Sophomore Level Self-Teaching Webbook for

Computational and Algorithmic Linear Algebra and n-Dimensional Geometry

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PREFACE

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The Importance of the Subject

The subject of **linear algebra** originated more than 2000 years ago in an effort to develop tools for modeling real world problems in the form of systems of linear equations, and to solve and analyze these problems using those models. Linear algebra and its twentieth century extensions, **linear and integer programming**, are the most useful and most heavily used branches of mathematics. A thorough knowledge of the most fundamental parts of linear algebra is an essential requirement for anyone in any technical job these days in order to carry out job duties at an adequate level. In this internet age with computers playing a vital and increasing role in every job, making the most effective use of computers requires a solid background in at least the basic parts of linear algebra.

Recent Changes in the Way Linear Algebra is Taught

Until about the 1970's there used to be a full semester course on computational linear algebra at the sophomore level in all undergraduate engineering curricula. That course used to serve the purpose of providing a good background in the basics of linear algebra very well. However, due to increasing pressure to introduce more and more material in the undergraduate mathematics sequence of engineering curricula, beginning in the 1970's many universities started combining this first level linear algebra course with the course on differential equations and shaping it into a combined course on differential equations and linear algebra. On top of this, linear algebra is being squeezed out of this combined course gradually, with the result that now-a-days most engineering undergraduates get only about four weeks of instruction on linear algebra. The net effect of all this is that many undergraduates of science and engineering are not developing the linear algebra skills that they need.

The Goals of This Book

There are many outstanding books on linear algebra, but most of them have their focus on undergraduates in mathematics departments. They concentrate on developing theorem-proving skills in the readers, and not so much on helping them develop mathematical modeling, computational, and algorithmic skills. Students who miss a few classes, particularly those already not doing well in the class, often find it very difficult to catch up with the help of such text books. The net result after about half the term is an unhealthy situation in which only the top half of the class is comfortably following the teachers lectures, with the rest falling more and more behind.

Also, with mounting pressure to pack more and more material into undergraduate mathematics courses, it is highly unlikely that more class time can be found to allocate to linear algebra. So, an important component of a solution strategy to this problem is to see whether students can be helped to learn these concepts to a large extent from their own efforts.

An incident providing evidence of self-learning: As evidence of self-learning of higher mathematics, I will mention the strangest of mathematical *talks* ever given, by Cole, in October 1903 at an AMS meeting, on Mersenne's claim that $2^{67} - 1$ is a prime number. This claim fascinated mathematicians for over 250 years. Not a single word was spoken at the seminar by anyone, but everyone there understood the result obtained by Cole. Cole wrote on the blackboard: $2^{67} - 1 = 147573952589676412927 = a$. Then he wrote 761838257287, and underneath it 193707721. Without uttering a word, he multiplied the two numbers together to get a, and sat down to the wild applause from the audience. $\bowtie$

In the same way, I strongly believe that students can learn the concepts of numerical and algorithmic matrix algebra mostly on their own using a carefully designed webbook with illustrative examples, exercises, figures and well placed summaries. My primary goal is to make this such a book.

We have all seen prices of paper copies of textbooks going up every year. In the USA, the average undergraduate paper textbook costs at least \$70 these days, and those that do not enjoy high sales volumes cost even more. To make paper for printing these books, vast areas of beautiful forests are being cleared, creating an unreasonable demand on the small part of nature not yet destroyed by us. So, I have decided to make this a **webbook** (also called an **electronic** or **e-textbook**). The main intent in preparing this book is to use information technology and web-based tools to create an e-text that will enable students to comfortably operate in the happy mode of self-teaching and self-learning. Our focus is to help the students develop the mathematical modeling, computational, and algorithmic skills that they need to bring linear algebra tools and facts to solving real world problems. Proofs of simple results are given without being specially labeled as proofs. Proofs of more complex facts are left out, for the reader to look up in any of the traditional textbooks on the subject, some of which are listed in the bibliography.

This webbook can also be used to supplement students' classes and traditional textbooks and to reinforce concepts with more examples and exercises, in order to assist them in fully understanding the material presented in classes. The self-teaching environment in the e-text makes it possible for those students who feel that certain material was covered too quickly during class, to go back and review on the screen the carefully developed version in the e-text at their own pace. In the same way those students who want to move forward with the coursework at a faster pace than the class will be able to do so on their own in the webbook learning environment.

Importance of Learning the Details of the Methods

Students often ask me why it is important for them to learn the details of the methods for solving systems of linear equations these days when computers and high quality software for solving them are so widely available everywhere, and so easy to use. I keep telling them that many complex real world applications demand deeper knowledge than that of pressing a button to solve a given system of linear equations. First, someone has to construct the system of equations from information available on the real world problem: this part of the job is called mathematical modeling. If the model constructed yields a strange and unrealistic solution, or if it turns out to have no solution, one has to analyze the model and decide what changes should be made in it to make sure that it represents the real world problem better. This type of analysis can only be carried out efficiently by someone who has a good understanding of the methods of linear algebra. We have all seen the total helplessness of some grocery store checkout persons who cannot add two simple numbers when their computerized cash register fails. A person without the proper knowledge of linear algebra methods will be in the same position when he/she has to analyze and fix an invalid model giving strange results to the real world problem. Fortunately for grocery store checkout persons, the occurrence of cash register terminal failures is rare. However, for technical people trying to model their problems, the need to analyze and fix an invalid model is a daily event. Typically, models for real world problems are revised several times before they produce an acceptable solution to the problem.

For these reasons, I believe that it is very important for all undergraduates to learn the basics of linear algebra well.

The Screen Version With Weblinks

We are posting this book on the web with size 12 letters for comfortable reading on the screen. The book will always be available on the web at this address for anyone to access, so it is very convenient to read most of the book on the screen itself. Much of the explanatory material and the examples would probably be read only a few times, so there is no reason to print them on paper. **I hope the readers will only print the most essential portions of the book that they feel they need to reference frequently on paper.**

With weblinks it becomes very easy to search in each chapter. Clicking on the light shiny blue portion in the table of contents, index, or the list of important topics at the end of each chapter, takes the screen to the appropriate page. I hope the readers will find it very convenient to read the book on the computer screen itself with these links, without having to print it on paper.

The Poetic Beauty of the Subject

In classical Indian literature in Sanskrit, Telugu, etc., whenever they wanted to praise something or someone, they would compare them to the mighty ocean. In an effort to create an even higher pedestal of excellence, they developed the concept of **"kshiira saagar"** (literally "milk ocean", the mightiest ocean of pure milk goodness), and if they wanted to say that something is very good, they would say that it is like kshiira saagar. Actually, in terms of its applicability, sheer elegance and depth, linear algebra is very much like kshiira saagar. In this webbook you will get acquainted with the most basic tools and facts of linear algebra that are called for most frequently in applications. Those of you who have developed a taste for the beauty of the subject can delve deeper into the subject by looking up some of the other books mentioned in the bibliography.

Distinct Features of this Book

Chapter 1 explains the main difference between the classical subject of linear algebra (study of linear equations without any linear inequalities), and the modern 20th century subjects of linear programming (study of linear constraints including linear inequalities) and integer programming and combinatorial optimization (study of linear constraints and integer restrictions on some variables). This important distinction is rarely explained in other linear algebra books. When trying to handle a system of linear constraints including linear inequalities and integer requirements on some variables by classical linear algebra techniques only, we essentially have to ignore the linear inequalities and integer requirements. Chapter 1 deals with constructing mathematical models of systems of linear equations for real world problems, and the Gauss-Jordan (GJ) and Gaussian (G) elimination methods for solving these systems and analyzing the solution sets for such systems, including practical infeasibility analysis techniques for modifying an infeasible system into a feasible one. To keep the new terminology introduced to a minimum, the concept of a "matrix" is not even used in Chapter 1. All the algorithmic steps are stated using the fundamental tool of row operations on the detached coefficient tableau for the system with the variables entered in a top row in every tableau. This makes it easier for the readers to see that the essence of these methods is to take linear combinations of constraints in the original system to get an equivalent but simpler system from which a solution can be read out. Chapter 1 shows many instances of what sets this book apart from other mathematics books on linear algebra. In the descriptions of GJ, G methods in most of these books, the variables are usually left out. Also, they state the termination condition to be that of reaching the RREF (reduced row echelon form) or the REF (row echelon form). A tableau is defined to be in RREF [REF] if it contain a full set of unit vectors [vectors of an upper triangular matrix] in proper order at the left end.

OR people have realized that it is not important that all unit vectors (or vectors of the upper triangular matrix) be at the left end of the tableau (they can be anywhere and can be scattered all over); also it is not important that they be in proper order from left to right. They use the very simple **data structure** (this phrase means a strategy for storing information generated during the algorithm, and using it to improve the efficiency of that algorithm) of associating the variable corresponding to the rth unit vector (or the rth vector of the upper triangular matrix) in the final tableau as the rth basic variable or basic variable in the rth row; and they store these basic variables in a column on the tableau as the algorithm progresses. This data structure makes it easier to read the solution directly from the final tableau of the GJ method by making all nonbasic variables = 0; and the rth basic variable = the rth RHS constant, for all r. We present the GJ, G methods using this data structure. Students find it easier to understand the main ideas behind the elimination methods using it. It also opens the possibility of pivot column selection strategies instead of always selecting the leftmost eligible column in these methods.

I believe it is George Dantzig who introduced the data structure of storing basic variables in his pioneering paper, the 1947 paper on the simplex method. He called the final tableau the *canonical tableau* to distinguish it from the mathematical concepts RREF, REF.

Another important difference that appears in Chapter 1, between this book and all the other books on linear algebra, is worth mentioning. A system of linear equations is infeasible (i.e., has no solution) iff the fundamental inconsistent equation "$0 = 1$" can be expressed as a linear combination of equations in the system. Thus all the linear algebra books state that if the equation "$0 = a$" where a is a nonzero number, shows up in one of the tableaus during the application of the GJ or G methods on the system, then the system is infeasible. As an example, consider the system

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 2 \\
-x_1 - x_2 - x_3 &= -1 \\
2x_1 + 4x_2 + 6x_3 &= 12
\end{aligned}
$$

The linear combination of equations in this system with coefficients (1, 1, 0) is the equation $0x_1 + 0x_2 + 0x_3 = 1$ or $0 = 1$, hence this system is infeasible.

It is possible for the equation "$0 = 1$" to appear in the GJ or G methods due to computational errors. In hand computation these may be human errors, in digital computation on a computer these may be due to accumulation of roundoff errors. So, whenever the equation "$0 = a$" for some $a \neq 0$ appears in the GJ or G methods, it is also helpful for the methods to produce the row vector of coefficients in a linear combination of constraints in the original system that yields this "$0 = a$" equation. In the above example that vector of coefficients is $(1, 1, 0)$. This vector has been called by the obscure name **supervisor principle for infeasibility** (the idea behind this name is that if your supervisor is suspicious about your claim of infeasibility of the system, he/she can personally verify the claim by obtaining the linear combination of constraints in the system with coefficients in this vector and verify that it is "$0 = a$" where $a \neq 0$). But we will call this vector of coefficients by the simpler name **evidence (or certificate) of infeasibility**. Given this evidence one can easily check whether it leads to the inconsistent equation "$0 = a$" where $a \neq 0$.

In the same way when an equation "$0 = 0$" appears during the GJ or G methods, we know that the corresponding original constraint is a redundant constraint that can be deleted from the system without changing its set of solutions. This constraint is redundant becuse it can be obtained as a linear combination of other constraints on which pivot steps have been carried out so far. The vector of coefficients in this linear combination is called the **evidence (or certificate) of redundancy** for this redundant constraint.

Other books do not discuss how these evidences can be obtained. These evidences are obtained easily as byproducts of the GJ, G methods as explained in Section 1.12, if the memory matrix is added to the system at the beginning of these methods. Also, in the computationally efficient versions of the GJ and G methods discussed in Section 4.11, we show that these evidences are automatically obtained without any additional expense.

Chapter 2 then introduces the concept of a matrix, the evolution of matrix arithmetic, and the study of determinants of square matrices.

Chapter 3 introduces the fundamentals of n-dimensional geometry starting with the representation of vectors as points using a coordinate frame of reference and the connections between algebra and geometry that this representation makes possible. We include a thorough treatment of important geometric objects such as subspaces, affine spaces, convex sets, hyperplanes, half-spaces, straight lines, half-lines, directions, rays, cones, and orthogonal projections that appear most frequently in courses that the student is likely to take following this study, and in real world applications.

Chapter 4 on numerical linear algebra discusses the fundamental concepts of linear dependence and independence, rank, inverse, factorizations; and efficient algorithms to check or compute these. Major differences in the mathematical properties of linear constraints involving equations only, and those involving some inequalities; come to the surface in the study of linear optimization problems involving them. This topic, not discussed in other books, is the subject of Section 4.10. In this section we also discuss the duality theorem for linear optimization subject to linear equality constraints (a specialization of the duality theorem of linear programming) and its relationship to the theorem of alternatives for linear equation systems. In Section 4.11 we discuss the efficient memory matrix versions of the GJ, G methods (based on the ideas introduced by George Dantzig in his revised simplex method for linear programming), these versions have the additional advantage of producing evidences of redundancy or infeasibility automatically without any additional work.

Chapter 5 deals with representing quadratic functions using matrix notation, and efficient algorithms to check whether a given quadratic function is convex, concave, or neither. The treatment of positive (negative) (semi)definiteness and indefiniteness of square matrices, and diagonalization of quadratic forms is included.

Chapter 6 includes the definitions and review of the main properties of eigen values and eigen vectors of square matrices, and their role in matrix diagonalizations.

Finally in the brief Chapter 7, we review the features of commercial software resources available these days to solve many of the problems discussed in earlier chapters.

Acknowledgements

MATLAB computations in Chapter 7: Jeanne Whalen helped me with the MATLAB computations in Chapter 7 on a Marian Sarah Parker Scholarship during Summer 2001. My thanks to her and to the University of Michigan Marian Sarah Parker Scholarship Administration (Women in Engineering Office at the University of Michigan College of Engineering).

Weblinks: On a Marian Sarah Parker Scholarship during Summer 2003, Shital Thekdi helped me to link and make the PDF file of each chapter completely searchable. My thanks to her and to the University of Michigan Marian Sarah Parker Scholarship Administration.

Suggestions, corrections, and many other kinds of help have been received from several others too numerous to mention by name, and I express my heartfelt thanks to all of them.

Contributions Requested

Everyone using this book is requested to send a contribution of about US$10 to $15 to the author at the address given on the front page. If you are an instructor using this webbook in one of your classes, please remind your students in the first week of classes to send in their contributions. These funds will be used to keep the webbook updated, and to improve it to be more user-friendly.

About the Results and Excercises

Exercises are numbered serially beginning with number 1 in each section. Exercise $i.j.k$ refers to the kth exercise in Section $i.j$. In the same way Result $i.j.k$ refers to the kth result in Section $i.j$.

About the Numerical Exercises

At this level most students cannot really appreciate the details of an algorithm until they actually solve several numerical problems by themselves by hand. That's why I included many carefully constructed numerical exercises classified by the final outcome (infeasible system, system with unique solution, system with many solutions, redundant constraints, etc.) so that they know what to expect when they solve each exercise.

Given proper instructions, a digital computer can execute an algorithm on any data without complaining. But hand computation becomes very tedious if complicated fractions show up. That's why the numerical exercises are carefully constructed so that a pivot element of ± 1 is available for most of the steps of the algorithms. I hope that this makes it more enjoyable to understand the algorithms by working out the numerical exercises without encountering the tedium brought on by ugly fractions.

Each Chapter in a Separate file

In paperbooks all chapters are always put together between two covers and bound. But for a webbook we feel that it is convenient to have each chapter in a separate file to make it easier for users to download. That's why we are putting each chapter as a separate file, beginning with its individual table of contents, and ending with its own index of terms defined in it.

Invitation for Feedback

The web affords the wonderful opportunity of frequent revisions of the text, subject only to the limitation of the authors time and judgement. Feedback from users on possible improvements of the text or suggestions for interesting exercises to include to make the webbook more useful will be greatly appreciated, these can be sent by e-mail.

Katta G. Murty
Anna Arbor, MI, December 2001, July 2003.

Glossary of Symbols and Abbreviations

R^n	The n-dimensional real Euclidean vector space. The space of all vectors of the form $x = (x_1, \ldots, x_n)^T$ (written either as a row vector as here, or as a column vector) where each x_j is a real number.
$=, \geq, \leq$	Symbols for equality, greater than or equal to, less than or equal to, which must hold for each component.
x^T, A^T	Transpose of vector x, matrix A.
(a_{ij})	Matrix with a_{ij} as the general element in it.
$\backslash$	Set difference symbol. If D, E are two sets, $D \backslash E$ is the set of all elements of D which are not in E.
A^{-1}	Inverse of the nonsingular square matrix A.
$< u, v >$.	Dot (inner) product of vectors u, v.
$\text{diag}(a_1, \ldots, a_n)$	The $n \times n$ square matrix whose diagonal elements are $a_1, \ldots, a_n$ in that order, and all off-diagonal entries are 0.
$A_{i.}, A_{.j}$	The ith row vector, jth column vector of matrix A.
$\det(A)$	Determinant of square matrix A
$\|\|x\|\|$	Euclidean norm of vector $x = (x_1, \ldots, x_n)$, it is $\sqrt{x_1^2 + \ldots + x_n^2}$. Euclidean distance between two vectors x, y is $\|\|x - y\|\|$.
$\text{rank}(A)$	Rank of a matrix A, same as the rank of its set of row vectors, or its set of column vectors.
WRT	With respect to.
RHS	Right hand side.
PC, PR	Pivot column, pivot row.
RC , IE	Redundant constraint, inconsistent equation identified.
RI, CI	Row interchange, column interchange.
BV	Basic variable selected for that row.
GJ method	The Gauss-Jordan elimination method for solving systems of linear equations.
GJ pivot	The Gauss-Jordan pivot step on a tableau or a matrix.
G pivot	The Gaussian pivot step on a tableau or a matrix.

G method	The Gaussian elimination method for solving systems of linear equations.
PSD, NSD	Positive semidefinite, negative semidefinite.
PD, ND	Positive definite, negative definite.
Scalar	A real number. “Scalar multiplication” means multiplication by a real number.
$\bowtie$	This symbol indicates the end of the present portion of text (i.e., example, exercise, comment etc.). Next line either resumes what was being discussed before this portion started, or begins the next portion.

A Poem on Linear Algebra

To put you in a receptive mood as you embark on the study of this wonderful subject, I thought it fitting to begin this book with a lovely Telugu poem on the virtues of linear algebra composed by Dhulipalla V. Rao with help from my brother Katta Sai, daughter Madhusri Katta and myself. The original poem in roman transliteration first, followed by the English translation.

- -

OhO lIniyar AljIbrA nIkivE mAjOhArlu!

sAgaramagAdhamani caduvukunnaM
nIlOtiMkAminnani telusukunnAM

aMdAniki marOpEru araviMdaM
gaNitulaku nIiMpu marimari aMdaM

jIvanAniki maMchinIru eMtO muKyaM
mAnavALi pragatiki nuvvaMtakannA muKyaM

maMciruciki baMginipalli mAmiDi maharAju
gaNitameriginavAriki nuvvE rArAju

paMTalaku tEneTIgalostAyi tODu
nuvulEkuMTE mAnavajIvitamavunu bIDu

medaDuku padunu nI adhyayanaM
pragatiki sOpAnaM nI paTanaM.
- -

Oh, linear algebra! You are
Deep as the mighty ocean
As exciting as the pink lotus is beautiful
As satisfying to my brain as the sweet banginapalli* mango is to my tongue
As essential to science as water is to life
As useful in applications as bees are for the cultivation of crops+
I cannot describe in words how much richer human life has become because we discovered you!.

---

* A variety of mango grown in Andhra Pradesh, India, with crisp sweet flesh that has no fibrous threads in it, that is considered by some to be the most delicious among fruits.
+ Bees carry out the essential task of pollinating the flowers in fruit, vebetable, and some other crops. Without bees, we will not have the abundance of fruits and vegetables we enjoy today.