Sphere Methods Using No Matrix Inversions for LPs, and Extension to NLP

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Abstract

Existing software implementations for solving Linear Programming (LP) models are all based on full matrix inversion operations involving every constraint in the model in every step. This **linear algebra component** in these systems makes it difficult to solve dense models even with moderate size, and it is also the source of accumulating roundoff errors affecting the accuracy of the output.

We present new versions of the Sphere method, SM-5, 6, for LP not using any pivot steps; and its extensions to NLP and 0-1 integer linear programming models. These methods are currently undergoing computational tests.

Key words: Linear Programming (LP), Interior point methods (IPMs) , solving LPs by descent methods without using matrix inversions.

1 Brief Historical Review

Linear equation models were used for decision making by the Chinese, Indians and other human societies over 2000 years ago. For example, the Chinese text *Chiu-Chang Suanshu (Nine*

Chapters on the Mathematical Art), composed over 2000 year ago, discusses the problem of determining the yield of of an alcoholic drink from 3 types of grains- inferior, medium, superior- given the yield data from 3 experiments each using a separete combination of the 3 types of grains; with a system of 3 linear equations in 3 variables [see, http://www-groups.dcs.st-and.ac.uk/ history/HistTopics/Nine-chapters.html]. Similarly the Indian text *Bakshali Manuscript* dating to the same period describes 2, 3 equation models in the same number of variables for similar applications. And they both described the Elimination method for solving a square systems of linear equations, using pivot steps now known as **Gauss-Jordan** or **Gaussian pivot steps**.

After a long gap, mathematical models consisting of systems of linear inequalities started appearing in the literature around the 18th century, from studies in mechanics. From mid-18th century Linear Programming models (LPs) involving the optimization of a linear objective function subject to linear inequality constraints started appearing in published literature. Then based on the work of many researchers, Fourier, De la Valle Poussin, Kantorovich, and others; in 1947 the simplex method for solving systems of linear constraints (equations, inequalities), and LPs was developed by Dantzig. The basic computaional tool used by the simplex method is also the pivot step, for various matrix inversion operations in the method. The simplex method for LP is known as a **boundary point method**, as it follows a path contained on the boundary of the feasible region, leading to an optimum solution

In the 2nd half of the 20th century, Karmarkar pioneered an **interior point method, IPM** (following a path in the interior of the feasible solution set) for solving LPs, and showed that intrinsically interior point methods can lead to significantly better computational performance. In many talks he gave in 1980s he repeatedly emphasized that descent steps taken from the centralmost part of the feasible region in the direction of descent of the objective function, have the ability to take longer steps before being stopped by the boundary, and thus have the potential of being much more efficient than boundary methods on large scale models. In fact claims were made that IPMs can be hundreds of times more efficient than boundary methods. Encouraged by his results, several interior point methods have been developed to solve LPs. See the books [2, 3, 4, 12] for detailed discussion of these methods, and references on them. Most of these methods are based on the study of properties of ellipsoids contained inside the feasible region. Existing commercial software systems for solving LPs are all based on these interior point methods.

Like the simplex method, these IPMs are also based on matrix inversion operations involving every constraint in the model in every step, using the fundamental computational tool, the pivot step. In large scale applications, these pivot steps limit the ability of these algorithms to only those with very sparse coefficient matrices. Typically, the effectiveness of these algorithms fades as the density of the coefficient matrix increases.

Spheres are easier to manipulate and study than ellipsoids. In 2006, Sphere methods for LP, IPMs based on the properties of spheres (instead of ellipsoids like in other IPMs) were introduced in Murty [2006a, b]. The initial version of the sphere method also needed pivot steps for matrix inversions, but these pivot steps only involve a subset of constraints in the original LP. After some other versions, in this paper we describe two versions of the sphere method for LP, SM-5, 6, not involving any pivot steps. We also discuss extensions to solve nonlinear programs (NLPs) in which all constraints are linear; and present results on computational performance of these methods.

2 Sphere Methods, SM-5, 6, for LP

SMs consider LPs in the form:

$$\begin{aligned} \min \quad & z = cx \\ \text{subject to} \quad & Ax \geq b \end{aligned} \tag{1}$$

where A is an $m \times n$ data matrix; with a known interior feasible solution x^1 (i.e., satisfying $Ax^1 > b$). LPs in any other form can be directly transformed into this form, see [Murty 2009a, b]. Here is some basic notation that we will use.

- **Notation for rows and columns of A:** $A_{i.}, A_{.j}$ denote the i^{th} row, and j^{th} column of A.

- **Feasible region and its interior:** K denotes the set of feasible solutions of (1), and $K^0 = \{x : Ax > b\}$ is its interior.

- **Facetal hyperplanes, and their half-spaces containing K:** $FH_i = \{x : A_{i.}x = b_i\}$, the i-th facetal hyperplance of K for $i = 1$ to m. Also, $FH_i^+ = \{x : A_{i.}x \geq b_i\}$ is the half-space of FH_i containing K.

- **IFS:** Interior feasible solution, a point $x \in K^0$

In order to get good performance from sphere methods on this problem, from Karmarkar's original observation, we see that descent steps should be taken from the center of a largest inscribed sphere within K. A point like that, the center of a largest insphere of K, is called a **ball center** of K, to distinguish it from centers used in other IPMs for LP. In order to develop a method for computing a ball center or a good approximation for it, we first need to determine, for any given interior point $x \in K$, the radius of the largest insphere of K with x as center. This is given in Theorem 1 below.

Theorem 1: For any $\bar{x} \in K^0$, the radius of the largest insphere of K with $\bar{x}$ as center is minimum$\{(A_{i.}\bar{x} - b_i)/||A_{i.}|| : i = 1 \text{ to } m\}$.

Proof: Consider any $i \in \{1, ..., m\}$. The nearest point on FH_i to $\bar{x}$ is its orthogonal projection $\bar{x}^i = \bar{x} - A_{i.}^T[(A_{i.}\bar{x} - b_i)/||A_{i.}||^2]$ of $\bar{x}$ on FH_i; and its distance from $\bar{x} = ||\bar{x} - \bar{x}^i|| = (A_{i.}\bar{x} - b_i)/||A_{i.}||$. Therefore the sphere $\{x : ||x - \bar{x}|| \leq (A_{i.}\bar{x} - b_i)/||A_{i.}||\}$ with $\bar{x}$ as center, is completely contained in the half-space FH_i^+. Also FH_i is a tangent plane to this sphere with $\bar{x}^i$ as their touching point, see Figure 1. Any move from $\bar{x}$, on the halfline joining it to $\bar{x}^i$, beyond $\bar{x}^i$, will lead to points outside the half-space FH_i^+. So this sphere is the largest with $\bar{x}$ as center inside FH_i^+.

Figure 1: $\bar{x}^i$ is the orthogonal projection of $\bar{x}$ on FH_i. The sphere shown with $\bar{x}$ as center and $\bar{x}^i$ as a boundary point, is the largest sphere with $\bar{x}$ as center in the half-space FH_i^+. FH_i is a tangent plane to it, and $\bar{x}^i$ is their touching point.

Since the above holds for all $i = 1$ to m, and K is the intersection of all the half-spaces FH_i^+ for $i = 1$ to m, we conclude that the largest sphere with $\bar{x}$ as center inside K, has radius $=$ minimum$\{(A_i\bar{x} - b_i)/||A_{i.}|| : i = 1 \text{ to } m\}$. Call this sphere $B(\bar{x}, K)$, and let $T(\bar{x}, K)$ denote the set of all indices i which tie for the minimum in $\{(A_i\bar{x} - b_i)/||A_{i.}|| : i = 1 \text{ to } m\}$. Also from the above arguments we conclude that FH_i for $i \in T(\bar{x}, K)$ are all the facetal hyperplanes of K which touch the sphere $B(\bar{x}, K)$. See Figure 2. $\bowtie$

We will use "$\bowtie$" to indicate the end of a proof, or an argument.

Figure 2: $\bar{x} \in K^0$, and the ball shown is $B(\bar{x}, K)$, the largest ball inside K with $\bar{x}$ as center. Facetal hyperplanes FH_1, FH_2 corresponding to indices 1, 2 are tangent planes to this ball, with their touching points $\bar{x}^1, \bar{x}^2$ respectively. So, $T(\bar{x}, K)$ here is $\{1, 2\}$, known as the touching constraint index set at $\bar{x}$.

We will use the terms "ball, sphere" interchangebly. Here is the additional notation that we will use.

- $\delta(x, K)$: Defined for $x \in K$, it is the radius of the largest ball inside K with x as center. From Theorem 1, we know that $\delta(x, K) = \text{minimum}\{\frac{A_{i.}x - b_i}{||A_{i.}||} : i = 1, ..., m\}$. For any point x on the boundary of K , i.e., satisfying at least one of the constraints in (1) as an equation, $\delta(x, K) = 0$ by this definition.

- **Largest ball inscribed in K with a given IFS x as center:** $B(x, K) = \{y : ||y - x|| \leq \delta(x, K)\}$ is that largest inscribed ball in K with x as its center.

- **Touching constraint index set at a given IFS:** $T(x, K)$ defined for $x \in K^0$, is the set of all indices i satisfying: $\frac{A_{i.}x - b_i}{||A_{i.}||} = \text{Minimum}\{\frac{A_{p.}x - b_p}{||A_{p.}||} : p = 1 \text{ to } m\} = \delta(x, K)$. The facetal hyperplane $FH_i = \{x : A_{i.}x = b_i\}$ is a tangent plane to $B(x, K)$ for each $i \in T(x, K)$, that's why $T(x, K)$ is called the **index set of touching constraints in (1) defining K**, at x.

- **Profitable direction at an IFS $x^0 \in K$:** A direction g (i.e., vector $g \neq 0$) is said to be a **profitable direction** at $x^0 \in K$, if $\delta(x^0 + \alpha g)$ is increasing at $\alpha = 0$, as α increases from 0.

Figure 3: Moving the point $x^0 \in K$ in the direction indicated by the arrow, traces the point $x^0 + \alpha g$ as α increases from 0. g is a profitable direction at $x^0 \in K$ since $\delta(x^0 + \alpha g)$ is increasing as α increases from 0.

How to Compute A Ball Center of K

As defined above, a ball center of K is a point $x \in K$ which maximizes $\delta(x, K) = \{\frac{A_{i.}x - b_i}{||A_{i.}||} : i = 1 \text{ to } m\}$. So, if x^* is a ball center of K, then $(\delta(x^*, K), x^*)$ is an optimum solution of the LP

$$\max \quad \delta \tag{2}$$
$$\text{subject to} \quad \delta ||A_{i.}|| \leq A_{i.}x - b_i, \quad i = 1, ..., m$$

This is another LP about the same size as the original LP. We will now state Theorem 2, which will be used in Sphere method 5 (SM-5) to be discussed later.

Theorem 2: Let Δ be a positive number, and let $b(\Delta) = (b_i(\Delta) = b_i - \Delta ||A_{i.}|| : i = 1, ..., m)$ and let $K(\Delta)$ denote the set of feasible solutions of

$$\min \quad z = cx \tag{3}$$
$$\text{subject to} \quad Ax \geq b(\Delta)$$

Our original set of feasible solutions for (1), K, is $K(0)$ in this notation. Clearly, $K \subset K(\Delta)$ for $\Delta \geq 0$. Every ball center of K is a ball center of $K(\Delta)$ for $\Delta \geq 0$ and vice versa. Also, for all $x \in K$, $\Delta \geq 0$, $\delta(x, K(\Delta)) = \delta(x, K) + \Delta$.

Proof: Since any vector x which is feasible to (1), automatically satisfies (3) for $\Delta \geq 0$, it is clear that $K \subset K(\Delta)$. Let $(\bar{\delta}, \bar{x})$ be an optimum solution of (2). So, $\bar{x}$ is a ball center of K, and $\bar{\delta} = \delta(\bar{x}, K) = \text{maximum } \{\delta(x, K) : x \in K\}$ = radius of the largest ball insaide K.

Let $\Delta > 0$. For each $x \in K$, we have seen above that since $K \subset K(\Delta)$, $x \in K(\Delta)$, and $\delta(x, K(\Delta)) = \text{minimum}\{((A_{i.}x - b_i)/(||A_{i.}||)) + \Delta : i = 1, ..., m\} = \delta(x, K) + \Delta$. So, the radius of the largest ball inside $K(\Delta) \geq \bar{\delta} + \Delta$.

Let $\breve{x}$ = a ball center of $K(\Delta)$, and $\breve{\delta}$ = radius of a largest ball inside $K(\Delta)$. So, $\breve{\delta} = \text{minimum}\{(A_{i.}\breve{x} - b_i(\Delta))/(||A_{i.}||) : i = 1, ..., m\}$. Therefore we have

$\frac{A_{i.}\breve{x} - b_{i.}(\Delta)}{||A_{i.}||} \geq \breve{\delta}$ for all $i = 1, ..., m$.

i.e., $\frac{A_{i.}\breve{x} - b_{i.}}{||A_{i.}||} + \Delta \geq \breve{\delta} \geq \bar{\delta} + \Delta$ for all $i = 1, ..., m$

i.e., $A_{i.}\breve{x} - b_i \geq \bar{\delta}||A_{i.}|| \geq 0 \qquad$ for all $i = 1, ..., m$.

Therefore, $\breve{x} \in K$, and hence $\delta(\breve{x}, K(\Delta)) = \delta(\breve{x}, K) + \Delta \leq \bar{\delta} + \Delta$. All these facts together imply $\breve{\delta} = \bar{\delta} + \Delta$, and that every ball center of K is a ball center of $K(\Delta)$ and vice versa. $\bowtie$

Figure 4: K, the original set of feasible solutions for (1) is shown in solid lines. Its ball center is unique, it is $\bar{x}$. $B(\bar{x}, K)$, the largest ball inscribed in K, is the smaller ball with $\bar{x}$ as center in the figure. Each facet of K is moved parallel to itself outward by a distance of $\Delta > 0$, leading to $K(\Delta)$ shown with dashed lines. $\bar{x}$ is also the unique ball center of $K(\Delta)$. $B(\bar{x}, K), \quad B(\bar{x}, K(\Delta))$ are concentric.

Figure 5: K, the original set of feasible solutions for (1) is the rectangle in solid lines; and $K(\Delta)$ is the rectangle in dashed lines obtained by moving every facet of K outward by a distance of $\Delta > 0$. Every point on the dotted line inside, is a ball center of both K and $K(\Delta)$.

In all the sphere methods so far, each iteration begins with an initial IFS, and consists of two major steps. The first called the **Centering cycle**, has the aim of finding a **center for**

this iteration, a good approximation for the ball center of the current feasible solution under consideration, using some computationally efficient strategy. The second is the **Descent cycle**, in which descent steps are carried out from the Center obtained in the Centering cycle in several descent directions. The best output point from these descent steps, will be the initial IFS for the next iteration of the method.

In this paper we will discuss two recently developed versions of the sphere method for LP, Sphere methods 5, 6 (SM-5, 6). Both of them do not use any pivot steps. In every iteration of both SM-5, 6, we face a problem of finding the interval of values of a real parameter γ say, satisfying a given system of linear inequalities in the parameter. Now we give the procedure, we will call it **Subroutine 1** for computing this interval.

Subroutine 1: Let the system of inequalities in γ be

$$a_t + g_t\gamma \geq 0, \qquad t = 1, ..., \ell \tag{4}$$

In systems like this that we encounter in SM-5, 6; for any t if $g_t = 0$, a_t will be ≥ 0, and hence that constraint is a redundant constraint in the system. Let

$$\begin{aligned} \gamma^1 &= \text{maximum}\{(-a_t/g_t) : \text{over all } t \text{ satisfying } g_t > 0\} \\ \gamma^2 &= \text{minimum}\{(-a_t/g_t) : \text{over all } t \text{ satisfying } g_t < 0\} \end{aligned}$$

Here define the maximum [minimum] in the empty set to be $-\infty[+\infty]$ respectively. If $\gamma^1 > \gamma^2$ system (4) has no solution. Otherwise the required interval for γ feasible to this system is $\gamma^1 \leq \gamma \leq \gamma^2$.

In the next two sections, we will describe SM-5, 6 respectively. This will be followed by a discussion of their computational performance.

3 Sphere Method 5 (SM-5)

In this method, as in Sphere method 2 [Murty 2009a, b], the set of feasible solutions under consideration in each iteration, is updated by the objective value of the initial IFS for that iteration, as described below; and hence gets smaller with each iteration in the method.

In each iteration, the centering cycle finds the center, an approximate ball center of the current updated set of feasible solutions, using a series of line search steps in profitable directions as in Sphere methods 1, 2 [Murty 2009a, b]; but the profitable directions for these line search steps are computed by procedures described in [Xie, Snoeyink, Xu 2006, Clarkson 2010] not using any pivot steps.

We will now describe a general iteration in SM-5 beginning with the given initial IFS for it, $\underline{\mathrm{x}}$, say. Here are the various steps in the centering cycle in this iteration.

3.1: Centering Cycle in this Iteration

Step 1: Update the set of feasible solutiuons under consideration by the objective value at the current initial IFS: This updated set of feasible solutions, denoted by $\underline{\mathrm{K}} = \{x : Ax \geq b, \;\; A_{m+1.}x \geq b_{m+1}\}$, where $A_{m+1.} = -c, \;\; b_{m+1} = -c\underline{\mathrm{x}} - \epsilon$, where ϵ is a small positive tolerance.

Figure 6: K is the original set of feasible solutions of the original LP being solved. $\underline{\mathrm{x}}$ is the initial IFS of K for the current iteration. The current set of feasible solutions under consideration for this iteration is $\underline{\mathrm{K}}$. The ball shown is the largest ball inside $\underline{\mathrm{K}}$, the aim of this centering cycle is to compute a good approximation for its center.

Step 2: The centering Cycle

Here we will use the peocedure discussed by Xie, Snoeyink, Xu [2006] to compute a good approximation to the ball center of $\underline{\mathrm{K}}$ efficiently, beginning with the current IFS $\underline{\mathrm{x}}$. This procedure generates a sequence of points $\underline{x}^k$, $k = 1, 2, ...$ beginning with $\underline{x}^1 =$ the initial IFS $\underline{x} \in \underline{K}$, converging to an approximate ball center of $\underline{K}$. We will now discuss the work in this procedure in a general step when the current point in the sequence is $\underline{x}^k \in \underline{K}$.

3.1.1: Translate the Origin to $\underline{x}^k$

Define new variables $y = x - \underline{x}^k$. In terms of these new variables y,

$$\begin{aligned} \underline{K} &= \{y : A_{i.}(y + \underline{x}^k) \geq b_i\}, \quad \text{for } i = 1 \text{ to } m+1 \\ &= \{y : A_{i.}y \geq b_i - A_{i.}\underline{x}^k\}, \quad \text{for } i = 1 \text{ to } m+1 \end{aligned}$$

Since $\underline{x}^k \in K^0$, we know that $b_i - A_{i.}\underline{x}^k < 0$ for all $i = 1$ to m, and it also holds for $i = m+1$ from the definition of the $(m+1)$th constraint. Therefore dividing both sides of the i-th constraint in the representation of $\underline{K}$ given above by $b_i - A_{i.}\underline{x}^k$, we get

$\underline{K} = \{y : \underline{A}^k y \leq e\}$, where e is a column vector of all 1 entries in R^{m+1}, and $\underline{A}^k_{i.} = A_{i.}/(b_i - A_{i.}\underline{x}^k)$ for $i = 1, ..., m+1$.

If for any i, $|b_i - A_{i.}\underline{x}^k|$ is too small for this division; we can replace $b = (b_i : i = 1, ..., m+1)$ in the representation of $\underline{K}$ by $b(\Delta) = (b_i - \Delta||A_{i.}|| : i = 1, ..., m+1)$ for some positive number Δ , to make this division easier. By Theorem 2, we know that the ball center we want to approximate is unaffected by this change in the RHS constants. Then when the algorithm moves to determine the profitable direction to move at $\underline{x}^k$ (Section 3.1.3); we change all the RHS constants back to the original b_is for all the work in the sequel. However, we will describe this step below using the same symbols b_i for the RHS constants.

Define the set of points in the y-space: $S^k = \{P_i = (\underline{A}^k_{i.})^T : i = 1 ..., m+1\}$.

3.1.2: Compute the MES (Minimum Enclosing Sphere) of the Set of Points S^k in the y-space:

Figure 7: Given the set of points, each marked by an $\times$, the MES (Minimum Enclosing Sphere) containing all these points is shown. Q is its center.

Let Q^k denote the center of the MES of the set of points S^k defined above. This point Q^k in the y-space corresponds to the point $Q^k + \underline{x}^k$ in the original x-space. Results in Xie, Snoeyink, Xu [2006] indicate that if $\underline{x}^k$ is not a ball center of $\underline{K}$, then the straight line joining $\underline{x}^k$, $Q^k + \underline{x}^k$ is likely to contain points x satisfying $\delta(x, \underline{K}) > \delta(\underline{x}^k, \underline{K})$. Since our goal is to find an x maximizing $\delta(x, \underline{K})$, we will follow the strategy of computing Q^k, and then maximizing $\delta(x)$ on the line joining $\underline{x}^k$, $Q^k + \underline{x}^k$ to get a point x better than the current $\underline{x}^k$ for this goal.

Here we describe a simple iterative scheme from Clarkson [2010] for computing Q^k approximately. The scheme generates a sequence of points Q_t, $t = 0, 1, ...$ beginning with $Q_0 =$ average of points in S^k. Given Q_t, obtain the next element in the sequence, Q_{t+1} by the formula $Q_{t+1} = (1-a)Q_t + aP^{t+1}$, where $a = 2/(t+3)$ and P^{t+1} is the farthest point to Q_t by Euclidean distance in the set S^k.

Terminate this scheme when it reaches a point Q_{t+1} in the sequence satisfying $||Q_{t+1} - Q_t|| \leq \epsilon$, and take the final point Q_{t+1} as an approximation of Q^k.

3.1.3: Determining a profitable direction g to move at $\underline{\mathbf{x}}^k \in \underline{\mathbf{K}}$

In [Murty 2006 a, b] it has been proved that a direction g is a profitable direction to move in $\underline{K}$ at the IFS $\underline{x}^k$, iff $A_{i.}g > 0$ for all $i \in T(\underline{x}^k, \underline{K})$.

The straight line joining $\underline{x}^k$, $Q^k + \underline{x}^k$ consists of two half-lines, one from $\underline{x}^k$ in the direction Q^k, and another from $\underline{x}^k$ in the direction $-Q^k$. From the above, this implies that at most one of these two half-lines can contain a point x satisfying $\delta(x, \underline{K}) > \delta(\underline{x}^k, \underline{K})$. Hence, do the following:

1. If $A_{i.}Q^k > 0$ for all $i \in T(\underline{x}^k, \underline{K})$, define g to be $= Q^k$ and go to 3.1.4 below to maximize $\delta(\underline{x}^k + \alpha g, \underline{K})$ over $\alpha \geq 0$.

2. If $A_{i.}Q^k < 0$ for all $i \in T(\underline{x}^k, \underline{K})$, define g to be $= -Q^k$ and go to 3.1.4 below to maximize $\delta(\underline{x}^k + \alpha g, \underline{K})$ over $\alpha \geq 0$.

Also notice that due to the $(m+1)$th constraint in the definition of the current feasible set $\underline{K}$, any profitable direction g at this stage is likely to be a descent direction for the objective function cx in the original problem (1), i.e., cg will be < 0.

If neither of the conditions in 1, 2 above hold, it is an indication that $\underline{x}^k$ corresponds to the maximum value of $\delta(x, \underline{K})$ on the line joining $\underline{x}^k$, $\underline{x}^k + Q^k$. In this case terminate this centering cycle, accept $\underline{x}^k$ as the approximate ball center of $\underline{K}$, and go to the Descent Cycle (Step 3) below with $\underline{x}^k$ as the center for this iteration.

3.1.4: Maximizing $\delta(\underline{\mathbf{x}}^k + \alpha g, \underline{\mathbf{K}})$ over $\alpha \geq 0$

From Theorem 1, we know that $\delta(\underline{x}^k + \alpha g, \underline{K}) = \text{minimum}\{[A_{i.}(\underline{x}^k + \alpha g) - b_i]/||A_{i.}|| : i = 1$ to $m + 1\}$. In the problem of maximizing $\delta(\underline{x}^k + \alpha g, \underline{K})$ over $\alpha \geq 0$, the only variables are α, δ. So, this problem is the following 2-variable LP with δ, α as the two variables.

$$\begin{array}{lll} & \max \quad \delta & \\ \text{subject to} & \delta + \alpha \dfrac{[-A_{i.}g]}{||A_{i.}||} \leq \dfrac{A_{i.}\underline{x}^k - b_i}{||A_{i.}||} & i = 1, ..., m + 1 \\ & \delta, \alpha \geq 0 & \end{array} \tag{5}$$

The RHS vector in this LP is > 0 as $\underline{x}^k$ is an IFS of $\underline{K}$. We will consider 3 cases that can occur in this problem.

1. If the coefficient vector of α in this LP (5) is < 0, then clearly the maximum value of δ in this LP is $+\infty$, which implies that the minimum value of $z = cx$ in the original LP (1) is $-\infty$. Define for $\alpha \geq 0$

$$\delta(\alpha) = \text{minimum} \quad \{\frac{(A_{i.}(\underline{x}^k + \alpha g) - b_i)}{||A_{i.}||} : \qquad i = 1, ..., m + 1\} \tag{6}$$

So $\delta(\alpha) \to +\infty$ as $\alpha \to +\infty$. Then the point $x(\alpha) = \underline{x}^k + \alpha g - \delta(\alpha)c^T/||c||$ is feasible for all $\alpha \geq 0$, and its objective value $cx(\alpha) \to -\infty$ as $\alpha \to +\infty$.

In fact, since g is a profitable direction at $\underline{x}^k$ for $\underline{K}$, as noted above $cg < 0$. And, since $\delta(\alpha)$ defined in (6) tends to infinity as α tends to infinity, we know that $A_{i.}(\underline{x}^k + \alpha g) - b_i \geq 0$ for all i $= 1$ to $m+1$, and hence $\underline{x}^k + \alpha g$ is feasible to (1) for all $\alpha \geq 0$, and clearly $c(\underline{x}^k + \alpha g) = c\underline{x}^k + \alpha cg$

tends to $-\infty$ since $cg < 0$. Thus $\{\underline{x}^k + \alpha g : \alpha \geq 0\}$ is a half-line in $\underline{K}$ along which the objective function in (1) diverges to $-\infty$. Terminate the algorithm with this conclusion.

Figure 8: Set of feasible solutions of the system: $\delta - \alpha \leq 4$, $\delta - 2\alpha \leq 1$, $\delta - 1.5\alpha \leq 2$, $\delta, \alpha \geq 0$. Maximum value of δ in this set is ∞.

2. If the coefficient vector of α in the LP (5) is ≤ 0, but not < 0, then $\underline{\delta}^k$ = minimum $\{(A_{i.}\underline{x}^k - b_i)/||A_{i.}|| : i = 1, ..., m+1$ such that $A_{i.}g = 0\}$ is an upper bound for δ in the LP (5). By definition, we know that $\delta(\underline{x}^k, \underline{K})$ is finite and $T(\underline{x}^k, \underline{K}) \neq \emptyset$. Also, since g is a profitable direction at $\underline{x}^k$, we know that the $\delta(\alpha)$ defined in (6) has value at $\alpha = 0$, $\delta(0) < \underline{\delta}^k$,and hence at least one coefficient of α in (5) must be < 0.

Therefore in this case the optimum value of α in (5) is the value of α which makes $\delta(\alpha) = \underline{\delta}^k$. To find this value of α do the following: Let $I = \{i : 1 \leq i \leq m+1$ satisfying $A_{i.}g > 0$ and $\frac{(A_{i.}\underline{x}^k - b_i)}{||A_{i.}||} < \underline{\delta}^k\}$.

Then, one value of α which makes $\delta(\alpha) = \underline{\delta}^k$ is

$$\underline{\alpha}^k = \text{maximum}\{(A_{i.}\underline{x}^k - b_i)/(-A_{i.}g) : \text{over } i \in I\}.$$

So, take $(\underline{\delta}^k, \underline{\alpha}^k)$ as an optimum solution of the 2-variable LP (5) in this case.

Figure 9: Set of feasible solutions of the system: $\delta - 2\alpha \leq 1$, $\delta - \alpha \leq 4$, $\delta \leq 6$, $\delta, \alpha \geq 0$. Maximum value of δ in this set is 6.

3. Consider the last case in which there is an i satisfying $-A_{i.}g > 0$, i.e., at least one coefficient of α in the 2-variable LP (5) is > 0. This problem can be solved by anyone of several efficient algorithms for solving 2-variable LPs in the literature. We will discuss one simple approach based on the line search algorithm used in nonlinear programming (NLP) literature: Since δ is a nonnegative variable in (5),

$$\beta = \text{minimum}\{(A_{i.}\underline{\mathrm{x}}^k - b_i)/(-A_{i,}g) : \text{over } i = 1 \text{ to } m+1 \text{ satisfying } A_{i.}g < 0\}$$

is an upper bound for the variable α in (5), i.e., $0 \leq \alpha \leq \beta$. Then the LP (5) is equivalent to the NLP in the single vartiable α

$$\text{maximize } \delta(\alpha) \text{ (equivalently minimize } -\delta(\alpha)) \quad \text{subject to} \quad 0 \leq \alpha \leq \beta$$

This problem can be solved by one of the line search algorithms in NLP, for example the "Quadratic fit line search method" (see the textbook Murty[1995]). Let $\underline{\alpha}^k$ be an optimum solution of this problem. Then $(\underline{\delta}^k = \delta(\underline{\alpha}^k), \underline{\alpha}^k)$ is an optimum solution of the 2-variable LP (5). The point in $\underline{\mathrm{K}}$ corresponding this solution of (5) is:

$$\underline{\mathrm{x}}^{k+1} = \underline{\mathrm{x}}^k + \underline{\alpha}^k g, \quad \text{and} \quad \underline{\delta}^{k+1} = \delta(\underline{\mathrm{x}}^{k+1}, \underline{\mathrm{K}}) = \text{minimum}\{(A_{i.}\underline{\mathrm{x}}^{k+1} - b_i)/(||A_{i.}||) : i = 1, ..., m+1\}.$$

Figure 10: Set of feasible solutions of the system: $\delta + \alpha \leq 7$, $\delta - 2\alpha \leq 1$, $\delta - 0.5\alpha \leq 3$, $\delta, \alpha \geq 0$. Maximum value of δ in this set is attained at the point $(\delta, \alpha) = (13/3, 8/3)$.

If $||\delta(\underline{\mathrm{x}}^{k+1}, \underline{\mathrm{K}}) - |\delta(\underline{\mathrm{x}}^{k}, \underline{\mathrm{K}})|| \leq \epsilon$, terminate this centering cycle , and take the final output point $\underline{\mathrm{x}}^{k+1}$ as the **center** for this iteration and go to the descent cycle (Step 3 in this iteration) with it. On the other hand if $||\delta(\underline{\mathrm{x}}^{k+1}, \underline{\mathrm{K}}) - |\delta(\underline{\mathrm{x}}^{k}, \underline{\mathrm{K}})|| > \epsilon$, with $\underline{\mathrm{x}}^{k+1}$ as the initial IFS go back to 3.1.

3.2 Descent Cycle in this Iteration

The center for this iteration: Let $\bar{x}$ denote the center, the output point in the centering cycle in this iteration.

3.2.1: Determining the Step Length in a General Descent Step

Here we describe how the step length is determined in a general descent step in this method. Consider a descent step from a point $\hat{x}$ in a descent direction d, i.e., a direction satisfying $cd < 0$.

In this method, the point $\hat{x}$ from which the descent step is taken, may or may not be in $\underline{\mathrm{K}}$, the current updated set of feasible solutions; i.e., the constraint $A_{i.}\hat{x} \geq b_i$ may or may not hold for some $i \in \{1, ..., m+1\}$. But we need to make sure that the output point from this descent step is a point in (in fact an IFS) of $\underline{\mathrm{K}}$, if it is going to be used as the initial feasible solution for the next iteration in this method SM-5. We consider these cases separately.

Case 1: $\hat{x}$, the point from which this descent step is taken, is an IFS of $\underline{\mathrm{K}}$

In this case, we take the maximum possible step length while keeping the output point at the end of this descent step as an IFS of $\underline{\mathrm{K}}$. The maximum possible step length, $\hat{\alpha}$ say, from $\hat{x}$ inside $\underline{\mathrm{K}}$ in direction d is the maximum value of α satisfying $A_{i.}(\hat{x} + \alpha d) \geq b_i$ for all $i = 1$ to $m+1$; which is minimum$\{(b_i - A_{i.}\hat{x})/(A_{i.}d)$: over all $i = 1$ to $m+1$ satisfying $A_{i.}d < 0\}$.

If $\{i : 1 \leq i \leq m+1, A_{i.}d < 0\} = \emptyset$, then this step length $\hat{\alpha} = +\infty$; we terminate the algorithm with the conclusion that the objective function cx in (1) is unbounded below in its set of feasible solutions, with $\{\hat{x} + \alpha d : \alpha \geq 0\}$ providing a feasible half-line along which cx diverges to $-\infty$.

If $\{i : 1 \leq i \leq m+1, A_{i.}d < 0\} \neq \emptyset$, Then the maximum step length is $\hat{\alpha}$ defined above. We take the actual step length to be $\hat{\gamma} = \hat{\alpha} - \epsilon$, leading to the output point $\hat{x} + \hat{\gamma}d$, with its objective value of $c(\hat{x} + \hat{\gamma}d)$.

Case 2: $\hat{x}$, the point from which this descent step is taken, is not an IFS of $\underline{\mathbf{K}}$, and may be outside $\underline{\mathbf{K}}$

Denote

M = the half-line $\{\hat{x} + \alpha d : \alpha \geq 0\}$.

In this case we need to make sure that M intersects $\underline{\mathrm{K}}$ in its interior, as otherwise this descent step cannot produce a desired output point no matter what the step length is. For this the following conditions must hold.

Condition 1: If there is an $i \leq i \leq m+1$ satisfying $A_{i.}d = 0$, we must have $A_{i.}\hat{x} - b_i > 0$.

Condition 2: $\text{Min}\{(b_i - A_{i.}\hat{x})/(A_{i.}d : \text{over all } i \text{ satisfying } A_{i.}d < 0\} > \text{Max}\{(b_i - A_{i.}\hat{x})/(A_{i.}d) :$ over all i satisfying $A_{i.}d > 0\}$. This can be seen as a requirement from Subroutine 1.

In both SM-5, 6, the point $\hat{x}$ from which the descent steps is taken, and the descent direction d will both be selected so that both these conditions will hold.

In this case if $\{i : 1 \leq i \leq m+1 \text{ and } A_{i.}d < 0\} = \emptyset$, the maximum step length in this descent step is $+\infty$, and hence the original objective function cx in (1) is unbounded below on its set of feasible solutions; and $\{\hat{x} + \alpha d : \alpha \geq \max\{0, \quad (b_i - A_{i.}\hat{x})/(A_{i.}d) : \text{over all } i \text{ satisfying } A_{i.}d > 0\}$ provides a feasible half-line along which cx divereges to $-\infty$, terminate the algorithm with this conclusion.

Figure 11: The point $\hat{x}$ outside of $\underline{\mathrm{K}}$ from which the descent step is taken, and the descent direction d indicated by the arrow, both together satisfy Conditions 1, 2; so the half-line M intersects the interior of $\underline{\mathrm{K}}$. Also, none of the coefficient vectors $A_{i.}$ in the definition of $\underline{\mathrm{K}}$ (defined in Step 1 in Section 3.1) satisfy $A_{i.}d < 0$; so the maximum step length from $\hat{x}$ in the descent direction d is ∞, and $M \cap \underline{\mathrm{K}}$ is a feasible half-line along which $cx \to -\infty$.

Otherwise, take the step length to be $\hat{\gamma} = \min\{(b_i - A_{i.}\hat{x})/(A_{i.}d)$: over all i satisfying $A_{i.}d < 0\} - \epsilon$. So, the output point of this descent step in this case is $\hat{x} + \hat{\gamma}d$ with the objective value at it $= c(\hat{x} + \hat{\gamma}d)$.

Additional notation that we will use in this Descent Cycle

- **Direction** c^i = the orthogonal projection of c^T on $\{x : A_{i.}x = 0\} = c^T - A_{i.}^T[(A_{i.}c^T)/(A_{i.}A_{i.}^T)]$ for each $1 \leq i \leq m+1$.
- **Ball** B. We will denote $B(\bar{x}, \delta(\bar{x}, \underline{\mathrm{K}}))$, the largest ball in $\underline{\mathrm{K}}$ with $\bar{x}$ as center, by B.
- **Touching point** $\bar{x}^i$ = the orthogonal projection of the center $\bar{x}$ on $FH_i = \{x : A_{i.}x = b_i\}$ for each $i \in T(\bar{x}, \underline{\mathrm{K}})$. This $\bar{x}^i = \bar{x} - A_{i.}^T[(A_{i.}\bar{x} - b_i)/(A_{i.}A_{i.}^T)]$ is the point where the ball B touches FH_i for $i \in T(\bar{x}, \underline{\mathrm{K}})$.
- **NTP (Near touching point)** $\hat{x}^i$:) Defined for $i \in T(\bar{x}, \underline{\mathrm{K}}) \cap \{1, ..., m\}$, it is the point $(1-\epsilon)\bar{x}^i + \epsilon\bar{x}$; ϵ-distance away from $\bar{x}^i$ on the line segment joining $\bar{x}^i$ to $\bar{x}$.
- $H(\hat{x})$ = For any point $\hat{x} \in \underline{\mathrm{K}}$, $\{x : cx = c\hat{x}\}$, the objective plane through $\hat{x}$.
- **The set** Γ. This set is used for storing the output points along with the value for the objective function cx at them, from various descent steps taken in this descent cycle in each iteration. In each iteration, initially this set is $\emptyset$. At the end of this descent cycle, the output point in this iteration, is taken as the point in Γ at that stage associated with the smallest objective value.

The following steps are carried out for each $i \in T(\bar{x}, \underline{\mathrm{K}})$.

3.2.2: Descent Step D5.1: This descent step was developed in [Murty 2006a, b; Murty and Oskoorouchi 2008] and gave good results in computational steps reported there.

It is carried out for each $i \in T(\bar{x}, \underline{\mathrm{K}})$, it takes a descent step from the NTP $\hat{x}^i$ in the descent direction $-c^i$. At the end, all the output points obtained along with their objective values are stored in Γ. If the step length in any of these descent steps turns out to be $+\infty$, the half-line generated by the descent step is a feasible half-line along which cx diverges to $-\infty$; and the algorithm is terminated with that conclusion.

Figure 12: $\bar{x}$ is the current center, $T(\bar{x}) = \{1, 2\}$. Descent direction $-c^T$ points down south, $-c^1$ = orthoginal projection of $-c^T$ on the facetal hyperplane of Constraint 1. $\bar{x}^1$ is the touching point on Constraint 1. $\hat{x}^1$ = NTP corresponding to Constraint 1. Descent steps from $\bar{x}$, $\hat{x}^1$ in descent direction $-c^1$ are shown, here descent step from $\hat{x}^1$ leads to higher reduction in objective value.

If no termination occurs, at the end of these descent steps, go to 3.2.3 for the next descent step.

3.2.3: Descent Step D2 in the Direction of the Path of Centers, and the Average GPTC Direction at the Center : For $i \in T(\bar{x}, K)$, the direction $-c^i$ is known as a GPTC (Gradient Projection on Touching Constraint) direction at the current center $\bar{x}$. From the current center $\bar{x}$, take a descent step in the direction of the average of $-c^i$ for $i \in T(\bar{x}, K)$.

Also from the current center $\bar{x}$, take a descent step in the direction of the average of vectors in the set $\{(A_{i.})^T : i \in T(\bar{x}, K)$ and satisfying $c(A_{i.})^T < 0\} \cup \{(-A_{i.})^T : i \in T(\bar{x}, K)$ and satisfying

$c(A_{i.})^T > 0\}$.

This descent step D2 is taken from the current center in the direction ($\bar{x}$ = center in this iteration − center in the previous iteration). It is not used in Iteration 1, but carried out from Iteration 2 onwards.

As in all descent steps in this descent cycle, store the output points obtained in these descent steps, along with their objective values, in the set Γ.

3.2.4: Descent Steps D5.7: Start with the objective plane through the current center $\bar{x}$, $\{x : cx = c\bar{x}\}$, and move it parallel to itself in the direction $-c^T$ until it becomes a tangent plane to the ball B. The point where the objective plane in its new position touches B is

$$\bar{\bar{x}} = \bar{x} - c^T \delta(\bar{x}, \underline{\mathrm{K}})/||c||.$$

If this $\bar{\bar{x}}$ is a boundary point of $\underline{\mathrm{K}}$, i.e., satisfies $A_{i.}\bar{\bar{x}} = b_i$ for some $1 \leq i \leq m$; then $H(\bar{\bar{x}})$ must be the same as $\{x : A_{i.}x = b_i\}$; so $\bar{\bar{x}}$ is an optimum solution for the original LP (1), terminate the algorithm with this conclusion. Otherwise $\bar{\bar{x}}$ is an IFS of $\underline{\mathrm{K}}$, continue.

Figure 13: When the objective plane $H(\bar{x})$ through the center of B is moved parallel to itself in the direction $-c^T$until it becomes a tangent plane to B, touching it at a point $\bar{\bar{x}}$; if $\bar{\bar{x}}$ is a boundary point of $\underline{\mathrm{K}}$, it is an optimum solution of the original LP (1), and $H(\bar{\bar{x}})$ is a facetal hyperplane of $\underline{\mathrm{K}}$ and K.

For each $i \in T(\bar{x}, \underline{\mathrm{K}})$, compute the orthogonal projection $\bar{\bar{x}}^i = \bar{x}^i - [c^T(c\bar{x}^i - c\bar{\bar{x}})]/cc^T$, of the touching point $\bar{x}^i$ on $H(\bar{\bar{x}})$. Notice that $\bar{\bar{x}}^i$ may, or may not, be in $\underline{\mathrm{K}}$. See Figure 14.

INSERT FIGURE 14 HERE.

Figure 14: The ball B with center $\bar{x}$ has 3 touching facets numbered 1, 2, 3 with touching points $\bar{x}^1, \quad \bar{x}^2, \quad \bar{x}^3$ respectively. The objective plane $H(\bar{x})$ is moved parallel to itself in the direction $-c^T$ until it becomes a tangent plane to B, let $\bar{\bar{x}}$ be its touching point with B. We will illustrate Steps 3.2.5, 3.2.6 with the touching facetal hyperplane corresponding to $i = 1$. $\bar{\bar{x}}^1$ is the orthogonal projection of $\bar{x}^1$ on $H(\bar{\bar{x}})$, and the straight line joining $\bar{\bar{x}}, \quad \bar{\bar{x}}^1$ is L_1 (in this 2-dimensional figure it is the same as $H(\bar{\bar{x}})$. In higher dimensions $H(\bar{\bar{x}})$ will be a hyperplane and L_1 will be a straight line on it.). $x^{11}, \quad x^{12}$ are the two boundary points of $\underline{\mathrm{K}}$ on $L_1 \cap \underline{\mathrm{K}}$. All points x on L_1 satisfying the property that the descent line from it in the direction $-c^T$ intersects $\underline{\mathrm{K}}$, are those between $x^{01}(\alpha_{11})$ and $x^{01}(\alpha_{12})$. Minimizing $f^1(\alpha)$ over $\alpha_{11} \leq \alpha \leq \alpha_{12}$ leads to the point $x^{01}(\bar{\alpha})$.

3.2.5: Finding Line Segments Intersecting the Interior of $\underline{\mathrm{K}}$ on the Objective Plane $H(\bar{\bar{x}})$

Now carry out the following work for each $i \in T(\bar{x}, \underline{\mathrm{K}})$.

Define the straight line joining $\bar{\bar{x}}$ and $\bar{\bar{x}}^i$

$L_i = \{x^i(\gamma) = \bar{\bar{x}}^i + \gamma(\bar{\bar{x}} - \bar{\bar{x}}^i) : \gamma \text{ takes real values } \}$

in parametric representation with parameter γ, and contained on the objective plane $H(\bar{\bar{x}})$.

Since $x^i(\gamma = 1) = \bar{\bar{x}}$, an interior point of $\underline{\mathrm{K}}$; L_i passes through the interior of $\underline{\mathrm{K}}$, and so it must interesct $\underline{\mathrm{K}}$ at two of its boundarey points if $L_i \cap \underline{\mathrm{K}}$ is a line segment. To find those boundary points of intersection, we need to solve the following system of linear inequalities in the parameter γ.

$$\text{for all } p = 1 \text{ to } m+1 \qquad A_{p.}x^i(\gamma) - b_p = A_{p.}\bar{\bar{x}}^i - b_p + \gamma A_{p.}(\bar{\bar{x}} - \bar{\bar{x}}^i) \geq 0 \tag{7}$$

The interval of values of γ feasible to (7) can be found using Subroutine 1, suppose it is $\gamma_{i1} \leq \gamma \leq \gamma_{i2}$. There are three possibilities here, which we will discuss separately.

Possibility 1: $\gamma_{i1} = -\infty$. This possibility occurs if $A_{p.}(\bar{\bar{x}} - \bar{\bar{x}}^i) \leq 0$ for all $p = 1$ to $m+1$.

If $A_{p.}(\bar{\bar{x}} - \bar{\bar{x}}^i) < 0$ for all $p = 1$ to $m+1$, from Theorem 1 we verify that $\delta(x^i(\gamma), \underline{\mathrm{K}})$ diverges to $+\infty$ as $\gamma \to -\infty$. In this case the point $x^i(\gamma) - \delta(x^i(\gamma), \underline{\mathrm{K}})[c^T/||c||]$ is feasible for all $\gamma \to -\infty$, and objective value at it diverges to $-\infty$.

We know that $\delta(x^i(\gamma)) = \min\{A_{i.}x^i(\gamma) - b_i \;=\; A_{i.}(\bar{\bar{x}}^i + \gamma(\bar{\bar{x}} - \bar{\bar{x}}^i)) - b_i : i = 1 \text{ to } m+1\}$. This and the fact that $\delta(x^i(\gamma)) \to \infty$ as $\gamma \to -\infty$ implies that $A_{i.}(\bar{\bar{x}} - \bar{\bar{x}}^i) < 0$ for all $i = 1$ to $m+1$. So, if β is a small positive number $< \min\{-A_{i.}(\bar{\bar{x}} - \bar{\bar{x}}^i) : i = 1 \text{ to } m+1\}$, then the growth rate in $\delta(x^i(\gamma))$ as γ decreases is going to be larger than β. These facts show that the half-line $\{\bar{\bar{x}}^i + [(\bar{\bar{x}} - \bar{\bar{x}}^i) + \beta(-c^T)]\gamma : \gamma \leq \gamma_{i2}\}$ is feasible and the objective value diverges to $-\infty$ along it. So, we terminate the algorithm with this conclusion in this case.

If $A_{p.}(\bar{\bar{x}} - \bar{\bar{x}}^i) \leq 0$ for all $p = 1$ to $m+1$, but $= 0$ for some p, then carry out descent steps D5.1, D2 from $x^i(\gamma)$ for some $\gamma < 0$, and store the output points from this steps in the set Γ. Then with the best point in Γ by objective value at this stage as the initial IFS go to the next iteration in the algorithm.

Possibility 2: $\gamma_{i1} = +\infty$, which occurs if $A_{p.}(\bar{\bar{x}} - \bar{\bar{x}}^i) \geq 0$ for all $p = 1$ to $m+1$. All the conclusions are the same as in Possibility 1; with the change that $A_{p.}(\bar{\bar{x}} - \bar{\bar{x}}^i) \leq 0$ (or < 0) there need to be replaced by $A_{p.}(\bar{\bar{x}} - \bar{\bar{x}}^i) \geq 0$ (or > 0), and "γ negative " by "γ positive ".

Possibility 3: Both $\gamma_{i1} < \gamma_{i2}$ are finite.

In this case continue the algorithm by moving to Section 3.2.6.

3.2.6: Extending the line segments $L_i \cap \underline{\mathbf{K}}$ for Carrying Out Descent Steps

We found the line segment $L_i \cap \underline{\mathrm{K}} = \{x^i(\gamma) : \gamma_{I1} \leq \gamma \leq \gamma_{i2}$ on $H(\bar{\bar{x}})$ interesecting the interior of $\underline{\mathrm{K}}$ above.

Relabel the Line L_i: For simplicity we denote

$x^i(\gamma_{i1}) = x^{i1}, \qquad x^i(\gamma_{i2}) = x^{i2}.$

Then $L_i \cap \underline{\mathrm{K}}$ is the interval $[x^{i1}, \quad x^{i2}]$, the line segment joining x^{i1}, x^{i2}. We will now express L_i as the straight line joining x^{i1}, x^{i2} in terms of the new parameter α:

$L_i = \{x^{0i}(\alpha) = x^{i1} + \alpha(x^{i2} - x^{i1}) : \alpha \in R^1\}$

Then $L_i \cap \underline{\mathrm{K}} = \{x^{0i}(\alpha) : 0 \leq \alpha \leq 1\}$. Even when $\alpha \notin [0,1]$, it is possible that a descent step taken from $x^{0i}(\alpha)$ in the direction $-c^T$ leads to an output point in $\underline{\mathrm{K}}$. The condition to be satisfied for this, is that $x^{0i}(\alpha) + \lambda(-c^T) \in \underline{\mathrm{K}}$ for some $\lambda \geq 0$.

Determining the interval of values of α such that $x^{0i}(\alpha) + \lambda(-c^T) \in \underline{\mathbf{K}}$ for some $\lambda \geq 0$: We will Denote the interval by $\alpha_{i1} \leq \alpha \leq \alpha_{i2}$. This interval includes $[0,1]$, here we will determine α_{i1}, α_{i2}.

We consider three cases that can arise now. Define:

$I_1 = \{p : A_{p.}(-c^T) > 0\}, \qquad I_2 = \{p : A_{p.}(-c^T) < 0\}$

Case 1: $I_1 = \emptyset$**:** So $A_{p.}(-c^T) \leq 0$ for all $1 \leq p \leq m+1$. In this case for each p, the quantity $A_{p.}(x^{0i}(\alpha) + \lambda(-c^T))$ keeps decreasing as λ increases from 0. So, for this quantity to be $\geq b_i$ for some $\lambda \geq 0$, it must be $\geq b_i$ for $\lambda = 0$. So we need $A_{p.}x^{0i}(\alpha) - b_p \geq 0$ for all $1 \leq p \leq m+1$. So in this case the interval is $\alpha_{i1} = 0 \leq \alpha \leq \alpha_{i2} = 1$.

Case 2: $I_2 = \emptyset$**:** So, $A_{p.}(-c^T) \geq 0$ for $1 \leq p \leq m+1$. For any $0 \leq \alpha \leq 1$ the descent step from $x^{0i}(\alpha) \in \underline{\mathrm{K}}$ in the direction $-c^T$ has step length $= +\infty$ in this case, since $A_{p.}(x^{0i}(\alpha) - b_p +$

$\lambda A_{p.}(-c^T) \geq 0$ for all $\lambda \geq 0$, $1 \leq p \leq m+1$ in this case.

So for any $0 \leq \alpha \leq 1$, $\{x^{0i}(\alpha) + \lambda(-c^T) : \lambda \geq 0\}$ is a feasible half-line along which the objective function cx in (1) diverges to $-\infty$. So, we terminate the algorithm with this conclusion in this case.

Case 3: Both I_1, I_2 are nonempty:

From Subroutine 1, for a fixed value of α, for the system of inequalities

$$A_{p.}(x^{0i}(\alpha + \lambda(-c^T) - b_p = [A_{p.}x^{i1} - b_p + \alpha(A_{p.}(x^{i2} - x^{i1})] + \lambda(A_{p.}(-c^T)) \geq 0,$$

for all $1 \leq p \leq m+1$; to have a solution in λ for λ real; the condition to be satisfied is

$$\text{Max}\{-[A_{p.}x^{i1} - b_p + \alpha(A_{p.}(x^{i2} - x^{i1})]/(A_{p.}(-c^T)) : \text{over all } p \in I_1\} \quad \leq$$
$$\text{Min}\{-[A_{t.}x^{i1} - b_t + \alpha(A_{t.}(x^{i2} - x^{i1})]/(A_{t.}(-c^T)) : \text{over all } t \in I_2\}.$$

Further since we need this system to have a solution in $\lambda \geq 0$, we need in addition to the above, the following condition also to hold

$$\text{Min}\{-[A_{t.}x^{i1} - b_t + \alpha(A_{t.}(x^{i2} - x^{i1})]/(A_{t.}(-c^T)) : \text{over all } t \in I_2\} \geq 0.$$

Therefore $\{x^{0i}(\alpha)+\lambda(-c^T) : \lambda \geq 0\} \cap \underline{\text{K}} \neq \emptyset$, iff α satisfies the following system of inequalities:

$$-[A_{p.}x^{i1} - b_p + \alpha(A_{p.}(x^{i2} - x^{i1})]/(A_{p.}(-c^T)) + [A_{t.}x^{i1} - b_t + \alpha(A_{t.}(x^{i2} - x^{i1})]/(A_{t.}(-c^T)) \leq \quad 0 \quad \text{for all } p \in I_1, \quad t \in I_2, \text{ and}$$

$$-[A_{t.}x^{i1} - b_t + \alpha(A_{t.}(x^{i2} - x^{i1})]/(A_{t.}(-c^T)) \quad \geq \quad 0 \quad \text{for all } t \in I_2.$$

This is a system of linear inequalities in the parameter α, and we know that this system is feasible for all $0 \leq \alpha \leq 1$; using Subroutine 1 find the complete interval of values of α ;

$\alpha_{i1} \leq \alpha \leq \alpha_{i2}$ feasible to this system. We know that $\alpha_{i1} \leq 0$, and $\alpha_{i2} \geq 1$; and both are finite.

3.2.7: Carrying out a Line Serach Step over the Line Segment $\{x^{0i}(\alpha) = x^{i1} + \alpha(x^{i2} - x^{i1}) : \alpha_{i1} \leq \alpha \leq \alpha_{i2}\}$**:**

For $\alpha_{i1} \leq \alpha \leq \alpha_{i2}$, when a descent step is taken from $x^{0i}(\alpha)$ in the direction $-c^T$, the maximum step length is

$\lambda(\alpha) = -\epsilon + \text{minimum}\{(b_p - A_{p.}x^{0i}(\alpha))/(A_{p.}(-c^T))$ over all $1 \leq p \leq m+1$ satisfying $A_{p.}(-c^T) < 0\}$.

and the output point from this descent step is $x^{0i}(\alpha) + \lambda(\alpha)(-c^T)$. The objective value at this output point is

$$f^i(\alpha) = c[x^{0i}(\alpha) + \lambda(\alpha)(-c^T)]$$

Clearly $f^i(\alpha)$ defined over the interval $\alpha_{i1} \leq \alpha \leq \alpha_{i2}$ is piecewise linear convex. Carry out a line search step to

$$\text{Minimize} \quad f^i(\alpha) \quad \text{over } \alpha_{i1} \leq \alpha \leq \alpha_{i2}.$$

This line search step can be carried out by any of the algorithms discussed for it in NLP literature, for example, the Quadratic Fit Line Search Method, see the textbook, Murty[1995].

Let $\bar{\alpha}$ be the optimum value of α obtained in this line search step. Then the output point from this line search step is $x^{0i}(\bar{\alpha})$ with its objective value $f^i(\bar{\alpha}) = cx^{0i}(\bar{\alpha})$. Store the point $x^{0i}(\bar{\alpha})$, with its objective value in the set Γ.

When these line search steps are completed for all $i \in T(\bar{x}, \underline{\text{K}})$, this descent cycle is completed. This is also the termination of this iteration in SM-5

The output point in this iteration is the point in the set Γ associated with the smallest objective value at this stage. Suppose it is $\hat{x}$, with objective value $c\hat{x}$. With $\hat{x}$ as the initial IFS the algorithm now moves to the next iteration. The change in the objective value in this iteration is the difference in the objective values at the initial IFS and the final output point in this iteration, i.e., $c\underline{\mathrm{x}} - c\hat{x}$.

The algorithm is terminated in an iteration if the change in objective value attained in that iteration is $\leq \epsilon$, with the final output point in that iteration taken as the approximate optimum solution of (1).

4 Sphere Method 6 (SM-6)

SM-6 is only slightly different from SM-5. In SM-6 the set of feasible solutions under consideration is not updated as in Step 1 of the centering cycle in SM-5, it is the same as the original set of feassible solutions K throught the algorithm. Thus the centering step in SM-6 is a lot simpler than that in SM-5; but the descent steps in SM-6 are the same as in SM-5. As in SM-5, each iteration in SM-6 begins with an initial IFS of K, and terminates by producing an output IFS of K at the end. The first iteration begins with an initial IFS given in the input data for the problem. Subsequent iterations begin with the output IFS of the previous iteration.

We will now describe a general iteration in SM-6 beginning with the initial IFS of K, say $\bar{x}$, for it.

4.1: Centering Cycle in this Iteration of SM-6

$\bar{x}$ denotes the initial IFS of K with which this iteration begins.

Step C1: Finding an approximate center beginning with the initial IFS $\bar{x}$: Beginning with $\bar{x}$ carry out all the work described in Sections 3.2.4, 3.2.5 with $\underline{\mathrm{K}}$ replaced by the original set of feasible solutions K defined in (1) with m constraints and not $m+1$ (so, $m+1$ in those sections should be replaced by m for use here).

In this work in Step C1 here, if termination did not occur, we would have obtained line segments $L_i \cap K = \{x^i(\gamma) : \gamma_{i1} \leq \gamma \leq \gamma_{i2}\}$ contained inside K from one end of K to the other, for each $i \in T(\bar{x}, K)$, all lying on the objective plane $H(\bar{\bar{x}})$.

There are two strategies, 1 and 2, that can be used to select the center. Strategy 2 is simpler than Strategy 2, but the best among them has to be determined based on computational tests.

STRATEGY 1:

For each $i \in T(\bar{x}, K)$, compute $\delta((x^i(\gamma_{i1}) + x^i(\gamma_{i2})/2, K)$, and find an index $i \in T(\bar{x}, K)$ that corresponds to the maximum value of this quantity. Suppose this maximum is attained by an index r. Then the output point from this Step C1 beginning with the intial IFS $\bar{x}$ of K is $\bar{\bar{x}}^C = (x^r(\gamma_{r1}) + x^r(\gamma_{r2}))/2$, the mid-point of $L_r \cap K$.

STRATEGY 2: In this strategy select r to be the i that maximizes $||x^i(\gamma_{i1}) - x^i(\gamma_{i2})||$ among $i \in T(\bar{x}, K)$ and take $\bar{\bar{x}}^C = \bar{\alpha} x^i(\gamma_{i1}) + (1 - \bar{\alpha}) x^i(\gamma_{i2})$ where $\bar{\alpha}$ is an α that maximizes $\delta(\alpha x^r(\gamma_{r1}) + (1 - \alpha) x^r(\gamma_{r2})$ over $0 \leq \alpha \leq 1$.

Here we summarize the notation used in this step:

$\bar{x}$ = Initial IFS of K for this step.
$\bar{\bar{x}} = \bar{x} - \delta(\bar{x}, K) c^T / ||c||$, Bottom point of $B(\bar{x}, K)$ in the direction $-c^T$.
$\bar{x}^i = \bar{x} - A_{i.}^T[(A_{i.}\bar{x} - b_i)/(A_{i.}A_{i.}^T)]$, the point where FH_i touches $B(\bar{x}, K)$ for $i \in T(\bar{x}, K)$.
$\bar{\bar{x}}^i = \bar{x}^i - c^T[(c\bar{x}^i - c\bar{\bar{x}})/cc^T]$, the orthogonal projection of $\bar{x}^i$ on $H(\bar{\bar{x}})$ for $i \in T(\bar{x}, K)$.
$L_i = \{x^i(\gamma) = \bar{\bar{x}}^i + \gamma(\bar{\bar{x}} - \bar{\bar{x}}^i)$, Line joining $\bar{\bar{x}}^i$ and $\bar{\bar{x}}$ for $i \in T(\bar{x}, K)$. $L_i \cap K = \{x^i(\gamma) : \gamma_{i1} \leq \gamma \leq \gamma_{i2}\}$ for $i \in T(\bar{x}, K)$.
$\bar{\bar{x}}^C$: If Strategy 1 is used, r is an $i \in T(\bar{x}, K)$ that corresponds to the maximum value of $\delta((x^i(\gamma_{i1}) + x^i(\gamma_{i2}))/2, K)$ and this $\bar{\bar{x}}^C = x^r(\gamma_{r1}) + x^r(\gamma_{r2}))/2$. If Strategy 2 is used, r is an $i \in T(\bar{x}, K)$ that maximizes $||x^i(\gamma_{i1}) - x^i(\gamma_{i2})||$ and $\bar{\alpha}$ is an α that maximizes $\delta(\alpha x^r(\gamma_{r1}) + (1 - \alpha) x^r(\gamma_{r2}))$ over $0 \leq \alpha \leq 1$.
Go to Step C2 next.

Step C2: Finding an approximate center beginning with $\bar{\bar{x}}^C$, the output point from Step C1

Carry out Step C1 beginning with $\bar{\bar{x}}^C$ instead of $\bar{x}$, and suppose the output point obtained is $\bar{\bar{\bar{x}}}^C$. This point $\bar{\bar{\bar{x}}}^C$ is called the **Center** in this iteration in SM-6.

Here we summarize the notation used in this step:

$\bar{\bar{x}}^C$ = Initial IFS of K for this step.

$\bar{\bar{\bar{x}}} = \bar{\bar{x}}^C - \delta(\bar{\bar{x}}^C, K)c^T/||c||$, Bottom point of $B(\bar{\bar{x}}^C, K)$ in the direction $-c^T$.

$\bar{\bar{x}}^i = \bar{\bar{x}}^C - A_{i.}^T[(A_{i.}\bar{\bar{x}}^C - b_i)/(A_{i.}A_{i.}^T)]$, in this step, the point where FH_i touches $B(\bar{\bar{x}}^C, K)$ for $i \in T(\bar{\bar{x}}^C, K)$.

$\bar{\bar{\bar{x}}}^i = \bar{\bar{x}}^i - c^T[(c\bar{\bar{x}}^i - c\bar{\bar{\bar{x}}})/cc^T]$, the orthogonal projection of $\bar{\bar{x}}^i$ on $H(\bar{\bar{\bar{x}}})$ for $i \in T(\bar{\bar{x}}^C, K)$.

$L_i = \{x^i(\gamma) = \bar{\bar{\bar{x}}}^i + \gamma(\bar{\bar{\bar{x}}} - \bar{\bar{\bar{x}}}^i)$, Line joining $\bar{\bar{\bar{x}}}^i$ and $\bar{\bar{\bar{x}}}$ for $i \in T(\bar{\bar{x}}^C, K)$.

$L_i \cap K = \{x^i(\gamma) : \gamma_{i1} \leq \gamma \leq \gamma_{i2}\}$ for $i \in T(\bar{\bar{x}}^C, K)$.

$\bar{\bar{\bar{x}}}^C$: If Strategy 1 is used, r is an $i \in T(\bar{\bar{x}}^C, K)$ that corresponds to the maximum value of $\delta((x^i(\gamma_{i1}) + x^i(\gamma_{i2}))/2, K)$ and this $\bar{\bar{\bar{x}}}^C = x^r(\gamma_{r1}) + x^r(\gamma_{r2}))/2$. If Strategy 2 is used, r is an $i \in T(\bar{\bar{x}}^C, K)$ that maximizes $||x^i(\gamma_{i1}) - x^i(\gamma_{i2})||$ and $\bar{\alpha}$ is an α that maximizes $\delta(\alpha x^r(\gamma_{r1}) + (1-\alpha)x^r(\gamma_{r2})$ over $0 \leq \alpha \leq 1$.

Now go to the Descent steps in this iteration.

4.2: Descent steps in this iteration in SM-6

Select a set Γ, initially $= \emptyset$, for storing the output points with their objective values (for $z = cx$) generated in each of the descent steps carried out below.

Descent step D5.1: The current center is $\bar{\bar{\bar{x}}}^C$. For each $i \in T(\bar{\bar{\bar{x}}}^C, K)$, compute the **touching point** x^i of $B(\bar{\bar{\bar{x}}}^C, K)$ with FH_i, $x^i = \bar{\bar{\bar{x}}}^C - A_{i.}^T[(A_{i.}\bar{\bar{\bar{x}}}^C - b_i)/||A_{i.}||^2]$. Then $\hat{x}^i = \epsilon\bar{\bar{\bar{x}}}^C + (1-\epsilon)x^i$ is called the **NTP (Near touching point)** of $B(\bar{\bar{\bar{x}}}^C, K)$ with FH_i. It is the point ϵ distance away from x^i on the line segment joining x^i to $\bar{\bar{\bar{x}}}^C$, where ϵ is a small positive tolerance.

For each $i \in T(\bar{\bar{\bar{x}}}^C, K)$, take a descent step from the NTP $\hat{x}^i$ in the descent direction $-c^i$. Store the output points from each of these descent steps along with their objective values in the set Γ.

Descent Step D5.7 in SM-6:

We obtained line segments $L_i \cap K = \{\bar{\bar{\bar{x}}}^i(\gamma) : \gamma_{i1} \leq \gamma \leq \gamma_{i2}\}$ on the objective plane $H(\bar{\bar{\bar{x}}})$ for $i \in T(\bar{\bar{x}}^C, K)$ in Step C2. On these line segments on $H(\bar{\bar{\bar{x}}})$ execute the work in Step 3.2.6, 3.2.7 of SM-5 replacing $\underline{\mathrm{K}}$ with K , to extend these line segments and carry out line search to minimize $f^i(\alpha)$ on them for all $i \in T(\bar{\bar{x}}^C, K)$. Store the output points with the objective values at them in the set Γ

Other Descent steps: From the center $\bar{\bar{\bar{x}}}^C$, take descent steps in the directions $\bar{\bar{\bar{x}}}^C - \bar{\bar{x}}^C$, $-c^T$, and the average of the GPTC directions at the current center $\bar{\bar{\bar{x}}}^C$.

Also from the current center $\bar{\bar{\bar{x}}}^C$, take a descent step in the direction of the average of vectors in the set $\{(A_{i.})^T : i \in T(\bar{\bar{\bar{x}}}^C, K)$ and satisfying $c(A_{i.})^T < 0\} \cup \{(-A_{i.})^T : i \in T(\bar{\bar{\bar{x}}}^C, K)$ and satisfying $c(A_{i.})^T > 0\}$.

Store the output points from each of these descent steps along with the objective values at them, in the set Γ.

After all these descent steps, select the point in Γ corresponding to the least objective value as the new $\bar{x}$, reset the set Γ to be the empty set, and with the new $\bar{x}$ go to the next iteration.

Terminate the method when the change in objective value in an iteration falls below a selected tolerance. In the final iteration, take the point in Γ corresponding to the least objective value as an approximate optimum solution of (1).

5 Sphere Methods for Linearly Constrained Nonlinear Programming (NLP) Models

We consider the NLP

$$\begin{aligned} \min \quad & \theta(x) \\ \text{subject to} \quad & Ax \geq b \end{aligned} \tag{8}$$

where A is a data matrix of order $m \times n$; with a known IFS x^1, and $\theta(x)$ is a continuously differentiable nonlinear function of $x \in R^n$. We will continue using the notation developed earlier. $\nabla\theta(x)$ denotes the gradient vector of $\theta(x)$ at x written as a row vector.

Here we will assume that K, the set of feasible solutions of the problem, is bounded; so, (8) has an optimum solution. Also in this section, we will consider the case where $\theta(x)$ is a **convex function**. Extension to the case when $\theta(x)$ is nonconvex, or when some of the variables in x are required to be binary will be discussed in the next section. We begin discussing two results that we will use.

Result 1: Let $\hat{x} \in K$ satisfy $\nabla\theta(\hat{x}) \neq 0$. Then there exists an optimum solution x^0 of (8) satisfying $\nabla\theta(\hat{x})x^0 \leq \nabla\theta(\hat{x})\hat{x}$.

Proof: Since $\theta(x)$ is a convex function, we know that it satifies the *gradient support inequality* [2009b] at $\hat{x}$

$$\theta(x) - \theta(\hat{x}) \geq \nabla\theta(\hat{x})(x - \hat{x}) \qquad \text{for all} \quad x \in R^n.$$

Let x^0 be an optimum solution of (8). Substituting $x = x^0$ in the above gradient support inequality, we get $\theta(x^0) - \theta(\hat{x}) \geq \nabla\theta(\hat{x})(x^0 - \hat{x})$. But since x^0 is an optimum solution of (8), we have $0 \geq \theta(x^0) - \theta(\hat{x}) \geq \nabla\theta(\hat{x})(x^0 - \hat{x})$; so $\nabla\theta(\hat{x})(x^0 - \hat{x}) \leq 0$. $\bowtie$

The next result is the basis for the first method in Constrained NLP, the Frank-Wolfe Method.

Result 2: If $\hat{x} \in K$ minimizes the linear function $\nabla\theta(\hat{x})x$ over $x \in K$, then $\hat{x}$ is an optimum solution of the original NLP (8).

Proof: Suppose $\hat{x}$ satisfies $\nabla\theta(\hat{x}) = 0$. Since $\theta(x)$ is a convex function, in this case $\hat{x}$ is an optimum solution of (8).

Now consider the case $\nabla\theta(\hat{x}) \neq 0$. Since $\hat{x}$ minimizes $\nabla\theta(\hat{x})x$ over $x \in K$, by the Duality Theorem of Linear Programming, there exists a dual vector, a row, $\hat{\pi} = (\hat{\pi}_1, ..., \hat{\pi}_m)$ such that $\hat{x}, \hat{\pi}$ together satisfy the dual feasibility and complementary slackness optimality conditions:

$$\nabla\theta(\hat{x}) - \hat{\pi}A = 0; \qquad \hat{\pi} \geq 0; \qquad \hat{\pi}_i(A_{i.}\hat{x} - b_i) = 0, \qquad i = 1, ..., m.$$

Together with $A\hat{x} \geq b$, these are the KKT optimality conditions for $\hat{x}$ to be an optimum solution of (8). $\bowtie$

5.1 Descent Steps Based on SM-5, 6 for LP

We will consider the general iteration in which the current solution is the initial IFS, $\bar{x}$, say. By Result 1, we know that we can get descent directions at the IFS $\bar{x}$ for our original problem (8), or show that $\bar{x}$ is itself an optimum solution of (8), by solving the LP (9) below.

$$\begin{array}{rll} \min & \nabla\theta(\bar{x})x & \\ \text{subject to} & A_{i.}x \geq b_i & i = 1, ..., m+1 \end{array} \tag{9}$$

where $A_{m+1.} = -\nabla\theta(\bar{x})$, $b_{m+1} = \nabla\theta(\bar{x})\bar{x} - \epsilon$, ϵ a small positive tolerance to guarantee that $\bar{x}$ is an IFS of the set of feasible solutions, $\bar{K}$ of (9). Now consider the LP (9).

Apply one iteration of SM-5 or SM-6 to solve this LP with the current solution $\bar{x}$ as the initial IFS. Notice that in carrying out this iteration of SM-5 on (9), due to the $(m+1)$th constraint in (9), the set of feasible solutions of this LP, $\bar{K}$, is already updated by the objective value at the current initial IFS $\bar{x}$, so it is not necesssary to begin the centering cycle with Step 1 in Section 3.1, it can begin directly with Step 2.

When this iteration of SM-5 or SM- 6 on (9) terminates, if we conclude that $\bar{x}$ itself is an optimum solution of (9), then by Result 2 we conclude that $\bar{x}$ is optimum to the original NLP (8), and we terminate. Otherwise we would have the set Γ = set of all output points of descent steps for (9) carried out in this iteration. So, at this stage, for each $\hat{x} \in \Gamma$, $\nabla\theta(\bar{x})\hat{x} < \nabla\theta(\bar{x})\bar{x}$, or $\nabla\theta(\bar{x})(\hat{x} - \bar{x}) < 0$; i.e., $(\hat{x} - \bar{x})$ is a descent direction for $\theta(x)$ at $\bar{x}$.

Set up a set ΓN for storing the output points along with the objective value of $\theta(x)$ at them, from the various descent steps that will be carried out in this iteration; this set ΓN is $\emptyset$ initially.

Now for each $\hat{x} \in \Gamma$, apply a line search step to minimize $\theta(\bar{x} + \alpha(\hat{x} - \bar{x}))$ over $0 \leq \alpha \leq 1$. If $\hat{\alpha}$ is the optimum value of α obtained in this step, store the output point $\bar{x} + \hat{\alpha}(\hat{x} - \bar{x})$, along with its objective value $\theta(\bar{x} + \hat{\alpha}(\hat{x} - \bar{x}))$ in the set ΓN.

5.2: Descent Step in the Trust Region Descent Direction

Let $Q(x)$ be the 2nd order Taylor Series Approximation for $\theta(x)$ at the current solution $\bar{x}$. The problem of minimizing $Q(x)$ over the ball $B(\bar{x}, \bar{K})$ is known in the literature as a trust region problem. Efficient software implementations of an algorithm for solving it are available (see http://galahad.rl.ac.uk/galahad-www/ , also Matlab has programs for solving it) whether $Q(x)$ is convex or not.

Let $\underline{\underline{x}}$ be the optimum solution obtained for this problem. If $\nabla\theta(\underline{\underline{x}}) = 0$, since $\theta(x)$ is convex, we conclude that $\underline{\underline{x}}$ is an optimum solution of the orginal problem (8), and we terminate the algorithm. Otherwise, assuming that $Q(x)$ is a goood approximation for $\theta(x)$ inside $B(\bar{x}, \bar{K})$, $\underline{\underline{x}} - \bar{x}$ will be a descent direction for $\theta(x)$ at $\bar{x}$.

Use Subroutine 1 to find, $\bar{\gamma}$, the maximum value of the parameter γ that keeps the point $\bar{x} + \gamma(\underline{\underline{x}} - \bar{x})$ feasible to the original NLP (8). Then apply a line search step to minimize $\theta(\bar{x} + \alpha(\underline{\underline{x}} - \bar{x}))$ over $0 \leq \alpha \leq \bar{\gamma} - \epsilon$. If $\bar{\alpha}$ is the optimum value of α obtained in this step, store the output point $\bar{x} + \bar{\alpha}(\underline{\underline{x}} - \bar{x})$, along with its objective value $\theta(\bar{x} + \bar{\alpha}(\underline{\underline{x}} - \bar{x}))$ in the set ΓN.

5.3: Descent Step Based of Facetal Normal Dirctions

Find directions d in the set $\{A_{i.}^T, -A_{i.}^T : i \in T(\bar{x}, K)\}$ [CHECK WHETHER CHANGING THIS TO $i = 1, ...m$ HERE GIVES BETTER PERFORMANCE] which are descent directions at $\bar{x}$ (i.e., satisfy $\nabla\theta(\bar{x})d < 0$), and let y be the average of these directions. Find the maximum value of γ for $\bar{x} + \gamma y$ to be $\in K$ using Subroutine 1, and suppose it is $\bar{\gamma}$.

Apply a line search step to minimize $\theta(\bar{x} + \alpha y)$ over $0 \leq \alpha \leq \bar{\gamma} - \epsilon$. If $\bar{\alpha}$ is the optimum value of α, store the output point $\bar{x} + \bar{\alpha} y$ along with its objective value $\theta(\bar{x} + \bar{\alpha} y)$ in the set ΓN.

5.4: Descent Steps Based on the Touching Set Of Constraints

When $\bar{x}$ is the current IFS this technique uses a direction y that is a solution of

$$A_{i.}y = 1 \qquad \text{for all } i \in T(\bar{x}, K) \tag{10}$$

that also satisfies $\nabla\theta(\bar{x})y < 0$. To get such a direction, it finds a basic solution for (10). In this process suppose the basic vector obtained for (10) is $(y_1, ..., y_p)$, eliminating any redundant constraints in (10) found. Let $c = \nabla(\bar{x})$, and include "$cy = 0$" in another row vector under those in (10).

We will solve this system of linear equations using the efficient GJ (Gauss-Jordan) Method with the explicit basis inverse (see, Section 1.2.2 in Murty[2010]), then it generates $\mathcal{B}^{-1}$, the inverse of the basis of (10) corresponding to the basic vector $(y_1, ..., y_p)$ for (10). Every column vector in the canonical tableau of (10) with respect to this basic vector can be calculated using the well known formulas (see [Murty 2009b]) and the data in the original system (10). However, we will describe the descent steps carried out here using the canonical tableau displayed below, with the basic variables priced out in the last row.

y_1	....	y_p	y_{p+1}	.....	y_n	RHS
1	....	0	$\bar{a}^1_{p+1}$	.....	$\bar{a}^1_n$	$\bar{y}_1$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	.....	$\vdots$	$\vdots$
0	....	1	$\bar{a}^p_{p+1}$	.....	$\bar{a}^p_n$	$\bar{y}_p$
0	....	0	$\bar{c}_{p+1}$	.....	$\bar{c}_n$	Δ

Then indices $i = 1$ to p corresponding to the top p rows in the above canonical tableau are contained in $T(\bar{x}, K)$ corresponding to rows in (10). There are two cases to consider here.

Case 1: If $\bar{c}_j = 0$ for all $j = p+1$ to n: Then the vector $c = \nabla\theta(\bar{x})$ is a linear combination of $\{A_{i.} : i \in T(\bar{x}, K)\}$, these indices $i \in T(\bar{x}, K)$ correspond to the top rows 1 to p in the above canonical tableau. There are two subcases to consider here.

Subcase 1.1: $\Delta > 0$: In this subcase , $\bar{y} = (\bar{y}_1, ..., \bar{y}_p, 0, ..., 0)^T$ is a descent direction for $\theta(x)$ at $\bar{x}$. Find $\alpha_1 =$ the value of α that minimizes $\theta(\bar{x} + \alpha\bar{y})$ within K. If $\nabla\theta(\bar{x} + \alpha_1 y) = 0$, the conclude that $\bar{x} + \alpha_1 y$ is an optimum solution of the original NLP (8) , and terminate the algorithm. Otherwise, continue.

Now if $\bar{x}+\alpha_1 y$ is an interior point of K, i.e., satisfies all the constraints in (8) as strict inequalities, then take it as the output point of this line search operation. On the other hand, if $\bar{x}+\alpha_1 y$ is a boundary point of K, i.e., satisfies at least one of l the constraints in (8) as an equation, then take $\bar{x}+(\alpha_1-\epsilon)y$ as the output point of this line search operation. Store the output point in this line search operation, along with the value of $\theta(x)$ at it in the set ΓN.

Subcase 1.2: $\Delta < 0$:

Denote $c_B=(c_1,...,c_p)$, the row vector of elements in $c=\nabla\theta(\bar{x})$ corresponding to the basic variables in $(y_1,...,y_p)$, and $\bar{\pi}_B=c_B-B^{-1}$, $\bar{\pi}=(\bar{\pi}_B,0,...,0)\in R^m$. Then, $c=\sum_{i=1}^{p}\bar{\pi}_i A_{i.}=\bar{\pi}A$.

At this stage if $\delta(\bar{x},K)$ is small, and the vector $\bar{\pi}$ computed above is ≥ 0, then "$c=(\sum_{i=1}^{p}\bar{\pi}_i A_{i.})=(\bar{\pi}A)$, $\bar{\pi}\geq 0$, $\bar{\pi}_i=0$ for $i\notin\{1,...,p\}$" are an approximate form of the KKT optimality conditions for (8) for $\bar{x}$. So we terminate with the conclusion that $\bar{x}$ is an approximate optimum to the original NLP (8).

Suppose $\bar{\pi}\ngeq 0$. In this subcase there is no descent profitable direction at $\bar{x}$; so take $\bar{x}$ as the approximate ball center in this iteration. Apply the descent step in the Trust region descent direction (Section 5.2) at $\bar{x}$, and continue.

Case 2: One or more of $\bar{c}_j$ are $\neq 0$: Let $J=\{j: p+1\leq j\leq n, \bar{c}_j\neq 0\}$. Carry out the following for each $j\in J$:

The solution to (10) obtained by keeping all nonbasic variables other than y_j at 0, while giving the nonzero value β to y_j in the canonical tableau corresponding to the basic vector $(y_1,...,y_p)$ is:

$$y(\beta)=(\bar{y}_1-\bar{a}_j^1\beta,...,\bar{y}_p-\beta\bar{a}_j^p,0...0,\beta,0,...,0)^T$$

in which β is the value given to the nonbasic variable y_j. We will call the interval of values of β, in which the value of cx at $x=y(\beta)$, which is $\beta\bar{c}_j-\Delta$ is <0 as the **descent range** for β. Let $\beta_1=\Delta/\bar{c}_j$. This range is unbounded and it is :

$\beta<\beta_1$ if $\bar{c}_j>0$, or $\beta>\beta_1$ if $\bar{c}_j<0$

Moving in the direction $y(\beta)$ from the current IFS $\bar{x}$ a nonnegative step length of α yields the point $\bar{x} + \alpha y(\beta)$. For each fixed value $\bar{\beta}$ of β in its descent range, the point $\bar{x} + \alpha y(\bar{\beta})$ will be feasible to the original NLP (8) iff

$$A_{i.}(\bar{x} + \alpha y(\bar{\beta})) \geq b_i \text{ for all } i = 1 \text{ to } m.$$

Solving this systems of inequalities in α by Subroutine 1 discussed in Section 2 gives the feasible interval of values of α, limits of this interval will depend on $\bar{\beta}$. Denote this interval by $\alpha_1(\bar{\beta}) \leq \alpha \leq \alpha_2(\bar{\beta})$.

For any $\bar{\beta}$ in the descent range of β, let $g(\bar{\beta})$ denote minimum value of $\{\theta(\bar{x}+\alpha y(\bar{\beta})) : \alpha_1(\bar{\beta}) \leq \alpha \leq \alpha_2(\bar{\beta})\}$. Even though the function $g(\beta)$ is not available in explicit functional form, its value $g(\bar{\beta})$ for any given value $\bar{\beta}$ can be obtained by performing a line search from its definition above. Using this we can now minimize the function $g(\beta)$ itself over the descent range for β using a direct search line search method that only uses function values at selected values of β; this will give the minimum value of $\theta(\bar{x} + \alpha y(\beta))$ over all feasible α, β values. If β^*, α^* are the values of β, α at this minimum , include in the set ΓN the point $\bar{x} + \alpha^* y(\beta^*)$ as the output point obtained corresponding to this index j in the set J.

When this work is completed for all $j \in J$, select the best point in the set ΓN by objective $(\theta(x))$ value , and call it $\hat{x}$ the new current IFS. With it as the initial IFS go to the next iteration.

The algorithm is terminated when the distance between consecutive elements in the sequence of initial IFSs x^r in the algorithm becomes small. The final point in this sequence is taken as an approximation to the optimum for the original NLP (8).

6 Sphere Methods for Linearly Constrained Nonlinear Programming (NLP) Models in Which Objective Function $\theta(x)$ Is Nonconvex, Or for 0-1 Interger Linear Programs (ILPs)

Consider the 0-1 ILP

$$\begin{array}{rrcll} \text{Minimize} & cx & & & \\ \text{S. to} & Ax & \geq & b & \\ & x_j & \in & \{0,1\} & \text{for all } j \in J_1 \end{array} \tag{11}$$

where A is an $m \times n$ matrix, and $J_1 \subset \{1, ..., n\}$. This problem is equivalent to finding the global minimum of the following nonconvex Quadrtic Program (QP).

$$\begin{array}{rrcll} \text{Minimize} & z_1(x) & = & cx + M \sum_{j \in J_1} x_j(1 - x_j) & \\ \text{S. to} & Ax & \geq & b & \\ & 0 \leq x_j & \leq & 1 & \text{for all } j \in J_1 \end{array} \tag{12}$$

in which M is a large positive penalty parameter, which is a special case of (8) with $z_1(x)$ being a concave function. In this section we will consider the NLP (8) with $\theta(x)$ being nonconvex, which includes this problem.

In this nonconvex case the main differences are: Result 1 may not be valid, also if we get a point x satisfying $\nabla\theta(x) = 0$ during the algorithm, that point x is not guaranteed to be an optimum solution (i.e., global minimum) for (8), may not even be a local minimum. Also Result 2 is not valid as stated. So if $\hat{x}$ is a feasible solution satisfying: $\hat{x}$ is an optimum solution to the LP: minimize $\nabla\theta(\hat{x})x$ subject to $Ax \geq b$, we cannnot guarantee that $\hat{x}$ is a global optimum solution for (8) , but it is a KKT point for (8); and in real wirld applications of nonconvex nonlinear models, a KKT point is usually accepted as an approximation to an optimum solution.

So, starting with the initial IFS we apply the algorithm discussed in Section 5 in this nonconvex case also. Since the set ΓN generated by that algorithm contains the output points in descent steps carried out is all directions of the feasible region, at termination, the best point in ΓN is likely to be a good approximation to the global optimum. For solving the 0-1 ILP, at termination all binary variales are likely to have their values in the final solution very close to 0 or 1, they are then rounded off to ther nearest integer in determining the final solution.

7 Computational Results

Computational tests on these algorithms are being carried out by David Kaufman (davidlk@gmail.com) at the University of Michigan, Ann Arbor; Vincent F. Yu (vincent@mail.ntust.edu.tw) at the National Taiwan University of Science and Technology, and his students. The paper will be completed when the results of these tests become available.

8 Refereces

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