)$ be the cross-section areas (in m^2) of one-meter thick slice of shell and core respectively. We now present the formulation of the problem:

$$\begin{aligned} &\textit{Minimize} \quad C_{Shell}\ f_{Shell}(n_U, n_D, u) + C_{Core}\ f_{Core}(n_U, n_D, u) \quad \textit{subject to} \\ &FS_U(n_U, n_D, u) \ge 1.3 \\ &FS_D(n_U, n_D, u) \ge 1.5 \\ &u\ \varepsilon\ S, \textit{and}\ n_U, n_D\ \textit{are non-negative integers} \end{aligned}$$

Here the set S, the feasible region with respect to all constraints other than the factor of safety constraints.

7.3 Solving the Problem

If the problem is formulated in the above fashion allowing the n_U and n_D as decision variables, it becomes a complex problem to solve. As n_U and n_D will be small numbers, one approach is to solve the problem for different combinations of these two variables, and then choose the best among them. In practice, one may often be interested in finding a solution for fixed n_U and n_D. Therefore, it is proposed that the problem of finding an optimal design be treated as a two-stage optimization problem. In the first stage, optimal designs are obtained for each possible combination of fixed number of berms. Once this is done, the second stage optimization is that of picking up the best of the first stage optimal solutions.

Now consider the problem of finding an optimal design for a fixed number of berms. The function of factor of safety, involving the line integrals, has no closed form solution, and can only be evaluated using numerical methods. This turns out to be the major hurdle in solving the problem formulated above. The only solution to this problem is to use software such as MATLAB where a provision is given to solve such problems. The MATLAB has a function called *fmincon* which is used for finding solutions to optimization problems including nonlinear programming problems. Consider the optimization problem:

$$\text{Minimize } f(x) \text{ subject to } g(x) \leq 0 \text{ and } h(x) \leq 0, \quad (5)$$

where f, g and h are all nonlinear functions of the vector x of decision variables. To use *fmincon*, it is enough if there is computer program that provides the values of these functions at any given x. All that is required is that the computer program be written in MATLAB using its m-files. By writing $g(n_U, n_D, u) = 1.3 - FS_U(n_U, n_D, u)$, we can write an m-file for g and supply that for the upstream factor of safety constraint. Since this is the only option left to us to find an optimal solution to the above formulated problem, we developed a m-file that takes the input as the design parameters and produces a three dimensional vector (c, f_U, f_D) as output, where c is the cost of the material for the given design, f_U is the upstream factor of safety margin and f_D is the downstream factor of safety margin. An attempt was made to find out an optimal design for the Pendakal dam. It so happened that the *fmincon* was not producing any output. Failing to resolve this problem, an alternative approach was adopted to find out an optimal design. Though this approach does not assure optimal solution, it is found effective at least in improving an existing design. The method is summarized briefly in the steps below with the help of the Pendekal dam example in the next section. This approach uses optimizing one variable at a time, while keeping all other variables fixed at present values.

7.4 A Heuristic for Optimal Design

Step 1: Start with an initial design vector u. If u is feasible, then find out that component of u which yields a good profit (material cost reduction) and change only that variable with an appropriate margin. Evaluate the new design and if it feasible, repeat Step 1.

Step 2: If u is an infeasible design, identify the chord where the factor of safety is low and identify a component that is responsible for the weak factor of safety. Modify the component suitably and check for feasibility. If the modified design is feasible, go to Step 1. Otherwise repeat Step 2.

Repeat the above procedure until a satisfactory design is obtained.

7.5 Application to the Pendekal Dam

We shall now apply the procedure to analyze and improvise the Pendekal dam design. To facilitate the approach a table is designed to follow the steps in the algorithm. Before explaining the steps of the solution, let us formulate the problem for the Pendekal dam. This is explained with the help of the diagram in Fig 19. Pendekal dam is a 2-berm design. There are nine decision variables, in this design. Let u denote the vector of these decision variables. We shall call u as the design vector. The dam height is 12.5 m, dam top width is 6 m, core height is 9.5 m, core top width is 3 m, and FRL is 9 m. The height variables u_8 and u_9 cannot exceed FRL. The cost of core material for this dam is approximately 25 percent more than the cost of its shell material. The total cost of the material of the dam is about 2000 million rupees (that is, about $100,000). Let $A_S(u)$ and $A_C(u)$ be the areas of shell and core cross sections. Then the objective function is $A_S(u) + 1.25A_C(u)$. Since our approach starts with an initial design, the actual existing design in this case, at each stage we modify the decision variables within the allowable limits. With this in mind, the only constraints are the constraints of factor of safety and that u_8 and u_9 cannot exceed 9.5. The steps of our approach are listed below.

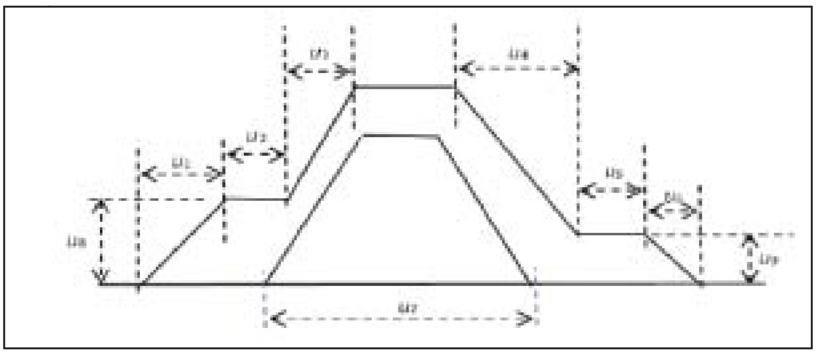

Fig. 19: Design variables of the Pendekal dam

Solution for Pendekal Design

1. Set up the Table for the approach. This is shown in Table-4. The items in the table are self explanatory.
2. Start with the first row and place the initial solution. The initial solution for this example is u^0 = (9, 3, 16, 19, 8, 6, 12.5, 4.5, 3).
3. Evaluate the design for its factors of safety. In this case, the factors of safety are: FS_U = 1.271 and FS_D = 1.541. The critical circle has its center at (12.84, 21.50) and has a radius of 25.11 m. The circle belongs to the Chord(1,33). Using this information, the natural way to increase the upstream factor of safety is by making the left side of the dam flatter. This can be done by increasing u_3^0.
4. Increase the u_3^0 from 16 to 18. The new design vector u^1 = (9, 3, 18, 19, 8, 6,12.5, 4.5, 3) is shown in the second row of the Table. This new design turns out to be feasible with FS_U = 1.366 and FS_D = 1.853. This is marked underthe column heading "Validity of the Design" in the Table. The cost of the design has gone up from 443.41 to 460.41. This means an increase of 3.83% over the original cost (see the last column of Table-4), but the new design is feasible where as the original one is not.
5. Next, it is examined if a jump of 2 units (16 to 18) was really necessary. So, the design u^2 = (9, 3, 17, 19, 8, 6, 12.5, 4.5, 3), shown in row 3 of the Table, is checked for its feasibil ity. As FS_U = 1.349 and FS_D = 1.850 for this design is also feasible and its cost is 1.92% more than the original cost.
6. Since the downstream factor of safety is more than required, a new design may be pos sible with lesser cost. A new design is explored by reducing u_4^2 from 19 to 15. The new design u^3 = (9, 3, 17, 15, 8, 6, 12.5, 4.5, 3), shown in row 4 of Table-4, is explored. The new design is feasible and has a cost reduction (savings) of 5.08%. Further exploration in this direction of reducing u_4^3 are shown in the subsequent rows of the table.
7. When u_4^3 is reduced to 10, the design becomes infeasible as the factor of safety fails (in this case, both factors fall below their allowable limits). The last row of Table-4 shows that when u_4^6 = 12 the design is feasible and the overall reduction in the cost is 10.32%. As the factors of safety are close to their permissible limits, the search for optimal design is stopped at the 7th iteration and the design given below is taken as a satisfactory design.

$$u^6 = (9, 3, 17, 12, 8, 6, 12.5, 4.5, 3) \qquad (6)$$

Thus, starting from an existing infeasible design, we are able to produce a satisfactory design with substantial reduction in the material cost. Each iteration took less than two minutes of running time where as using *fmincon* of MATLAB no output was seen even after two hours of computer running time.

8. Computer Programs Prepared for Decision Support

As a part of this work, a computer program is developed in Microsoft Excel and also in MATLAB. The programs are developed using the procedures described above. The MATLAB program is developed for using it as a tool to optimize the design using *fmincon* function and m-files of MATLAB. On the other hand, the Excel program is more like a decision support system. Besides determining the factors of safety, the program can be effectively used for studying a host of slip circles, evaluating the factor of safety on a given chord, finding out the chord to which a slip circle belongs to, finding out the factor of safety with respect to a slip circle which is at a given distance from a given chord and so on. This program has turned out to be extremely useful in analyzing the existing or any given designs. Interested readers may contact G S R Murthy at murthygsr@gmail.com for further details on the Excel program and a copy of it.

The excel program has six sheets and it is a macro-based program. The sheets are titled 'Specs', 'SlipCirlce', 'ChordAnalysis', 'DamAnalyzer', 'Optimizor' and 'Parameters'. Of these, the last sheet titled 'Parameters' is meant for the program and has no relevance to the user interaction and hence can be ignored by the user. The functions of the other sheets are briefly described below.

Specs: The Excel program is meant for analyzing the factor of safety of a given design. This program can be used effectively to analyze earth dams having up to four berms on either side. This sheet is meant for providing all the necessary inputs to compute the factors of safety. It is designed in such a way that the user can look at the graph of the design and provide the inputs against the dimensions in the graph. The user must specify: (i) number of berms (it takes the same number of berms for upstream and downstream but by specifying negligible berm widths one can analyze designs having different number of berms on upstream and downstream sides), (ii) FRL height, core height, dam height, (iii) densities for driving force and resistance, coefficients of cohesion and friction for different zones (iv) all widths and heights of berms and slants, and (v) widths of core top and bottom and foundation height. Once the data are provided, the user should update the data by clicking on the 'Yellow Push Button.' User can modify the data partially or fully to analyze different designs. This is particularly useful to study the effect of changing design variables while searching for optimal designs.

SlipCircle: This sheet is meant for finding the safety over a given slip circle. The circle is identified by chord parameters and the distance from the center of the circle to the chord. The reason for designing the inputs this way is that the user can get the factors of safety of a number of slip circles whose common chord is the chord specified. It takes five numbers as input, namely, x_1, x_2, d, δd (Delta d) and number of circles required (say n). Here x_1, x_2 are the chord parameters of Chord(x_1, x_2), d is the distance from center to the chord. After giving these inputs the user should click on the push button titled 'Slip Circle Analysis'. Upon clicking, n slip circles are analyzed and the following outputs are displayed for each of these circles: (i) distance from the center to the chord (= $d_i = d + (i - 1)\,\delta d$ for the i^{th} circle), (ii) center coordinates and the radius (r_i) of the circle, (iii) driving force, cohesive force, resistance, factor of safety and the parameter $\lambda_i = r_i/d_i$. If the input d is less than the smallest possible value for a valid slip circle, then the program will produce the output starting with smallest possible d. In this sheet an option is provided to find out the chord parameters for a given slip circle whose center coordinates and the radius are known. With this option, the user can analyze the slip circles whose parameters are known in either format (through chord and distance or center coordinates and radius).

ChordAnalysis: This sheet provides the option of finding the minimum factor of safety of all circles associated with the chord. The program uses the procedure described in Section 6.3. It must be remembered that the minimum factor of safety determined here is an approximation. The inputs for this module are just the chord parameters. The output is displayed for all the circles encountered in the procedure and the output format is similar to that of 'SlipCircle' sheet format.

DamAnalyzer: This sheet is meant for finding out the upstream and downstream factors of safety of the dam. The user is asked for four numbers as inputs, namely, L, R, DeltaX1 (for δx_1) and DeltaX2 (for δx_2) can specify the range of chord parameters optionally. The program will explore the chords with parameters between L and s_R for the upstream and the chords with parameters between s_L and R for the downstream with increments δx_1 and δx_2 using the procedure described in Section 6.5 (see the steps in that procedure).

Optimizer: This sheet provides a convenient format for searching for an optimal solution manually. The table in this sheet is self explanatory and the user can change one or more of the design variables simultaneously and evaluate.

9. Summary

In this paper we have looked the problem of designing earth dams as an optimization problem. Majority of the dams constructed are earth dams as they are most economical. Factor of safety is an important and mandatory aspect of designing earth dams. The factor of safety is evaluated using slip circle method. This method is most widely used and recommended method for evaluating the factor of safety. As per the mandatory recommendations, the factors of safety should be evaluated under different conditions. In this paper we have examined only one of these conditions.

It is understood that there is no good software available to the engineers to design earth dams. At present, couple of excel based software programs are being widely used for evaluating the manually developed designs. These programs are not interactive and they only evaluate at some fixed slip circles and the results for those circles are displayed. It is not possible to use the software to evaluate the factor of safety for a slip circle that an engineer wants to explore. Toward this end, the software developed under this work is highly interactive and can be used to explore various possibilities.

The problem of finding the critical circle is a complex one and involves exploring infinitely many slip circles. In this work we have introduced a new approach, the chord based approach, using which the search process is reduced to that of finding the critical chord instead of critical slip circle. In this approach, for each chord, the factor of safety is approximately a convex function of the parameter λ which is the ratio of distance from the center of slip circle to the chord to the radius of the slip circle. Using this observation, the critical circle with respect to each chord is efficiently determined.

The problem of designing the dam is formulated as nonlinear program with nonlinear objective function and linear and nonlinear constraints. The constraints pertaining to the minimum factors of safety have no closed forms and have to be evaluated using only numerical methods and computer programs. An attempt has been made to solve the design optimization problem using the *fmincon* function of MATLAB but with no success. As an alternative, a heuristic approach has been adopted and this appears to be promising. In Section 7 we have discussed a way of solving the problem approximately. But compared to the present methods of computing factor of safety, our approach is far superior in that it explores a huge number of slip circles.

The methodology is illustrated with a live example using Pendekal dam. It is found through the analysis using the software developed that the actual design falls marginally short of upstream factor of safety. The optimal design for this dam determined using the heuristic proposed in this work exhibited approximately 10% savings.

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