Contents

Chapter 2

Matrices, Matrix Arithmetic, Determinants

This is Chapter 2 of "Sophomore Level Self-Teaching Webbook for Computational and Algorithmic Linear Algebra and n-Dimensional Geometry" by Katta G. Murty.

2.1 A Matrix as a Rectangular Array of Numbers, Its Rows and Columns

A **matrix** is a rectangular array of real numbers. If it has m rows and n columns, it is said to be an $m \times n$ matrix, or a **matrix of order** or **size** $m \times n$. Here is a 3×5 matrix with three rows and five columns.

$$\begin{pmatrix} -1 & 7 & 8.1 & 3 & -2 \\ 0 & -2 & 0 & 5 & -7 \\ 1 & -1 & 3 & 0 & -8 \end{pmatrix}$$

The size or order of a matrix is always a pair of positive numbers, the first being the number of rows of the matrix, and the second the number of columns.

In mathematical literature, matrices are usually written within brackets of (...) or [...] type on the left and right.

History of the name "matrix": The word "matrix" is derived from the Latin word for "womb" or "a pregnant animal". In 1848 J. J. Sylvester introduced this name for a rectangular array of numbers, thinking of the large number of subarrays that each such array can generate (or "give birth to"). A few years later A. Cayley introduced matrix multiplication and the basic theory of matrix algebra quickly followed.

Initially, the concept of a matrix was introduced mainly as a tool for storing the coefficients of variables in a system of linear equations, in that case it is called the **coefficient matrix** or **matrix of coefficients** of the system. In 1858 A. Cayley wrote a memoir on the theory of linear transformations and matrices. With arithmetical operations on matrices defined, we can think of these operations as constituting an *algebra* of matrices, and the study of **matrix algebra** by Cayley and Sylvester in the 1850's attracted a lot of attention that led to important progress in matrix and determinant theory. $\bowtie$

Consider the following system of linear equations.

$$\begin{array}{rcl} 3x_2 - 4x_4 + x_1 - x_3 & = & -10 \\ 2x_1 + 5x_3 - 2x_4 + 6x_2 & = & -5 \\ 4x_4 - x_1 + 8x_3 - 3x_2 & = & 7 \end{array}$$

Since this is a system of 3 equations in 4 variables, its coefficient matrix will be of order 3×4. Each row of the coefficient matrix will correspond to an equation in the system, and each of its columns corresponds to a variable. So, in order to write the coefficient matrix of this system, it is necessary to arrange the variables in a particular order, say as in the column vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

and write each equation with the terms of the variables arranged in this particular order x_1, x_2, x_3, x_4. This leads to

$$\begin{array}{rcl} x_1 + 3x_2 - x_3 - 4x_4 & = & -10 \\ 2x_1 + 6x_2 + 5x_3 - 2x_4 & = & -5 \\ -x_1 - 3x_2 + 8x_3 + 4x_4 & = & 7 \end{array}$$

Now the coefficient matrix of the system $A = (a_{ij} : \quad i = 1, \dots, m; j = 1, \dots, n)$, where a_{ij} is the coefficient of x_j in the ith equation, can be easily written. It is

$$\begin{pmatrix} 1 & 3 & -1 & -4 \\ 2 & 6 & 5 & -2 \\ -1 & -3 & 8 & 4 \end{pmatrix}$$

For $i = 1$ to 3, **row** i **of this matrix** A corresponds to the ith equation in the system, and we denote it by the symbol $A_{i.}$. For $j = 1$ to 4, **column** j **of this matrix** A, denoted by the symbol $A_{.j}$, corresponds to, or is associated with, the variable x_j in the system.

Thus the coefficient matrix of a system of linear equations is exactly the array of entries in the detached coefficient tableau representation of the system without the RHS constants vector. The rows of the coefficient matrix are the rows of the detached coefficient tableau without the RHS constant, its columns are the columns of the detached coefficient tableau.

In general, an $m \times n$ matrix D can be denoted specifically as $D_{m\times n}$ to indicate its order; it will have m row vectors denoted by $D_{1.}, \dots, D_{m.}$; and n column vectors denoted by $D_{.1}, \dots, D_{.n}$. The matrix D itself can be viewed as the array obtained by putting its column vectors one after the other horizontally as in

$$D_{m\times n} = (D_{.1} \quad D_{.2} \dots D_{.n})$$

or putting its row vectors one below the other vertically as

$$D_{m\times n} = \begin{pmatrix} D_{1.} \\ D_{2.} \\ \vdots \\ D_{m.} \end{pmatrix}$$

A row vector by itself can be treated as a matrix with a single row. Thus the row vector

$$(3, -4, 7, 8, 9)$$

can be viewed as a matrix of order 1×5.

In the same way, a column vector can be viewed as a matrix with a single column. So, the column vector

$$\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

can be viewed as a matrix of order 3×1.

The entry in a matrix D in its ith row and jth column is called its (i, j)th entry, and denoted by a symbol like d_{ij}. Then the matrix D itself can be denoted by the symbol (d_{ij}). So, the $(2, 3)$th entry of the coefficient matrix A given above, $a_{23} = 5$.

Let b denote the column vector of RHS constants in a system of linear equations in which the coefficient matrix is A of order $m \times n$. Then the matrix $\mathcal{A}$ obtained by including b as another column at the right of A, written usually in a **partitioned** form as in

$$\mathcal{A} = (A|b)$$

is known as the **augmented matrix** of the system of equations.

A **partitioned matrix** is a matrix in which the columns and/or the rows are partitioned into various subsets. The augmented matrix $\mathcal{A}$ is written as a partitioned matrix with the columns associated with the variables in one subset of columns, and the RHS constants column vector by itself in another subset of columns.

As an example, consider the following system of 3 equations in 4 variables.

$$\begin{aligned} 3x_2 - 4x_4 - x_3 &= -10 \\ 2x_1 - 2x_4 + 6x_2 &= -5 \\ 4x_4 - x_1 + 8x_3 &= 7 \end{aligned}$$

Arranging the variables in the order x_1, x_2, x_3, x_4, the system is

$$\begin{aligned} 3x_2 - x_3 - 4x_4 &= -10 \\ 2x_1 + 6x_2 - 2x_4 &= -5 \\ -x_1 + 8x_3 + 4x_4 &= 7 \end{aligned}$$

So, for this system, the coefficient matrix A, and the augmented matrix are

$$A = \begin{pmatrix} 0 & 3 & -1 & -4 \\ 2 & 6 & 0 & -2 \\ -1 & 0 & 8 & 4 \end{pmatrix}, \mathcal{A} = \left(\begin{array}{rrrr|r} 0 & 3 & -1 & -4 & -10 \\ 2 & 6 & 0 & -2 & -5 \\ -1 & 0 & 8 & 4 & 7 \end{array}\right)$$

2.2 Matrix−Vector Multiplication

The concept of the product of a matrix times a column vector in this specific order, is formulated so that we can write the system of equations with coefficient matrix A, column vector of variables x, and RHS constants vector b, in matrix notation as $Ax = b$. We define this concept next.

Definitions: Matrix−vector products: Consider a matrix D of order $m \times n$, and a vector p. The product Dp, written in this specific order, is only defined if p is a column vector in R^n (i.e., if the number of column vectors of D is equal to the number of rows in the column vector p), i.e., if $p = (p_1, \ldots, p_n)^T$. Then the product Dp is itself a column vector given by

$$\begin{aligned} Dp &= \begin{pmatrix} D_{1.}p \\ \vdots \\ D_{m.}p \end{pmatrix} \\ &= \sum_{j=1}^{n} p_j D_{.j} \end{aligned}$$

Also, if q is a vector, the product qD, written in this specific order, is only defined if q is a row vector in R^m (i.e., if the number of columns in q is equal to the number of rows in D), i.e., if $q = (q_1, \ldots, q_m)^T$, and the product qD is itself a row vector given by

$$\begin{aligned} qD &= (qD_{.1}, \ldots, qD_{.n}) \\ &= \sum_{i=1}^{m} q_i D_{i.} \end{aligned}$$

We have given above two ways of defining each matrix-vector product; both ways yield the same result for the product in each case.

In this chapter we use superscripts to number various vectors in a set of vectors (for example, q^1, q^2, q^3 may denote three different vectors in R^n). We always use subscripts to indicte various entries in a vector, for example for a row vector $p \in R^3$, the entries in it will be denoted by p_1, p_2, p_3, so, $p = (p_1, p_2, p_3)$.

Example 1: Let

$$q^1 = (1, 2, 1); \quad q^2 = (1, 2); \quad q^3 = (2, 2, 2, 1)$$

$$A = \begin{pmatrix} 1 & 3 & -1 & -1 & -4 \\ 2 & 6 & 5 & 5 & -2 \\ -1 & -3 & 8 & 8 & 4 \end{pmatrix}, p^3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, p^4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$p^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}, p^2 = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

The products $Ap^3, Ap^4, Aq^1, Aq^2, Aq^3, p^1A, p^2A, p^3A, p^4A, q^2A, q^3A$ are not defined. The reader is encouraged to explain the reasons for each. Here are the row vectors of matrix A.

$$\begin{aligned} A_{1.} &= (1, 3, -1, -1, -4) \\ A_{2.} &= (2, 6, 5, 5, -2) \\ A_{3.} &= (-1, -3, 8, 8, 4) \end{aligned}$$

The column vectors of matrix A are:

$$A_{.1} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, A_{.2} = \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}, A_{.3} = \begin{pmatrix} -1 \\ 5 \\ 8 \end{pmatrix}$$

$$A_{.4} = \begin{pmatrix} -1 \\ 5 \\ 8 \end{pmatrix}, A_{.5} = \begin{pmatrix} -4 \\ -2 \\ 4 \end{pmatrix}$$

We have

$$Ap^1 = \begin{pmatrix} A_{1.}p^1 \\ A_{2.}p^1 \\ A_{3.}p^1 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \\ 7 \end{pmatrix}.$$

We also have

$$Ap^1 = 1A_{.1} + 0A_{.2} + 1A_{.3} - 1A_{.4} + 2A_{.5} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \\ 8 \end{pmatrix}$$

$$- \begin{pmatrix} -1 \\ 5 \\ 8 \end{pmatrix} + 2 \begin{pmatrix} -4 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \\ 7 \end{pmatrix}$$

giving the same result. In the same way verify that both definitions give

$$Ap^2 = \begin{pmatrix} 0 \\ 13 \\ 7 \end{pmatrix}$$

Similarly

$$\begin{aligned} q^1A = (q^1A_{.1}, q^1A_{.2}, q^1A_{.3}, q^1A_{.4}, q^1A_{.5}) &= (4, 12, 17, 17, -4) \\ &= 1A_{1.} + 2A_{2.} + A_{3.}. \end{aligned}$$

Some Properties of Matrix-Vector Products

Let A be an $m \times n$ matrix, $u^1, \ldots, u^r$ be all column vectors in R^n, and $\alpha_1, \ldots, \alpha_r$ real numbers. Then

$$A(u^1 + \ldots + u^r) = \sum_{t=1}^{r} Au^t$$

$$A(\alpha_1 u^1 + \ldots + \alpha_r u^r) = \sum_{t=1}^{r} \alpha_t Au^t$$

We will provide a proof of the second relation above. (The first relation is a special case of the second obtained by taking $\alpha_t = 1$ for all t.) Let u denote a general column vector in R^n. From the definition

$$Au = \begin{pmatrix} A_{1.}u \\ \vdots \\ A_{m.}u \end{pmatrix}$$

From Result 1.7.1 of Section 1.7, $A_{i.}u$ is a linear function of u, and hence from the definition of linear functions (see Section 1.7)

$$A_{i.}(\alpha_1 u^1 + \ldots + \alpha_r u^r) = \sum_{t=1}^{r} \alpha_t A_{i.}u^t$$

Using this in the above equation

$$Au = \begin{pmatrix} \sum_{t=1}^{r} \alpha_t A_{1.}u^t \\ \vdots \\ \sum_{t=1}^{r} \alpha_t A_{m.}u^t \end{pmatrix} = \sum_{t=1}^{r} \alpha_t \begin{pmatrix} A_{1.}u^t \\ \vdots \\ A_{m.}u^t \end{pmatrix} = \sum_{t=1}^{r} \alpha_t Au^t$$

proving the second relation above.

Also, if $v^1, \ldots, v^r$ are all row vectors in R^m, then we get the following results from similar arguments

$$(v^1 + \ldots + v^r)A = \sum_{t=1}^{r} v^t A$$

$$(\alpha_1 v^1 + \ldots + \alpha_r v^r)A = \sum_{t=1}^{r} \alpha_t v^t A$$

Exercises

2.2.1: Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & 2 & 1 \end{pmatrix}, C = \begin{pmatrix} -2 & 1 \\ 1 & -2 \\ -2 & -3 \end{pmatrix}.$$

$$p^1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, p^2 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, p^3 = \begin{pmatrix} 1 \\ 17 \\ -8 \\ 9 \end{pmatrix}.$$

$$q^1 = (-3, 2), \quad q^2 = (-2, -1, 2), \quad q^3 = (8, -7, 9, 15).$$

Mention which of Ax, Ax, Cx, yA, yB, yC are defined for $x, y \in \{p^1, p^2, p^3, q^1, q^2, q^3\}$ and which are not. Compute each one that is defined.

2.3 Three Ways of Representing the General System of Linear Equations Through Matrix notation

Consider the general system of m equations in n variables in detached coefficient tableau form.

x_1	$\dots$	x_j	$\dots$	x_n	RHS constants
a_{11}	$\dots$	a_{1j}	$\dots$	a_{1n}	b_1
$\vdots$		$\vdots$		$\vdots$	$\vdots$
a_{i1}	$\dots$	a_{ij}	$\dots$	a_{in}	b_i
$\vdots$		$\vdots$		$\vdots$	$\vdots$
a_{m1}	$\dots$	a_{mj}	$\dots$	a_{mn}	b_m

Let

$A = (a_{ij})$	the $m \times n$ coefficient matrix
$b = (b_1, \ldots, b_m)^T$	the column vector of RHS constants
$x = (x_1, \ldots, x_n)^T$	the column vector of variables

in this system.

Representing the System as a Matrix Equation

Clearly, the above system of equations can be represented by the **matrix equation**

$$Ax = b$$

The number of rows of the coefficient matrix A is the number of equations in the system, and the number of columns in A is the number of variables in the system.

From now on, we will use this notation to represent the general system of linear equations whenever it is convenient for us to do so.

Example 2: Consider the following system in detached coefficient form.

x_1	x_2	x_3	
1	−2	−1	−5
−3	1	2	−7

Denoting the coefficient matrix, RHS constants vector, column vector of variables, respectively by A, b, x, we have

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix}, b = \begin{pmatrix} -5 \\ -7 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and we verify that this system of equations is: $Ax = b$ in matrix notation.

Representing the System by Equations

The ith row of the coefficient matrix A is $A_{i.} = (a_{i1}, \ldots, a_{in})$. Hence the equation by equation representation of the system is

$$A_{i.}x = b_i \qquad i = 1 \text{ to } m$$

Representing the System as a Vector Equation

Another way of representing the system of equations $Ax = b$ as a vector equation involving the column vectors of the coefficient matrix A and the RHS constants vector b is

$$x_1 A_{.1} + x_2 A_{.2} + \ldots + x_n A_{.n} = b$$

i.e., the values of the variables in a solution of the system x are the coefficients in an expression of b as a linear combination of the column vectors of the coefficient matrix A.

As an example, consider the system of equations $Ax = b$ in Example 2, where

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix}, b = \begin{pmatrix} -5 \\ -7 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

One representation of this system as a set of equations that should hold simultaneously is

$$\begin{array}{lll} A_{1.}x = b_1 & \text{which is} & x_1 - 2x_2 - x_3 = -5 \\ A_{2.}x = b_2 & \text{which is} & -3x_1 + x_2 + 2x_3 = -7 \end{array}$$

Another representation of this system as a vector equation is

$$x_1 A_{.1} + x_2 A_{.2} + x_3 A_{.3} = b$$

$$\text{i.e.} \qquad x_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ -7 \end{pmatrix}$$

The basic solution of this system corresponding to the basic vector of variables (x_1, x_3) obtained by the GJ pivotal method is $\bar{x} = (17, 0, 22)^T$. It satisfies this vector equation because

$$17\begin{pmatrix} 1 \\ -3 \end{pmatrix} + 0\begin{pmatrix} -2 \\ 1 \end{pmatrix} + 22\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ -7 \end{pmatrix}$$

The general solution of this system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 17 - 3\bar{x}_2 \\ \bar{x}_2 \\ 22 - 5\bar{x}_2 \end{pmatrix}$$

where $\bar{x}_2$ is an arbitrary real valued parameter, and it can be verified that this satisfies the above vector equation.

Cautionary Note: Care in writing matrix products: This is for those students who are still not totally familiar with matrix manipulations. In real number arithmetic, the equation

$$5 \times 4 = 20$$

can be written in an equivalent manner as

$$5 = \frac{20}{4}$$

Carrying this practice to equations involving matrix products will most often result in drastic blunders. For example, writing

$$Ax = b \quad \text{as} \quad A = \frac{b}{x} \quad \text{or as} \quad x = \frac{b}{A} \quad \text{is a blunder}$$

because division by vectors is not defined, inverse of a rectangular matrix is not defined; and even if the matrix A has an inverse to be defined later, it is not denoted by $\frac{1}{A}$ but by a different symbol.

Therefore deducing that $x = \frac{b}{A}$ from $Ax = b$ is always a drastic blunder.

Exercises

2.3.1: For A, b given below, write the system of linear equations $Ax = b$ as a vector equation.

$$(i) \quad A = \begin{pmatrix} 1 & 2 & 0 & 9 \\ -1 & 0 & 1 & 1 \\ 0 & 3 & 4 & -7 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}.$$

$$(ii) \quad A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 2 & 1 & 2 \\ 1 & -2 & 1 & -2 \\ 0 & -1 & -2 & 0 \end{pmatrix}, b = \begin{pmatrix} 9 \\ -7 \\ 0 \\ 4 \end{pmatrix}.$$

2.4 Scalar Multiplication and Addition of Matrices, Transposition

Let $A = (a_{ij})$ be a matrix of order $m \times n$. Scalar multiplication of this matrix A by a scalar α involves multiplying each element in A by α. Thus

$$\alpha A = \alpha(a_{ij}) = (\alpha a_{ij})$$

For instance

$$6 \begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -12 & -6 \\ -18 & 6 & 12 \end{pmatrix}.$$

Given two matrices A and B, their sum is only defined if both matrices are of the same order. So, if $A = (a_{ij})$ and $B = (b_{ij})$ are both of order $m \times n$, then their sum $A + B$ is obtained by summing up corresponding elements, and is equal to $(a_{ij} + b_{ij})$. For instance

$$\begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{is not defined.}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & -10 & 18 \\ 8 & -7 & 20 \end{pmatrix} = \begin{pmatrix} 4 & -12 & 17 \\ 5 & -6 & 22 \end{pmatrix}$$

The operation of transposition of a matrix changes its row vectors into column vectors and vice versa. Thus if A is an $m \times n$ matrix, its transpose denoted by A^T is an $n \times m$ matrix obtained by writing the row vectors of A as column vectors in A^T in their proper order. For instance

$$\begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & -3 \\ -2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 9 \\ 18 & -33 \end{pmatrix}^T = \begin{pmatrix} 3 & 18 \\ 9 & -33 \end{pmatrix}.$$

Earlier in Section 1.4 we defined the transpose of a row vector as a column vector and vice versa. That definition tallies with the definition of transpose for matrices given here when those vectors are looked at as matrices with a single row or column.

Some Properties of Matrix Addition and Transposition

Let $A^t = (a^t_{ij})$ be a matrix of order $m \times n$ for $t = 1$ to r. The sum, and a linear combination with multipliers $\alpha_1, \ldots, \alpha_r$ respectively, of these r matrices are defined similarly. We have

$$A^1 + \ldots + A^r = (\sum_{t=1}^{r} a^t_{ij})$$

$$\alpha_1 A^1 + \ldots + \alpha_r A^r = (\sum_{t=1}^{r} \alpha_t a^t_{ij})$$

In finding the sum of two or more matrices, these matrices can be written in the sum in any arbitrary order, the result does not depend on this order. For instance

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 8 & -7 \end{pmatrix} + \begin{pmatrix} 3 & 6 & -3 \\ -4 & -2 & -1 \end{pmatrix} + \begin{pmatrix} -1 & -8 & -3 \\ -2 & -4 & 13 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -8 & -3 \\ -2 & -4 & 13 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 3 \\ 0 & 8 & -7 \end{pmatrix} + \begin{pmatrix} 3 & 6 & -3 \\ -4 & -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -4 & -3 \\ -6 & 2 & 5 \end{pmatrix}$$

Let A be an $m \times n$ matrix, $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots v_n)^T$. Then it can be verified that

$$(Au)^T = u^T A^T \qquad (vA)^T = A^T v^T$$

So, the transpose of a matrix vector product is the product of their transposes in reverse order. For example, let

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 8 & -7 \end{pmatrix}, u = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, v = (5, 10).$$

Then

$$Au = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 8 & -7 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 23 \end{pmatrix}$$

$$vA = (5, 10) \begin{pmatrix} 1 & -2 & 3 \\ 0 & 8 & -7 \end{pmatrix} = (5, 70, -55)$$

$$u^T A^T = (3, 2, -1) \begin{pmatrix} 1 & 0 \\ -2 & 8 \\ 3 & -7 \end{pmatrix} = (-4, 23)$$

$$A^T v^T = \begin{pmatrix} 1 & 0 \\ -2 & 8 \\ 3 & -7 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 5 \\ 70 \\ -55 \end{pmatrix}$$

So, we verify that $(Au)^T = u^T A^T$ and $(vA)^T = A^T v^T$.

Exercises

2.4.1: Write the transposes of the following matrices. Also, mention for which pair U, V of these matrices is $5U - 3V$ defined, and compute the result if it is.

$$A = \begin{pmatrix} 1 & -3 & 0 \\ -2 & 2 & -2 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}, C = \begin{pmatrix} 0 & 3 \\ -1 & -2 \\ 1 & -3 \end{pmatrix},$$

$$D = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}.$$

2.5 Matrix Multiplication & Its Various Interpretations

Rules for matrix multiplication evolved out of a study of transformations of variables in systems of linear equations. Let $A = (a_{ij})$ be the coefficient matrix of order $m \times n$ in a system of linear equations, b the RHS constants vector, and $x = (x_1, \ldots, x_n)^T$ the column vector of variables in the system. Then the system of equations is $Ax = b$ in matrix notation. Here are all the equations in it.

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ \vdots \quad &\quad \vdots \\ a_{i1}x_1 + \ldots + a_{in}x_n &= b_i \\ \vdots \quad &\quad \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

Now suppose we transform the variables in this system using the transformation $x = By$ where $B = (b_{ij})$ is an $n \times n$ matrix, and $y = (y_1, \ldots y_n)$ are the new variables. So the transformation is:

$$\begin{aligned} x_1 &= b_{11}y_1 + \ldots + b_{1n}y_n \\ \vdots \quad & \quad \vdots \\ x_j &= b_{j1}y_1 + \ldots + b_{jn}x_n \\ \vdots \quad & \quad \vdots \\ x_n &= b_{n1}y_1 + \ldots + b_{nn}y_n \end{aligned}$$

Substituting these expressions for $x_1, \ldots, x_n$ in the original system of equations and regrouping terms will modify it into the transformed system in terms of the new variables y. Suppose the transformed system is $Cy = b$ with C as its coefficient matrix. Alternately, substituting $x = By$ in $Ax = b$ gives $ABy = b$; hence we define the matrix product AB to be C.

We will illustrate this with a numerical example. Let

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix}, b = \begin{pmatrix} -5 \\ -7 \end{pmatrix}, x = (x_1, x_2, x_3)^T$$

and consider the system of equations $Ax = b$, which is

$$x_1 - 2x_2 - x_3 = -5$$

$$-3x_1 + x_2 + 2x_3 = -7$$

Suppose we wish to transform the variables using the transformation

$$\begin{aligned} x_1 &= 10y_1 + 11y_2 + 12y_3 \\ x_2 &= 20y_1 + 21y_2 + 22y_3 \\ x_3 &= 30y_1 + 31y_2 + 32y_3 \end{aligned}$$

or $x = By$ where

$$B = \begin{pmatrix} 10 & 11 & 12 \\ 20 & 21 & 22 \\ 30 & 31 & 32 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Substituting the expressions for x_1, x_2, x_3 in terms of y_1, y_2, y_3 in the two equations of the system above, we get

$$1(10y_1+11y_2+12y_3)-2(20y_1+21y_2+22y_3)-1(30y_1+31y_2+32y_3) = -5$$

$$-3(10y_1+11y_2+12y_3)+1(20y_1+21y_2+22y_3)+2(30y_1+31y_2+32y_3) = -7$$

Regrouping the terms, this leads to the system

$$(1,-2,-1)(10,20,30)^T y_1 + (1,-2,-1)(11,21,31)^T y_2$$
$$+(1,-2,-1)(12,22,32)^T y_3 = -5$$

$$(-3,1,2)(10,20,30)^T y_1 + (-3,1,2)(11,21,31)^T y_2$$
$$+(-3,1,2)(12,22,32)^T y_3 = -7$$

or $Cy = b$ where $y = (y_1, y_2, y_3)^T$ and

$$C = \begin{pmatrix} A_{1.}B_{.1} & A_{1.}B_{.2} & A_{1.}B_{.3} \\ A_{2.}B_{.1} & A_{2.}B_{.2} & A_{2.}B_{.3} \end{pmatrix}$$

So, we should define the matrix product AB, in this order, to be the matrix C defined by the above formula. This gives the reason for the definition of matrix multiplication given below.

Matrix Multiplication

If A, B are two matrices, the product AB in that order is defined only if

> the number of columns in A is equal to the number of rows in B

i.e., if A is of order $m \times n$, and B is of order $r \times s$, then for the product AB to be defined n must equal r; otherwise AB is not defined.

If A is of order $m \times n$ and B is of order $n \times s$, then $AB = C$ is of order $m \times s$, and the (i,j)th entry in C is the dot product $A_{i.}B_{.j}$.

Therefore matrix multiplication is **row by column multiplication**, and the (i,j)th entry in the matrix product AB is the dot product of the ith row vector of A and the jth column vector of B.

Examples of Matrix Multiplication

$$\begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{is not defined}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 10 & 11 & 12 \\ 20 & 21 & 22 \\ 30 & 31 & 32 \end{pmatrix} = \begin{pmatrix} -60 & -62 & -64 \\ 50 & 50 & 50 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -5 \\ 2 & -3 & -6 \end{pmatrix} \begin{pmatrix} -2 & -4 \\ -6 & -4 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -11 & 10 \\ -4 & 16 \end{pmatrix}$$

The important thing to remember is that matrix multiplication is **not commutative**, i.e., the order in which matrices are multiplied is very important. When the product AB exists, the product BA may not even be defined, and even if it is defined, AB and BA may be of different orders; and even when they are of the same order they may not be equal.

Examples

Let

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 10 & 11 & 12 \\ 20 & 21 & 22 \\ 30 & 31 & 32 \end{pmatrix},$$

$$D = \begin{pmatrix} -2 & -4 \\ -6 & -4 \\ 3 & -2 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

AB is computed above, and BA is not defined.

$$AD = \begin{pmatrix} 7 & 6 \\ 6 & 4 \end{pmatrix}, \quad DA = \begin{pmatrix} 10 & 0 & -6 \\ 6 & 8 & -2 \\ 9 & -8 & -7 \end{pmatrix}$$

$$BE = \begin{pmatrix} 12 & 10 & 11 \\ 22 & 20 & 21 \\ 32 & 30 & 31 \end{pmatrix}, \quad EB = \begin{pmatrix} 20 & 21 & 22 \\ 30 & 31 & 32 \\ 10 & 11 & 12 \end{pmatrix}$$

So, AD and DA are of different orders. BE and EB are both of the same order, but they are not equal.

Thus, when writing a matrix product, one should specify the order of the entries very carefully.

We defined matrix-vector multiplication in Section 2.2. That definition coincides with the definition of matrix multiplication given here when the vectors in the product are treated as matrices with a single row or column respectively.

In the matrix product AB we say that A **premultiplies** B; or B **postmultiplies** A.

Column-Row Products

Let $u = (u_1, \ldots, u_m)^T$ be a column vector in R^m, and $v = (v_1, \ldots, v_n)$ a row vector in R^n. Then the product uv in that order, known as the **outer product of the vectors** u, v is the matrix product uv when u, v are treated as matrices with one column and one row respectively, given by

$$uv = \begin{pmatrix} u_1v_1 & u_1v_2 & \ldots & u_1v_n \\ u_2v_1 & u_2v_2 & \ldots & u_2v_n \\ \vdots & & & \vdots \\ u_mv_1 & u_mv_2 & \ldots & u_mv_n \end{pmatrix}$$

As an example

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} (3, 4) = \begin{pmatrix} 3 & 4 \\ 0 & 0 \\ -3 & -4 \\ 6 & 8 \end{pmatrix}$$

Notice the difference between the inner (or dot) product of two vectors defined in Section 1.5, and the outer product of two vectors defined here. The inner product is always a real number, and it is only defined for a pair of vectors from the same dimensional space. The outer product is a matrix, and it is defined for any pair of vectors (they may be from spaces of different dimensions).

Matrix Multiplication Through a Series of Matrix-Vector Products

Let A, B be matrices of orders $m \times n$, $n \times k$ respectively. We can write the product AB as

$$\begin{aligned} AB &= A(B_{.1} \vdots \ldots \vdots B_{.k}) \\ &= (AB_{.1} \vdots \ldots \vdots AB_{.k}) \end{aligned}$$

where for each $t = 1$ to k, $B_{.t}$ is the tth column vector of B, and $AB_{.t}$ is the column vector that is the result of a matrix-vector product as defined earlier in Section 2.2. Thus the product AB can be looked at as the matrix consisting of column vectors $AB_{.t}$ in the order $t = 1$ to k.

Another way of looking at the product AB is

$$AB = \begin{pmatrix} A_{1.} \\ \vdots \\ A_{m.} \end{pmatrix} B = \begin{pmatrix} A_{1.}B \\ \vdots \\ A_{m.}B \end{pmatrix}$$

From these interpretations of matrix multiplication, we see that the

jth column of the product AB is $AB_{.j}$

ith row of the product AB is $A_{i.}B$

Matrix Product as Sum of a Series of Column-Row Products

Let $A = (a_{ij}), B = (b_{ij})$ be two matrices such that the product AB is defined. So

the number of columns in A = number of rows in $B = n$, say.

Let $A_{.1}, \ldots, A_{.n}$ be the column vectors of A; and $B_{1.}, \ldots, B_{n.}$ be the row vectors in B. For the sake of completeness suppose A is of order $m \times n$, and B is of order $n \times k$.

From the definition of the matrix product AB we know that its (i, j)th element is

$$\sum_{t=1}^{n} a_{it}b_{tj} = a_{i1}b_{1j} + \ldots + a_{in}b_{nj}$$

So,

$$AB = \sum_{t=1}^{n}(\text{matrix of order } m \times k \text{ whose } (i,j)\text{th element is } a_{it}b_{tj})$$

However, the matrix whose (i, j)th element is $a_{it}b_{tj}$ is the outer product $A_{.t}B_{t.}$. Hence

$$AB = \sum_{t=1}^{n} A_{.t}B_{t.} = A_{.1}B_{1.} + \ldots + A_{.n}B_{n.}$$

This is the formula that expresses AB as a sum of n column-row products. As an example let

$$\begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 10 & 11 & 12 \\ 20 & 21 & 22 \\ 30 & 31 & 32 \end{pmatrix}$$

Then

$$A_{.1}B_{1.} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}(10, 11, 12) = \begin{pmatrix} 10 & 11 & 12 \\ -30 & -33 & -36 \end{pmatrix}$$

$$A_{.2}B_{2.} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}(20, 21, 22) = \begin{pmatrix} -40 & -42 & -44 \\ 20 & 21 & 22 \end{pmatrix}$$

$$A_{.3}B_{3.} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}(30, 31, 32) = \begin{pmatrix} -30 & -31 & -32 \\ 60 & 62 & 64 \end{pmatrix}$$

So

$$A_{.1}B_{1.} + A_{.2}B_{2.} + A_{.3}B_{3.} = \begin{pmatrix} -60 & -62 & -64 \\ 50 & 50 & 50 \end{pmatrix}$$

which is the same as the product AB computed earlier.

Exercises

2.5.1 Let A, B be $m \times n, n \times k$ matrices respectively. Show that

each column of AB is a linear combination of the columns of A

each row of AB is a linear combination of the rows of A.

The Product of a Sequence of 3 Or More Matrices

Let $A_1, A_2, \ldots, A_t$ be $t \geq 3$ matrices. Consider the matrix product

$$A_1 A_2 \ldots A_t$$

in this specific order. This product is defined iff

the product of every consecutive pair of matrices in this order is defined,

i.e., for each $r = 1$ to $t-1$, the number of rows in A_r is equal to the number of columns in A_{r+1}.

When these conditions are satisfied, this product of t matrices is computed recursively, i.e., take any consecutive pair of matrices in this order, say A_r an A_{r+1} for some $1 \leq r \leq t-1$, and suppose $A_r A_{r+1} = D$. Then the above product is

$$A_1 \ldots A_{r-1} D A_{r+2} \ldots A_t$$

.

There are only $t-1$ matrices in this product, it can be reduced to a product of $t-2$ matrices the same way, and the same procedure is continued until the product is obtained as a single matrix.

When it is defined, the product $A_1 A_2 \ldots A_t$ is of order $m_1 \times n_t$ where

m_1 = number of rows in A_1, the leftmost matrix in the product

n_t = number of columns in A_t, the rightmost matrix in the product.

The product $A_1 A_2 \ldots A_t$ is normally computed by procedures known as **string computation**, of which there are two.

The **left to right string computation** for $A_1 A_2 \ldots A_t$ computes $D_r = A_1 \ldots A_r$ for $r = 2$ to t in that order using

$$\begin{aligned} D_2 &= A_1 A_2 \\ D_{r+1} &= D_r A_{r+1}, \qquad \text{for } r = 2 \text{ to } t-1 \end{aligned}$$

to obtain the final result D_t.

The **right to left string computation** for $A_1 A_2 \ldots A_t$ computes $E_r = A_r \ldots A_t$ for $r = t-1$ to 1 in that order using

$$\begin{aligned} E_{t-1} &= A_{t-1} A_t \\ E_{r-1} &= A_{r-1} E_r, \qquad \text{for } r = t-1 \text{ to } 2 \end{aligned}$$

to obtain the final result E_1, which will be the same as D_t obtained by the previous recursion.

These facts follow by repeated application of the following result.

Result 2.5.1: Product involving three matrices: *Let A, B, C be matrices of orders $m \times n, n \times p, p \times q$ respectively. Then $A(BC) = (AB)C$.*

To prove this result, suppose $BC = D$. Then from one of the interpretations of two-matrix products discussed earlier we know that

$$D = BC = (BC_{.1} \vdots \ldots \vdots BC_{.q})$$

So,

$$A(BC) = AD = A(BC_{.1} \vdots \ldots \vdots BC_{.q}) = (A(BC_{.1}) \vdots \ldots \vdots A(BC_{.q}))$$

Also, let $C = (c_{ij})$. Then, for each $j = 1$ to q, $BC_{.j} = \sum_{i=1}^{p} c_{ij} B_{.i}$. So

$$A(BC_{.j}) = A(\sum_{i=1}^{p} c_{ij} B_{.i}) = \sum_{i=1}^{p} c_{ij}(AB_{.i})$$

So, $A(BC)$ is an $m \times q$ matrix whose jth column is $\sum_{i=1}^{p} c_{ij}(AB_{.i})$ for $j = 1$ to q.

Also, from another interpretation of two-matrix products discussed earlier we know that

$$AB = (AB_{.1} \vdots \ldots \vdots AB_{.p})$$

So,

$$(AB)C = (AB_{.1} \vdots \ldots \vdots AB_{.p}) \begin{pmatrix} C_{1.} \\ \vdots \\ C_{p.} \end{pmatrix} = (AB_{.1})C_{1.} + \ldots + (AB_{.p})C_{p.}$$

Notice that for each $i = 1$ to p, $(AB_{.i})C_{i.}$ is an outer product (a column-row product) that is an $m \times q$ matrix whose jth column is $c_{ij}(AB_{.i})$ for $j = 1$ to q. Hence, $(AB)C$ is an $m \times q$ matrix whose jth column is $\sum_{i=1}^{p} c_{ij}(AB_{.i})$ for $j = 1$ to q; i.e., it is the same as $A(BC)$ from the fact established above. Therefore

$$A(BC) = (AB)C$$

establishing the result.

Properties Satisfied By Matrix Products

Order important: The thing to remember is that the order in which the matrices are written in a matrix product is very important. If the order is changed, the product may not be defined, and even if it is defined, the result may be different. As examples, let

$$A = \begin{pmatrix} -3 & 0 & 1 \\ -2 & -6 & 4 \end{pmatrix}, B = \begin{pmatrix} -5 & -10 \\ 20 & 30 \\ 1 & 2 \end{pmatrix}, C = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Then

$$AC = \begin{pmatrix} -4 \\ -10 \end{pmatrix}, \qquad \text{and } CA \text{ is not defined}$$

$$AB = \begin{pmatrix} -14 & -28 \\ -106 & -152 \end{pmatrix}, \quad BA = \begin{pmatrix} 35 & 60 & -45 \\ -120 & -180 & 140 \\ -7 & -12 & 9 \end{pmatrix}$$

so, AB and BA are not even of the same order.

Associative law of multiplication: If A, B, C are matrices such that the product ABC is defined, then $A(BC) = (AB)C$.

Left and Right distributive laws: Let B, C be matrices of the same order. If A is a matrix such that the product $A(B+C)$ is defined, it is $= AB + AC$. If A is a matrix such that the product $(B+C)A$ is defined, it is $= BA + CA$. So, matrix multiplication distributes across matrix addition.

If α, β are scalars, and A, B two matrices such that the Product AB is defined, then, $(\alpha + \beta)A = \alpha A + \beta A$; and $\alpha AB = (\alpha A)B = A(\alpha B)$.

The product of two nonzero matrices may be zero (i.e., a matrix in which all the entries are zero). For example, for 2×2 matrices A, B both of which are nonzero, verify that $AB = 0$.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

If A, B, C are matrices such that the products AB, AC are both defined; and $AB = AC$, we cannot, in general, conclude that $B = C$. Verify this for the following matrices A, B, C, we have $AB = AC$, even though $B \neq C$.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, C = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}$$

Matrix multiplication and transposes: If A, B are two matrices such that the product AB is defined, then the product $B^T A^T$ is defined and $(AB)^T = B^T A^T$. This can be verified directly from the definitions.

In the same way if the matrix product $A_1 A_2 \ldots A_r$ is defined, then $(A_1 A_2 \ldots A_r)^T = A_r^T A_{r-1}^T \ldots A_2^T A_1^T$.

Exercises

2.5.2: For every pair of matrices U, V among the following, determine whether UV is defined, and compute it if it is.

$$A = \begin{pmatrix} 1 & -3 & 2 & 0 \\ -1 & 2 & 3 & 4 \\ 2 & 1 & 1 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & 2 \\ -1 & -3 & 0 \\ -2 & -7 & 6 \end{pmatrix},$$

$$C = \begin{pmatrix} 4 & -5 & 2 \\ -1 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix}, D = \begin{pmatrix} 5 & -5 & 0 & 3 \\ -1 & -1 & -1 & -1 \\ 1 & 2 & 2 & 1 \\ 0 & -2 & -2 & -1 \end{pmatrix},$$

$$E = \begin{pmatrix} 2 & -3 \\ 4 & -2 \end{pmatrix}, F = \begin{pmatrix} 1 & -2 & 3 & -4 \\ -1 & 3 & -2 & 2 \end{pmatrix}.$$

2.5.3: Sometimes when A, B are a pair of matrices, both products AB, BA may exist. These products may be of different orders, but even when they are of the same order they may or may not be equal. Verify these with the following pairs.

(i): $A = \begin{pmatrix} 1 & 7 & -7 & 2 \\ 0 & -5 & 3 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$

(ii): $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}.$

(iii): $A = \begin{pmatrix} 8 & 3 \\ -4 & -2 \end{pmatrix}, B = \begin{pmatrix} 2 & 9 \\ 6 & 3 \end{pmatrix}.$

2.5.4: Compute the products $ADB, DBA, BAD, EFDB$ where the matrices A to F are given in Exercise 2.5.2.

2.6 Some Special Matrices

Square Matrices

A matrix of order $m \times n$ is said to be a

square matrix	if $m = n$
rectangular matrix that is not square	if $m \neq n$.

Hence a square matrix is a matrix of order $n \times n$ for some n. Since the number of rows and the number of columns in a square matrix are the same, their common value is called the **order of the square matrix**. Thus an $n \times n$ square matrix is said to be of order n. Here are examples of square matrices.

$$\begin{pmatrix} 2 & -3 \\ 1 & -10 \end{pmatrix}, \begin{pmatrix} -3 & -4 & -5 \\ 6 & 7 & 8 \\ -9 & 10 & 11 \end{pmatrix}$$

In a square matrix (a_{ij}) of order n, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ constitute its **diagonal entries** or its **main diagonal**. All the other entries in this matrix are called **off-diagonal entries**. Here is a picture

$$\begin{pmatrix} \boxed{a_{11}} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & \boxed{a_{22}} & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & \boxed{a_{n-1,n-1}} & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & \boxed{a_{nn}} \end{pmatrix}$$

A square matrix with its diagonal entries boxed.

Unit (Identity) Matrices

Unit matrices are square matrices in which all diagonal entries are "1" and all off-diagonal entries are "0". Here are the **unit matrices** (also called **identity matrices**) of some orders.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The reason for the name is that when a matrix A is multiplied by the unit matrix I of appropriate order so that the product is defined, the resulting product IA or AI is A itself. Here are some examples.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix}$$

Hence, in the world of matrices, the unit matrices play the same role as the number "1" does in the world of real numbers. This is the reason for their name.

The unit matrix is usually denoted by the symbol I when its order can be understood from the context. If the order has to be indicated specifically, the unit matrix of order n is usually denoted by the symbol I_n.

In Chapter 1 we defined unit vectors as column vectors with a single nonzero entry of "1". From this it is clear that the unit vectors are column vectors of the unit matrix. Thus if I is the unit matrix of order n, then its column vectors $I_{.1}, I_{.2}, \ldots, I_{.n}$ are the 1st, 2nd, ..., nth unit vectors in R^n.

Permutation Matrices

A square matrix of order n in which all the entries are 0 or 1, and there is exactly a single "1" entry in each row and in each column is called a **permutation matrix** or an **assignment**. A permutation matrix can always be transformed into the unit matrix by permuting (i.e., rearranging in some order) its rows, or its columns. Here are some permutation matrices.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Premultiplying any matrix A by a permutation matrix of approproate order so that the product is defined, permutes its rows. In the same way, postmultiplying any matrix A by a permutation matrix of approproate order so that the product is defined, permutes its columns. Here are some examples.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix}$$

The unit matrix is also a permutation matrix. Also, it can be verified that if P is a permutation matrix, then

$$PP^T = P^TP = I, \quad \text{the unit matrix of the same order.}$$

Diagonal Matrices

A square matrix with all its off-diagonal entries zero, and all diagonal entries nonzero is called a **diagonal matrix**. Here are some examples.

$$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -8 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Since all the off-diagonal entries in a diagonal matrix are known to be zero, if we are given the diagonal entries in it, we can construct the diagonal matrix. For this reason, a diagonal matrix of order n with diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$ is denoted by the symbol $\mathbf{diag}\{a_{11}, a_{22}, \ldots, a_{nn}\}$. In this notation, the three diagonal matrices given above will be denoted by $\text{diag}\{3, -2\}$, $\text{diag}\{-2, -1, -8\}$, $\text{diag}\{1, 2, 3, 4\}$ respectively.

Premultiplying a matrix A by a diagonal matrix of appropriate order so that the product is defined, multiplies every entry in the ith row of A by the ith diagonal entry in the diagonal matrix; this operation is called **scaling the rows of** A. Similarly, postmultiplying a matrix A by a diagonal matrix **scales the columns of** A.

$$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -6 & -3 \\ 6 & -2 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -8 \end{pmatrix} = \begin{pmatrix} -2 & 2 & 8 \\ 6 & -1 & -16 \end{pmatrix}$$

Upper and Lower Triangular Matrices, Triangular Matrices

We already discussed upper, lower triangular tableaus; and triangular tableaus in Section 1.18. The corresponding matrices are called upper triangular, lower triangular, or triangular matrices. We give formal definitions below.

A square matrix is said to be **upper triangular** if all entries in it below the diagonal are zero, and all the diagonal entries are nonzero. It is said to be **lower triangular** if all the diagonal entries are nonzero, and all the entries above the diagonal are zero. It is said to be **triangular** if its rows can be rearranged to make it upper triangular. Here are examples.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

The first matrix is upper triangular, the second is lower triangular, and the third is triangular.

Symmetric Matrices

A square matrix $D = (d_{ij})$ is said to be **symmetric** iff $D^T = D$, i.e., $d_{ij} = d_{ji}$ for all i, j.

A square matrix which is not symmetric is said to be an **asymmetric matrix**.

If D is an asymmetric matrix, $(1/2)(D + D^T)$ is called its **symmetrization**.

$$\begin{pmatrix} 6 & 1 & -1 & -3 \\ 1 & 0 & -1 & 0 \\ -1 & -1 & -2 & 1 \\ -3 & 0 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The first matrix is symmetric, the second is asymmetric, and the third is the symmetrization of the second matrix.

Submatrices of a Matrix

Let $A = (a_{ij})$ be a given matrix of order $m \times n$. Let $R \subset \{1, \ldots, m\}$, $C \subset \{1, \ldots, n\}$, in both of which the entries are arranged in increasing order. By deleting all the rows of A not in R, and all the columns of A not in C, we are left with a matrix which is known as the **submatrix of A determined by the subsets R, C of rows and columns** and denoted usually by A_{RC}. For example, let

$$A = \begin{pmatrix} 3 & -1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 4 & -2 & 0 & 1 & 1 \\ 5 & -3 & 0 & 0 & 2 \end{pmatrix}$$

Let $R = \{1, 3, 4\}$, $C = \{1, 2, 4, 5\}$. Then the submatrix corresponding to these subsets is

$$A_{RC} = \begin{pmatrix} 3 & -1 & 1 & 0 \\ 4 & -2 & 1 & 1 \\ 5 & -3 & 0 & 2 \end{pmatrix}$$

Principal Submatrices of a Square Matrix

Let $D = (d_{ij})$ be a given square matrix of order n. Let $Q \subset \{1, \ldots, n\}$, in which the entries are arranged in increasing order. By deleting all the rows and columns of D not in Q, we are left with a matrix which is known as the **principal submatrix of D determined by the subset Q of rows and columns** and denoted usually by D_{QQ}. For example, let

$$D = \begin{pmatrix} 3 & -1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 & 0 \\ 4 & -2 & 0 & 1 & 1 \\ 5 & -3 & 0 & 0 & 2 \\ 8 & -17 & -18 & 9 & 11 \end{pmatrix}$$

Let $Q = \{2, 3, 5\}$. Then the principal submatrix of D determined by the subset Q is

$$D_{QQ} = \begin{pmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \\ -17 & -18 & 11 \end{pmatrix}$$

Principal submatrices are only defined for square matrices, and the main diagonal of a principal submatrix is always a subset of the main diagonal of the original matrix.

Exercises

2.6.1: Let

$$A = \begin{pmatrix} -20 & 10 & 30 & 4 & -17 \\ 18 & 0 & 3 & 2 & -19 \\ 0 & 6 & -7 & 8 & 12 \\ 13 & -6 & 19 & 33 & 14 \\ 12 & -9 & 22 & 45 & 51 \end{pmatrix}$$

Write the submatrix of A corresponding to the subset of rows $\{2, 4\}$ and the subset of columns $\{1, 3, 5\}$. Also write the principal submatrix of A determined by the subset of rows and columns $\{2, 4, 5\}$.

2.7 Row Operations and Pivot Steps on a Matrix

Since a matrix is a 2-dimensional array of numbers, it is like the tableau we discussed earlier. Rows and columns of the matrix are exactly like the rows and columns of a tableau.

Row operations on a matrix are exactly like row operations on a tableau. They involve the following:

1. Scalar Multiplication: Multiply each entry in a row by the same nonzero scalar

2. Add a Scalar Multiple of a Row to Another: Multiply each element in a row by the same nonzero scalar and add the result to the corresponding element of the other row.

3. Row interchange: Interchange two rows in the matrix.

Example : Consider the matrix

$$A = \begin{pmatrix} -1 & 2 & 1 & 3 \\ 0 & -3 & 9 & 7 \\ 8 & 6 & -4 & -5 \end{pmatrix}$$

Multiplying $A_{3.}$ = Row 3 of A, by -2 leads to the matrix

$$A' = \begin{pmatrix} -1 & 2 & 1 & 3 \\ 0 & -3 & 9 & 7 \\ -16 & -12 & 8 & 10 \end{pmatrix}$$

Performing the row operation $A_{2.} - 2A_{1.}$ leads to the matrix

$$A'' = \begin{pmatrix} -1 & 2 & 1 & 3 \\ 2 & -7 & 7 & 1 \\ 8 & 6 & -4 & -5 \end{pmatrix}$$

A GJ (Gauss-Jordan) pivot step on a matrix is exactly like a GJ pivot step on a tableau. It is specified by choosing a row of the matrix as the pivot row, and a column of the matrix as the pivot column, with the element in the pivot row and pivot column called the pivot element which should be nonzero. The GJ pivot step then converts the pivot column into the unit column with a "1" entry in the pivot row and "0" entries in all other rows, by appropriate row operations.

A G (Gaussian) pivot step on a matrix is like the GJ pivot step, with the exception that it only converts all entries in the pivot column below the pivot row into zeros by appropriate row operations, but leaves all the rows above the pivot row and the pivot row itself unchanged.

2.8 Determinants of Square Matrices

History of determinants: Determinants are real valued functions of square matrices, i.e., associated with every square matrix is a unique real number called its **determinant**. Determinants are only defined for square matrices. There is no such concept for rectangular matrices that are not square. The definition of the determinant of a square array of numbers goes back to the the end of the 17th century in the works of Seki Takakazu (also called Seki Kowa in some books) of Japan and Gottfried Leibnitz of Germany. Seki arrived at the notion of a determinant while trying to find common roots of algebraic equations. To find common roots of polynomials $f(x), g(x)$ of small degrees Seki got determinant expressions and published a treatize in 1674. Lebnitz did not publish the results of his studies related to determinants, but in a letter to l'Hospital in 1693 he wrote down the determinant condition of compatiability for a system of three linear equations in two unknowns. In Europe the first publication mentioning determinants was due to Cramer in 1750 in which he gave a determinant expression for the problem of finding a conic through five given points (this leads to a system of linear equations).

Since then determinants have been studied extensively for their theoretical properties and their applications in linear algebra theory. Even though determinants play a very major role in theory, they have not been used that much in computational algorithms. Since the focus of this book is computational linear algebra, we will list all the important properties of determinants without detailed proofs. References for proofs of the results are provided for interested readers to pursue. $\bowtie$

There are several equivalent ways of defining determinants, but all these satisfy the following fundamental result.

Result 2.8.1: One set of properties defining a determinant: *The determinant of a square matrix of order n is the unique real valued function of the matrix satisfying the following properties.*

(a) *If $A = (a_{ij})$ of order n is lower or upper triangular, then the determinant of A is the product of the diagonal* *entries of A.*

(b) *$det(A^T) = det(A)$ for all square matrices A.*

(c) *If A, B are two square matrices of order n, then $det(AB) = det(A) det(B)$.*

For $n = 1, 2$, determinants of square matrices of order n can be defined very easily.

The determinant of a 1×1 matrix (a_{11}) is defined to be a_{11}. So, $\det(0) = 0$, $\det(-4) = -4$, $\det(6) = 6$, etc.

For a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\det(A)$ is defined to be $a_{11}a_{22} - a_{12}a_{21}$. For example

$$\det \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = 3 \times 4 - 2 \times 1 = 12 - 2 = 10$$

$$\det \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} = 1 \times 2 - 4 \times 3 = 2 - 12 = -10$$

$$\det \begin{pmatrix} 1 & 4 \\ -3 & -12 \end{pmatrix} = 1 \times (-12) - (-3) \times 4 = -12 + 12 = 0$$

The original concept of the determinant of a square matrix $A = (a_{ij})$ of order n for $n \geq 2$ consists of a sum of $n!$ terms, half with a coefficient of $+1$, and the other half with a coefficient of -1. We will now explain this concept.

Some preliminary definitions first. In linear programming literature, each permutation matrix of order n is referred to as an **assignment of order** n. Here, for example, is an assignment of order 4.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Hence an assignment of order n is a square matrix of order n that contains exactly one nonzero entry of "1" in each row and column. Verify that there are $n!$ distinct assignments of order n.

We number the rows and columns of an assignment in natural order, and refer to the (i, j)th position in it as **cell** (i, j) (this is in row i and column j). We will represent each assignment by the subset of cells in it corresponding to the "1" entries written in natural order of rows of the matrix. For example, the assignment of order 4 given above corresponds to the set of cells $\{(1, 2), (2, 4), (3, 1), (4, 3)\}$ in this representation.

So, in this notation a general assignment or permutation matrix of order n can be represented by the set of cells $\{(1, j_1), (2, j_2), (3, j_3), \dots, (n, j_n)\}$ where $(j_1, j_2, \dots, j_n)$ is a **permutation** of $(1, 2, \dots, n)$, i.e., an arrangement of these integers in some order. We will say that this permutation matrix and permutation correspond to each other. For example, the permutation matrix of order 4 given above corresponds to the permutation $(2, 4, 1, 3)$.

There are $6 = 3!$ different permutations of $(1, 2, 3)$. These are: $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$. In the same way there are $24 = 4!$ permutations of $(1, 2, 3, 4)$; and in general $n!$ permutations of $(1, 2, \dots, n)$.

Consider the permutation $(j_1, \dots, j_n)$ corresponding to the permutation represented by the set of cells $\{(1, j_1), \dots, (n, j_n)\}$. We will use the symbol p to represent either the permutation or the corresponding permutation matrix. The determinant of the square matrix $A = (a_{ij})$ of order n contains one term corresponding to the permutation p, it is

$$(-1)^{NI(p)} a_{1j_1} a_{2j_2} \dots a_{nj_n}$$

where $NI(p)$ is a number called the **number of inversions** in the permutations p, which we will define below. The determinant of A is the sum of all the terms corresponding to all the $n!$ permutations.

In the permutation $(j_1, \dots, j_n)$, consider a pair of distinct entries (j_r, j_s) where $s > r$. If $j_s < j_r$ $[j_s > j_r]$ we say that this pair contributes one [0] inversions in this permutation. The total number of inversions in this permutation is obtained by counting the contributions of all pairs of the form (j_r, j_s) for $s > r$ in it. It is equal to $\sum_{r=1}^{n-1} q_r$ where

$$q_r = \text{number of entries in } (j_{r+1}, \dots, j_n) \text{ which are } < j_r.$$

A permutation is called an **even (odd) permutation** if the number of inversions in it is an even (odd) integer. As an example consider the permutation $(6, 1, 3, 4, 5, 2)$. Here

$j_1 = 6$, all of j_2, j_3, j_4, j_5, j_6 are $< j_1$, so $q_1 = 5$.

$j_2 = 1$, no. entries among $(j_3, j_4, j_5, j_6) < j_2$ is 0, so $q_2 = 0$.

$j_3 = 3$, no. entries among $(j_4, j_5, j_6) < j_3$ is 1, so $q_3 = 1$.

$j_4 = 4$, no. entries among $(j_5, j_6) < j_4$ is 1, so $q_4 = 1$.

$j_5 = 5$, no. entries among $(j_6) < j_5$ is 1, so $q_5 = 1$.

So the number of inversions in this permutation is $5+0+1+1+1 = 8$, hence this permutation is an even permutation.

In the same way verify that the number of inversions in the permutation $(2, 4, 1, 3)$ is $1 + 2 + 0 = 3$, hence this is an odd permutation.

For example, for $n = 3$, there are $3! = 6$ permutations. These permutations, and the formula for the determinant of $A = (a_{ij})$ of order 3 are given below.

Permutation p	$NI(p)$	Term corresponding to p
$(1, 2, 3)$	0	$a_{11}a_{22}a_{33}$
$(1, 3, 2)$	1	$-a_{11}a_{23}a_{32}$
$(2, 3, 1)$	2	$a_{12}a_{23}a_{31}$
$(2, 1, 3)$	1	$-a_{12}a_{21}a_{33}$
$(3, 1, 2)$	2	$a_{13}a_{21}a_{32}$
$(3, 2, 1)$	3	$-a_{13}a_{22}a_{31}$
Determinant(A)		$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$

In the same way, the original formula for the determinant of any square matrix of any order can be derived. However, since it is in the form of a sum of $n!$ terms, this formula is not used for computing the determinant when $n > 2$. There are much simpler equivalent formulas that can be used to compute determinants far more efficiently.

Recursive Definition of the Determinant of an $n \times n$ Matrix

We will now give a recursive definition of the determinant of an $n \times n$ matrix in terms of the determinants of its $(n-1) \times (n-1)$ submatrices. This definition follows from the one given above. We can use this definition to compute the determinant of a 3×3 matrix using the values of determinants of its 2×2 submatrices which themselves can be computed by the formula given above. The determinant of a matrix of order 4×4 can be computed using the determinants of its 3×3 submatrices, and so on.

Given a square matrix $A = (a_{ij})$, we will denote by A_{ij} the submatrix obtained by deleting the ith row and the jth column from A, this submatrix is called a **minor** of A.

Example: Let

$$A = \left(\begin{array}{r|r|rr} 1 & -2 & 0 & 0 \\ -1 & 7 & 8 & 5 \\ \hline 3 & 4 & -3 & -5 \\ \hline 2 & 9 & -4 & 6 \end{array}\right)$$

$$A_{32} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 8 & 5 \\ 2 & -4 & 6 \end{pmatrix}$$

Given the square matrix $A = (a_{ij})$, the (i,j)th **cofactor** of A, denoted by C_{ij} is given by

$$C_{ij} = (-1)^{i+j}\text{determinant}(A_{ij})$$

where A_{ij} is the minor of A obtained by deleting the ith row and the jth column from A.

So, while the minor A_{ij} is a matrix, the cofactor C_{ij} is a real number.

Example: Let $A = \begin{pmatrix} 3 & 1 & 2 \\ -2 & -1 & 1 \\ 8 & 1 & 0 \end{pmatrix}$. Then

$$A_{11} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_{11} = (-1)^{1+1}\det\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = -1$$

$$A_{12} = \begin{pmatrix} -2 & 1 \\ 8 & 0 \end{pmatrix}, \quad C_{12} = (-1)^{1+2}\det\begin{pmatrix} -2 & 1 \\ 8 & 0 \end{pmatrix} = 8$$

$$A_{13} = \begin{pmatrix} -2 & -1 \\ 8 & 1 \end{pmatrix}, \quad C_{13} = (-1)^{1+3}\det\begin{pmatrix} -2 & -1 \\ 8 & 1 \end{pmatrix} = 6$$

Definition: Cofactor expansion of a determinant: For $n \geq 2$, the determinant of the $n \times n$ matrix $A = (a_{ij})$ can be computed by a process called the **cofactor expansion** (or also **Laplace expansion**) **across any row** or **down any column**. Using the notation developed above, this expansion across the ith row is

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

Example: Let

$$A = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 3 & -2 \\ 1 & -1 & 0 \end{pmatrix}$$

Using cofactor expansion down the third column, we have

$$\begin{aligned} \det(A) &= 0C_{13} - 2C_{23} + 0C_{33} \\ &= -2(-1)^{2+3}\det\begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix} = 2(2 \times -1 \text{ - } 4 \times 1) = -12 \end{aligned}$$

Using cofactor expansion across third row, we have

$$\begin{aligned}\det(A) &= 1C_{31} - 1C_{32} \\ &= (-1)^{3+1}\det\begin{pmatrix} 4 & 0 \\ 3 & -2 \end{pmatrix} -1(-1)^{3+2}\det\begin{pmatrix} 2 & 0 \\ 3 & -2 \end{pmatrix} \\ &= -8 - 4 = -12, \text{ same value as above.}\end{aligned}$$

If a row or column has many zero entries, the cofactor expansion of the determinant using that row or column has many terms that are zero, and the cofactors in those terms need not be calculated. So, to compute the determinant of a square matrix, it is better to choose a row or column with the maximum number of zero entries, for cofactor expansion.

Exercises

2.8.1 Find the determinants of the following matrices with cofactor expansion using rows or columns that involve the least amount of computation.

$$(i) \quad \begin{pmatrix} 1 & 13 & 18 & -94 \\ 0 & -2 & -11 & 12 \\ 0 & 0 & 7 & 14 \\ 0 & 0 & 0 & -8 \end{pmatrix}, \quad (ii) \quad \begin{pmatrix} 0 & 0 & -4 & 3 \\ 3 & 0 & 0 & 8 \\ 2 & 1 & -2 & 0 \\ -1 & 3 & 4 & 0 \end{pmatrix}$$

$$(iii) \quad \begin{pmatrix} 0 & -8 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad (iv) \quad \begin{pmatrix} 0 & 0 & -2 & -3 \\ 1 & 3 & 2 & -4 \\ 0 & 0 & 0 & -5 \\ 0 & -1 & -2 & 4 \end{pmatrix}$$

Some Results On Determinants

1. Determinants of upper (lower) triangular matrices: *The determinant of an upper triangular, lower triangular, or diagonal matrix is the product of its diagonal entries.*

Let P be a permutation matrix. Its determinant is $(-1)^{NI(P)}$ where $NI(P)$ is the number of inversions in the permutation corresponding to P.

2. Adding a scalar multiple of a row (column) to another row (column) leaves determinant unchanged: *If a square matrix B is obtained by adding a multiple of a row to another row (or by adding a multiple of a column to another column) in a square matrix A, then det(B) = det(A).*

3. Determinant is 0 if matrix has a 0-row or 0-column: *If a square matrix has a zero row or a zero column, its determinant is 0.*

4. Determinant is 0 if a row (column) is a scalar multiple of another row (column): *If a row in a square matrix is a multiple of another row, or if a column in this matrix is a multiple of another column, then the determinant of that matrix is zero.*

5. Interchanging a pair of rows (columns) multiplies determinant by -1: *If a square matrix B is obtained by interchanging any two rows (or interchanging any two columns) of a square matrix A, then det(B) = $-$ det(A).*

6. Multiplying a row (column) by a scalar also multiplies determinant by that scalar: *If a square matrix B is obtained by multiplying each element in a row or a column of a square matrix A by the same scalar α, then $det(B) = \alpha det(A)$.*

7. Multiplying all entries in a matrix of order n by scalar α, multiplies determinant by α^n: *If A is a square matrix of order n, and α is a scalar, then $det(\alpha A) = \alpha^n det(A)$.*

8. A matrix and its transpose have the same determinant: *For any square matrix A, we have $det(A^T) = det(A)$.*

9. Determinant of a product of two matrices is the product of their determinants: *If A, B are two square matrices of order n, then $det(AB) = (det(A))\ (det(B))$.*

An incomplete proof of this product rule for determinants was given by J. P. M. Binet in 1813. The proof was corrected by A. L. Cauchy in 1815. Hence this result is known as the **Cauchy-Binet theorem.**

10. The effect of a GJ pivot step on the determinant: *If the square matrix A' is obtained by performing a GJ pivot step on the square matrix A with a_{rs} as the pivot element, then $det(A') = (1/a_{rs})det(A)$.*

11. The effect of a G pivot step on the determinant: *If the square matrix A' is obtained by performing a G pivot step on the square matrix A, then*

$det(A') = det(A)$ if no row interchange was performed on A before this G pivot step;

$det(A') = -\ det(A)$ if a row interchange was performed on A before this G pivot step.

12. Linearity of the Determinant function in a single column or row of the matrix: *The determinant of a square matrix $A = (a_{ij})$ of order n is a linear function of any single row or any single column of A, when all the other entries in the matrix are fixed at specific numerical values.*

The columns of A are $A_{.j}$, $j = 1$ to n. Suppose, for some s, all the entries in $A_{.1}, \ldots, A_{.s-1}, A_{.s+1}, \ldots, A_{.n}$ are held fixed, and only the entries in $A_{.s}$ are allowed to vary over all possible real values. Then $\det(A)$ is a linear function of the entries in $A_{.s} = (a_{1s}, \ldots, a_{ns})^T$. The reason for this is the following. Let C_{ij} denote the (i, j)th cofactor of A. Then by cofactor expansion down the sth column, we have

$$\det(A) = a_{1s}C_{1s} + a_{2s}C_{2s} + \ldots + a_{ns}C_{ns}$$

The cofactors $C_{1s}, \ldots, C_{ns}$ depend only on entries in $A_{.1}, \ldots, A_{.s-1}$, $A_{.s+1}, \ldots, A_{.n}$ which are all fixed, and hence these cofactors are all constants here. Hence by the above equation and the definition of linear functions given in Section 1.7, $\det(A)$ is a linear function of $a_{1s}, \ldots, a_{ns}$, the entries in $A_{.s}$, which are the only variables here.

$A_{i.}$, $i = 1$ to n are the row vectors of A. A similar argument shows that if for some r, all the entries in $A_{1.}, \ldots, A_{r-1.}, A_{r+1.}, \ldots, A_{n.}$ are held fixed, and only the entries in $A_{r.}$ are allowed to vary over all possible real values, then $\det(A)$ is a linear function of the entries in $A_{r.} = (a_{r1}, \ldots, a_{rn})$.

This leads to the following result. Suppose in the square matrix A of order n, the column vectors $A_{.1}, \ldots, A_{.s-1}, A_{.s+1}, \ldots, A_{.n}$ are all fixed, and

$$A_{.s} = \beta b + \delta d$$

where b, d are two column vectors in R^n, and β, δ are two scalars. So, the column vectors of A are as given below

$$A = (A_{.1} \vdots \ldots \vdots A_{.s-1} \vdots \beta b + \delta d \vdots A_{.s+1} \vdots \ldots \vdots A_{.n})$$

Define two matrices B, D with column vectors as given below.

$$B = (A_{.1} \vdots \ldots \vdots A_{.s-1} \vdots b \vdots A_{.s+1} \vdots \ldots \vdots A_{.n}), \quad D = (A_{.1} \vdots \ldots \vdots A_{.s-1} \vdots d \vdots A_{.s+1} \vdots \ldots \vdots A_{.n})$$

So, the matrices B, D differ from A only in their sth column. Then by the linearity

$$\det(A) = \beta \det(B) + \delta \det(D)$$

Caution: On the linearity of a determinant: For a square matrix A, $\det(A)$ is a linear function when it is treated as a function of the entries in a single column, or a single row of A, while all the other entries in A are held fixed at specific numerical values. As a function of all the entries in A, $\det(A)$ is definitely not linear. That's why, if B, C are square matrices of order n, and $A = B + C$; these facts do not imply that $\det(A) = \det(B) + \det(C)$.

Example: Let

$$E = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}, \quad F = \begin{pmatrix} -1 & -2 \\ -2 & 4 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -2 \\ 1 & 4 \end{pmatrix}.$$

$$B = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & -1 \\ -2 & 2 \end{pmatrix}.$$

Then, $A_{.2} = E_{.2} = F_{.2}$ and $A_{.1} = E_{.1} + F_{.1}$. We have $\det(E) = 10$, $\det(F) = -8$, and $\det(A) = 2 = \det(E) + \det(F)$, illustrating the linearity result stated above.

We also have $\det(B) = 5$, $\det(C) = -4$. Verify that even though $A = B + C$, we have $\det(A) \neq \det(B) + \det(C)$.

13. The determinant of a square matrix A is a multilinear function of A.

Definition of a multilinear function: Consider a real valued function $f(x^1, \ldots, x^r)$ of many variables which are partitioned into vectors $x^1, \ldots, x^r$; i.e., for each $k = 1$ to r, x^k is itself a vector of variables in R^{n_k} say. $f(x^1, \ldots, x^r)$ is said to be a **multilinear function** under this partition of variables if, for each $t = 1$ to r, the function

$$f(\bar{x}^1, \ldots, \bar{x}^{t-1}, x^t, \bar{x}^{t+1}, \ldots, \bar{x}^r)$$

obtained by fixing $x^k = \bar{x}^k$ where $\bar{x}^k$ is any arbitrary vector in R^{n_k} for each $k = 1, \ldots, t-1, t+1, \ldots, r$; is a linear function of x^t. Multilinear functions are generalizations of bilinear functions defined in Section 1.7.

Thus, from the above result we see that the determinant of a square matrix A is a multilinear function of A, when the entries in A are partitioned into either the columns of A or the rows of A.

The branch of mathematics dealing with the properties of multilinear functions is called **multilinear algebra**. The multilinearity property of the determinant plays an important role in many advanced theoretical studies of the properties of determinants.

14. Singular, nonsingular square matrices:

The square matrix A of order n is said to be a

singular square matrix if $\det(A) = 0$

nonsingular square matrix if $\det(A) \neq 0$.

The concepts of singularity, nonsingularity are only defined for square matrices, but not for rectangular matrices which are not square.

15. The inverse of a square matrix:

Definition of the inverse of a square matrix: Given a square matrix A of order n, a square matrix B of order n satisfying

$$BA = AB = I$$

where I is the unit matrix of order n, is called the **inverse** of A. If the inverse of A exists, A is said to be **invertible** and the inverse is denoted by the symbol A^{-1}. $\bowtie$

If A is invertible, its inverse is unique for the following reason. Suppose both B, D are inverses of A. Then

$$BA = AB = I; \quad DA = AD = I.$$

So, $B = BI = BAD = ID = D$.

Please note that the concept of the inverse of a matrix is defined only for square matrices, there is no such concept for rectangular matrices that are not square,

Also, if a square matrix A is singular (i.e., $\det(A) = 0$), then its inverse does not exist (i.e., A is not invertible) for the following reason. Suppose A is a singular square matrix and the inverse B of A exists. Then, since $BA =$ the unit matrix I, from Result 9 above we have

$$\det(B) \times \det(A) = 1$$

and since $\det(A) = 0$, the above equation is impossible. Hence a singular square matrix is not invertible.

Every nonsingular square matrix does have an inverse, in fact we will now derive a formula for its inverse in terms of its cofactors. This inverse is the matrix analogue of the reciprocal of a nonzero real number in real number arithmetic.

Let $A = (a_{ij})$ be a square matrix of order n, and for $i, j = 1$ to n let C_{ij} be the (i, j)th cofactor of A. Then by Laplace expansion we know that for all $i = 1$ to n

$$a_{i1}C_{i1} + \ldots + a_{in}C_{in} \quad = \quad \det(A)$$

and for all j

$$a_{1j}C_{1j} + \ldots + a_{nj}C_{nj} \quad = \quad \det(A)$$

Now consider the matrix A' obtained by replacing the first column of A by its second column, but leaving everything else unchanged. So, by its columns, we have

$$A' = (A_{.2} \vdots A_{.2} \vdots A_{.3} \vdots \ldots \vdots A_{.n})$$

By Result 4 above we have $\det(A') = 0$. Since A' differs from A only in its first column, we know that C_{i1} is also the $(i, 1)$th cofactor of A' for all $i = 1$ to n. Therefore by cofactor expansion down its first column, we have

$$\det(A') = a_{12}C_{11} + a_{22}C_{21} + \ldots + a_{n2}C_{n1} = 0$$

Using a similar argumant we conclude that for all $t \neq j$

$$a_{1t}C_{1j} + a_{2t}C_{2j} + \ldots + a_{nt}C_{nj} = 0$$

and for all $i \neq t$

$$a_{i1}C_{t1} + a_{i2}C_{t2} + \ldots + a_{in}C_{tn} = 0$$

Now consider the square matrix of order n obtained by replacing each element in A by its cofactor in A, and then taking the transpose of the resulting matrix. Hence the (i, j)th element of this matrix is C_{ji}. This transposed matrix of cofactors was introduced by the French mathematician A. L. Cauchy in 1815 under the name *adjoint of* A. However, the term *adjoint* has acquired another meaning subsequently, hence this matrix is now called **adjugate** or **classical adjoint** of A, and denoted by $\mathbf{adj}(A)$. Therefore

$$\text{adj}(A) = \begin{pmatrix} C_{11} & C_{21} & C_{31} & \dots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \dots & C_{n2} \\ C_{13} & C_{23} & C_{33} & \dots & C_{n3} \\ \vdots & \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & C_{3n} & \dots & C_{nn} \end{pmatrix}$$

The equations derived above imply that adj(A) satisfies the following important property

$$\text{adj}(A)A = A(\text{adj}(A)) = \text{diag}\{\det(A), \det(A), \dots, \det(A)\}$$

where diag$\{\det(A), \det(A), \dots, \det(A)\}$ is the diagonal matrix of order n with all its diagonal entries equal to $\det(A)$. So, if A is nonsingular, $\det(A) \neq 0$, and

$$(\frac{1}{\det(A)})\text{adj}(A)A = A[(\frac{1}{\det(A)})\text{adj}(A)] = I$$

i.e.,

$$A^{-1} = (\frac{1}{\det(A)})\text{adj}(A) = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{21} & C_{31} & \dots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \dots & C_{n2} \\ C_{13} & C_{23} & C_{33} & \dots & C_{n3} \\ \vdots & \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & C_{3n} & \dots & C_{nn} \end{pmatrix}$$

This provides a mathematical formula for the inverse of an invertible matrix. This formula is seldom used to compute the inverse of a matrix in practice, as it is very inefficient. An efficient method for computing the inverse based on GJ pivot steps is discussed in Chapter 4.

This formula for the inverse is only used in theoretical research studies involving matrices.

Cramer's Rule for the solution of a square nonsingular system of linear equations:

Consider the system of linear equations

$$Ax = b.$$

It is said to be a square nonsingular system of linear equations if the coefficient matrix A in it is square nonsingular. In this case A^{-1} exists, and $x = A^{-1}b$ is the unique solution of the system for each $b \in R^n$. To show this, we see that $\bar{x} = A^{-1}b$ is in fact a solution because

$$A\bar{x} = AA^{-1}b = Ib = b$$

If $\hat{x}$ is another solution to the system, then $A\hat{x} = b$, and multiplying this on both sides by A^{-1} we have

$$A^{-1}A\hat{x} = A^{-1}b = \bar{x}, \quad \text{i.e.,} \quad \bar{x} = A^{-1}A\hat{x} = I\hat{x} = \hat{x}$$

and hence $\hat{x} = \bar{x}$. So, $\bar{x} = A^{-1}b$ is the unique solution of the system.

Let $\bar{x} = A^{-1}b = (\bar{x}_1, \ldots, \bar{x}_n)^T$. Using

$$A^{-1} = (\frac{1}{\det(A)})\text{adj}(A) = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{21} & C_{31} & \ldots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \ldots & C_{n2} \\ C_{13} & C_{23} & C_{33} & \ldots & C_{n3} \\ \vdots & \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & C_{3n} & \ldots & C_{nn} \end{pmatrix}$$

where C_{ij} = the (i, j)th cofactor of A, we get, for $j = 1$ to n

$$\bar{x}_j = (\frac{1}{\det(A)})(b_1C_{1j} + b_2C_{2j} + \ldots + b_nC_{nj})$$

For $j = 1$ to n, let $A_j(b)$ denote the matrix obtained by replacing the jth column in A by b, but leaving all other columns unchanged. So, by columns

$$A_j(b) = (A_{.1} \vdots A_{.2} \vdots \ldots \vdots A_{.j-1} \vdots b \vdots A_{.j+1} \vdots \ldots \vdots A_{.n})$$

By cofactor expansion down the column b in $A_j(b)$ we see that

$$\det(A_j(b)) = b_1 C_{1j} + b_2 C_{2j} + \ldots + b_n C_{nj}$$

and hence from the above we have, for $j = 1$ to n

$$\bar{x}_j = \frac{\det(A_j(b))}{\det(A)}.$$

This leads to **Cramer's rule**, which states that the unique solution of the square nonsingular system of linear equations $Ax = b$ is $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$, where

$$\bar{x}_j = \frac{\det(A_j(b))}{\det(A)}$$

for $j = 1$ to n; with $A_j(b)$ being obtained by replacing the jth column in A by b, but leaving all other columns unchanged.

Historical note on Cramer's rule: Cramer's rule which first appeared in an appendix of the 1750 book *Introduction a L'analysedes Lignes Courbes Algebriques* by the Swiss mathematician Gabriel Cramer, gives the value of each variable in the solution as the ratio of two determinants. Since it gives the value of each variable in the solution by an explicit determinantal ratio formula, Cramer's rule is used extensively in theoretical research studies involving square nonsingular systems of linear equations. It is not used to actually compute the solutions of systems in practice, as the methods discussed in Chapter 1 obtain the solution much more efficiently.

Example : Consider the following system of equations in detached coefficient form.

x_1	x_2	x_3	b
1	-1	1	-5
0	2	-1	10
-2	1	3	-13

Let A denote the coefficient matrix, and for $j = 1$ to 3, let $A_j(b)$ denote the matrix obtained by replacing $A_{.j}$ in A by b. Then we have

$$\det(A) = \det\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ -2 & 1 & 3 \end{pmatrix} = 9;$$

$$\det(A_1(b)) = \det\begin{pmatrix} -5 & -1 & 1 \\ 10 & 2 & -1 \\ -13 & 1 & 3 \end{pmatrix} = 18;$$

$$\det(A_2(b)) = \det\begin{pmatrix} 1 & -5 & 1 \\ 0 & 10 & -1 \\ -2 & -13 & 3 \end{pmatrix} = 27;$$

$$\det(A_3(b)) = \det\begin{pmatrix} 1 & -1 & -5 \\ 0 & 2 & 10 \\ -2 & 1 & -13 \end{pmatrix} = -36$$

So, by Cramer's rule, the unique solution of the system is

$$\bar{x} = (\frac{18}{9}, \frac{27}{9}, \frac{-36}{9})^T = (2, 3, -4)^T.$$

Example : Consider the following system of equations in detached coefficient form.

x_1	x_2	x_3	b
1	−1	0	30
0	2	2	0
−2	1	−1	−60

Here the determinant of the coefficient matrix is 0, hence it is singular. So, Cramer's rule cannot be applied to find a solution for it.

Example : Consider the following system of equations in detached coefficient form.

x_1	x_2	x_3	b
1	−1	0	20
10	12	−2	−40

This system has three variables but only two constraints, it is not a square system. Hence Cramer's rule cannot be applied to find a solution for it.

An Algorithm for Computing the Determinant of a General Square Matrix of Order n Efficiently

BEGIN

Step 1: Perforing G pivot steps on the matrix to make it triangular: Let A be a given matrix of order n. To compute its determinant, perform G pivot steps in each row of A beginning at the top. Select pivot elements using any of the strategies discussed in Step 2 of the G elimination method in Section 1.20. Make row interchanges as indicated there, and a column interchange to bring the pivot element to the diagonal position (i.e., if the pivot element is in position (p, q), interchange columns p and q before carrying out the pivot step so that the pivot element comes to the diagonal (p, p) position). After the pivot step, enclose the pivot element in a box. At any stage, if the present matrix contains a row with all its entries zero, then $\det(A) = 0$, terminate.

Step 2: Computing the determinant: If a pivot step is performed in every row, in the final matrix the diagonal elements are the boxed pivot elements used in the various pivot steps. For $t = 1$ to n, let $\hat{a}_{tt}$ be the boxed pivot element in the final matrix in row t. Let r denote the total number of row and column interchanges performed. Then

$$\det(A) = (-1)^r \prod_{t=1}^{n} [\hat{a}_{tt}]$$

END.

Example : Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 3 & 0 \\ 4 & -1 & -1 & 2 \\ 6 & 2 & 10 & 2 \\ -8 & 0 & -2 & -4 \end{pmatrix}$$

To compute det (A) using G pivot steps, we will use the partial pivoting strategy discussed in Section 1.20 for selecting the pivot elements, performing pivot steps in Columns 2, 3, 4, 1 in that order. We indicate the pivot row (PR), pivot column (PC), and put the pivot element in a box after any row and column interchanges needed are performed. The indication "RI to Rt" on a row means that at that stage that row will be interchanged with Row t in that tableau. Similarly "CI of Cp & Cq" indicates that at that stage columns p and q will be interchanged in the present tableau.

Carrying out Step 1

0	1	3	0		
4	−1	−1	2		
6	2	10	2	RI to R1	
−8	0	−2	−4		
	PC↑				
6	2	10	2	PR	CI of C1 & C2
4	−1	−1	2		
0	1	3	0		
−8	0	−2	−4		
	PC↑				
$\boxed{2}$	6	10	2	PR	
−1	4	−1	2		
1	0	3	0		
0	−8	−2	−4		
PC↑					

Step 1 continued

$\boxed{2}$	6	10	2		
0	7	4	3		CI of C2 & C3
0	-3	-2	-1		
0	-8	-2	-4		
		PC↑			
$\boxed{2}$	10	6	2		
0	$\boxed{4}$	7	3	PR	
0	-2	-3	-1		
0	-2	-8	-4		
	PC↑				
$\boxed{2}$	10	6	2		
0	$\boxed{4}$	7	3		
0	0	$\frac{1}{2}$	$\frac{1}{2}$		
0	0	$\frac{-9}{2}$	$\frac{-5}{2}$		RI to R3
			PC↑		
$\boxed{2}$	10	6	2		
0	$\boxed{4}$	7	3		
0	0	$\frac{-9}{2}$	$\frac{-5}{2}$	PR	CI of C3 & C4
0	0	$\frac{1}{2}$	$\frac{1}{2}$		
			PC↑		

Step 1 continued

$\boxed{2}$	10	2	6	
0	$\boxed{4}$	3	7	
0	0	$\boxed{\frac{-5}{2}}$	$\frac{-9}{2}$	PR
0	0	$\frac{1}{2}$	$\frac{1}{2}$	
		PC$\uparrow$		
$\boxed{2}$	10	2	6	
0	$\boxed{4}$	3	7	
0	0	$\boxed{\frac{-5}{2}}$	$\frac{-9}{2}$	
0	0	0	$\boxed{\frac{-4}{10}}$	

Carrying out Step 2: We made a total of 5 row and column interchanges. So, we have

$$\det(A) = (-1)^5[2 \times 4 \times \frac{-5}{2} \times \frac{-4}{10}] = -8.$$

Example : Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 & 1 \\ 1 & -1 & 3 & 3 \\ 4 & -2 & 12 & 10 \\ 4 & -1 & 13 & 18 \end{pmatrix}$$

To compute $\det(A)$ using G pivot steps, we use the same notation as in the example above. We use the same partial pivoting strategy for selecting the pivot element from the pivot column indicated.

Carrying out Step 1

1	1	3	1		
1	−1	3	3		
4	−2	12	10		RI to R1
4	−1	13	18		
	PC↑				
4	−2	12	10	PR	CI of C1 & C2
1	−1	3	3		
1	1	3	1		
4	−1	13	18		
	PC↑				
$\boxed{-2}$	4	12	10	PR	
−1	1	3	3		
1	1	3	1		
−1	4	13	18		
PC↑					
$\boxed{-2}$	4	12	10		
0	−1	−3	−2		
0	3	9	6		RI to R2
0	2	7	13		
	PC↑				
$\boxed{-2}$	4	12	10		
0	$\boxed{3}$	9	6	PR	
0	−1	−3	−2		
0	2	7	13		
	PC↑				
$\boxed{-2}$	4	12	10		
0	$\boxed{3}$	9	6		
0	0	0	0		
0	0	1	9		

Since all the entries in the third row in the last tableau are all zero, $\det(A) = 0$ in this example.

Importance of Determinants

For anyone planning to go into research in mathematical sciences, a thorough knowledge of determinants and their properties is essential, as determinants play a prominent role in theoretical research. However, determinants do not appear often nowadays in actual numerical computation. So, the study of determinants is very important for understanding the theory and the results, but not so much for actual computational purposes.

Exercises:

2.8.2: Let $A_{n\times n} = (a_{ij})$ where $a_{ij} = i + j$. What is $\det(A)$?

2.8.3: What is the determinant of the following $n \times n$ matrix?

$$\begin{pmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & \dots & \dots & n^2 \end{pmatrix}$$

2.8.4: For any real p, q, r prove that the determinant of the following matrix is 0.

$$\begin{pmatrix} p^2 & pq & pr \\ qp & q^2 & qr \\ rp & rq & r^2 \end{pmatrix}$$

2.8.5: The following square matrix of order n called the **Vandermonde matrix** is encoutered often in research studies.

$$V(x_1, \dots, x_n) = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix}$$

Prove that the determinant of the Vandermonde matrix, called the Vandermonde determinant, is $= \prod_{i>j}(x_i - x_j)$. (Use the results on determinants stated above to simplify the determinant and then apply Laplace's expansion across the 1st row).

2.8.6: The following square matrix of order n is called the **Frobinious matrix** or the **companion matrix** of the polynomial $p(\lambda) = \lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \ldots - a_1\lambda - a_0$.

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ a_0 & a_1 & a_2 & \ldots & a_{n-2} & a_{n-1} \end{pmatrix}$$

Prove that $\det(\lambda I - A) = p(\lambda)$, where I is the identity matrix of order n.

2.8.7: A square matrix $A = (a_{ij})$ of order n is said to be a **skew-symmetric matrix** if $a_{ij} = -a_{ji}$ for all i, j (i.e., $a_{ij} + a_{ji} = 0$ for all i, j).

Prove that the determinant of a skew-symmetric matrix of odd order is 0.

Prove that the determinant of a skew-symmetric matrix of even order does not change if a constant α is added to all its elements.

In a skew-symmetric matrix $A = (a_{ij})$ of even order, if $a_{ij} = 1$ for all $j > i$, find $\det(A)$.

2.8.8: Let $A = (a_{ij}), B = (b_{ij})$ be two square matrices of the same order n, where $b_{ij} = (-1)^{i+j}a_{ij}$ for all i, j. Prove $\det(A) = \det(B)$.

2.8.9: Let $A = (a_{ij})$ be a square matrix of order n, and $s_i = \sum_{j=1}^{n} a_{ij}$ for all i. So, s_i is the sum of all the elements in row i of A. Prove that the determinant of the following matrix is $(-1)^{n-1}(n-1)\det(A)$.

$$\begin{pmatrix} s_1 - a_{11} & \dots & s_1 - a_{1n} \\ \vdots & & \vdots \\ s_n - a_{n1} & \dots & s_n - a_{nn} \end{pmatrix}.$$

2.8.10: Basic minors of a matrix: There are many instances where we have to consider the determinants of square submatrices of a matrix, such determinants are called **minors** of the matrix. A minor of order p of a matrix A is called a **basic minor** of A if it is nonzero, and all minors of A of orders $\geq p+1$ are zero; i.e., a basic minor is a nonzero minor of maximal order, and its order is called the *rank of* A.

If the minor of A defined by rows $i_1, \dots, i_k$ and columns $j_1, \dots, j_k$ is a basic minor, then show that the set of row vectors $\{A_{i_1.}, \dots, A_{i_k.}\}$ is linearly independent, and all other rows of A are linear combinations of rows in this set.

2.8.11: The **trace** of a square matrix $A = (a_{ij})$ of order n is $\sum_{i=1}^n a_{ii}$, the sum of its diagonal elements. If A, B are square matrices of order n, prove that $\text{trace}(AB) = \text{trace}(BA)$.

2.8.12: If the sum of all the elements of an invertible square matrix A in every row is s, prove that the sum of all the elements in every row of A^{-1} is $1/s$.

2.9 Additional Exercises

2.9.1: A, B are two matrices of order $m \times n$. Mention conditions under which $(A+B)^2$ is defined. Under these conditions, expand $(A+B)^2$ and write it as a sum of individual terms, remembering that matrix multiplication is not commutative.

2. Consider the following matrices. Show that FE and EF both exist but they are not equal.

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}, E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. Is it possible to have a square matrix $A \neq 0$ satisfying the property that $A \times A = A^2 = 0$? If so find such matrices of orders 2 and 3.

4. Show that the product of two lower triangular matrices is lower triangular, and that the product of upper triangular matrices is upper triangular.

Show that the inverse of a lower triangular matrix is lower triangular, and that the inverse of an upper triangular matrix is upper triangular.

Show that the sum of symmetric matrices is symmetric. Show that any linear combination of symmetric matrices of the same order is also symmetric.

Construct numerical examples of symmetric matrices A, B of the same order such that the product AB is not symmetric. Construct numerical examples of symmetric matrices whose product is symmetric.

If A, B are symmetric matrices of the same order, prove that the product AB is symmetric iff $AB = BA$.

Show that the inverse of a symmetric matrix is also symmetric.

For any matrix A show that both the products AA^T, A^TA always exist and are symmetric.

Prove that the product of two diagonal matrices is also diagonal.

5. Let $C = AB$ where A, B are of orders $m \times n, n \times p$ respectively. Show that each column vector of C is a linear combination of the column vectors of A, and that each row vector of C is a linear combination of row vectors of B.

6. If both the products AB, BA of two matrices A, B are defined then show that each of these products is a square matrix. Also, if the sum of these products is also defined show that A, B must themselves be square matrices of the same order.

Construct numerical examples of matrices A, B where both products AB, BA are defined but are unequal square matrices of the same order.

Construct numerical examples of matrices A, B both nonzero such that $AB = 0$.

Construct numerical examples of matrices A, B, C where $B \neq C$ and yet $AB = AC$.

7. Find the matrix X which satisfies the following matrix equation.

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} X = \begin{pmatrix} 11 & -5 & -7 \\ -5 & -7 & 11 \\ -7 & 11 & -5 \end{pmatrix}.$$

8. Consider the following square symmetric matrix $A = (a_{ij})$ of order n in which $a_{11} \neq 0$. Perform a G pivot step on A with a_{11} as the pivot element leading to $\bar{A}$. Show that the matrix $\mathcal{A}$ of order $n-1$ obtained from $\bar{A}$ by striking off its first row and first column is also symmetric.

$$A = \begin{pmatrix} \boxed{a_{11}} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & \bar{a}_{22} & \dots & \bar{a}_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & \bar{a}_{n2} & \dots & \bar{a}_{nn} \end{pmatrix}.$$

9. A square matrix A is said to be **skew symmetric** if $A^T = -A$. Prove that if A is skew symmetric and A^{-1} exists then A^{-1} is also skew symmetric.

If A, B are skew symmetric matrices of the same order show that $A+B$, $A-B$ are also skew symmetric. Show that any linear combination of skew matrices of the same order is also skew symmetric.

Can the product of skew symmetric matrices be skew symmetric?

Using the fact that $A = (1/2)((A+A^T)+(A-A^T))$ show that every square matrix is the sum of a symmetric matrix and a skew symmetric matrix.

If A is a skew symmetric matrix, show that $I+A$ must be invertible.

10. List all the 24 permutations of $\{1, 2, 3, 4\}$ and classify them into even, odd classes.

Find the number of inversions in the permutation $(7, 4, 2, 1, 6, 3, 5)$.

11. Show that the determinant of the elementary matrix corresponding to the operation of multiplying a row of a matrix by α is α.

Show that the determinant of the elementary matrix corresponding to the operation of interchanging two rows in a matrix is -1.

Show that the determinant of the elementary matrix corresponding to the operation of adding a scalar multiple of a row of a matrix to another is 1.

Show that the determinant of the pivot matrix corresponding to a GJ pivot step with a_{rs} as the pivot element is $1/a_{rs}$.

12. Construct a numerical example to show that $\det(A+B)$ is not necessarily equal to $\det(A) + \det(B)$.

By performing a few row operations show that the following determinants are 0

$$\begin{vmatrix} 5 & 4 & 3 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}, \quad \begin{vmatrix} 2 & 3 & 0 & 0 \\ 3 & 2 & 2 & 0 \\ -5 & -5 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{vmatrix}.$$

If A is an integer square matrix and $\det(A) = \pm 1$, then show that A^{-1} is also an integer matrix.

Show that the following determinant is 0 irrespective of what values the a_{ij}s have.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & 0 & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{vmatrix}.$$

13. In a small American town there are three newspapers that people buy, NY, LP, CP. These people make highly predictable transitions from one newspaper to another from one quarter to the next as described below.

Among subscribers to NY: 80% continue with NY, 7% switch to CP, and 13% switch to LP.

Among subscribers to LP: 78% continue with LP, 12% switch to NY, 10% switch to CP.

Among subscribers to CP: 83% continue with CP, 8% switch to NY, and 9% switch to LP.

Let $x(n) = (x_1(n), x_2(n), x_3(n))^T$ denote the vector of the number of subscribers to NY, LP, CP in the nth quarter. Treating these as continuous variables, find the matrix A satisfying: $x(n+1) = Ax(n)$.

14. There are two boxes each containing different numbers of bags of three different colors as represented in the following matrix A (rows of A corespond to boxes, columns of A correspond to colors of bags) given in tableau form.

	White	Red	Black
Box 1	3	4	2
Box 2	2	2	5

Each bag contains different numbers of three different fruit as in the following matrix B given in tabular form.

	Mangoes	Sapotas	Bananas
White	3	6	1
Red	2	4	4
Black	4	3	3

Show that rows, columns of the matrix product $C = AB$ correspond to boxes, fruit respectively, and that the entries in C are the total number of various fruit contained in each box.

15. Express the row vector $(a_{11}b_{11} + a_{12}b_{21}, a_{11}b_{21} + a_{12}b_{22})$ as a product of two matrices.

16. A, B are two square matrices of order n. Is $(AB)^2 = A^2B^2$?

17. Plant 1, 2, each produce both fertilizers Hi-ph, Lo-ph simultaneously each day with daily production rates (in tons) as in the following matrix A represented in tableau form.

	Plant 1	Plant 2
Hi-ph	100	200
Lo-ph	200	300

In a certain week, each plant worked for different numbers of days, and had total output in tons of (Hi-ph, Lo-ph)$^T = b = (1100, 1800)^T$.

What is the physical interpretation of the vector x which is the solution of the system $Ax = b$?

18. The % of N, P, K in four different fertilizers is given below.

	N	P	K
Fertilizer 1	10	10	10
2	25	10	5
3	30	5	10
4	10	20	20

Two different mixtures are prepared by combining various quantities in the following way.

	Lbs in mixture, of			
	Fertilizer 1	2	3	4
Mixture 1	100	200	300	400
2	300	500	100	100

Derive a matrix product that expresses the % of N, P, K in the two mixtures.

19. Bacterial growth model: A species of bacteria has this property: after the day an individual is born, it requires a maturation period of two complete days, then on 3rd days it divides into 4 new individuals (after division the original individual no longer exists).

Let $x^k = (x_1^k, x_2^k, x_3^k)^T$ denote number of individuals for whom day k is 1st, 2nd, 3rd complete day after the day of their birth.

Find a square matrix A of order 3 which expresses x^{k+1} as Ax^k. For any k express x^k in terms of A and x^1.

20. Bactterial growth model with births and deaths: Consider bacterial population described in Exercise 19 above with two exceptions.

1. Let x_i^k be continuous variables here.

2. In 1st full day after their birth day each bacteria either dies with probability 0.25 or lives to 2nd day with probability 0.75.

Each bacteria that survives 1st full day of existence after birth, will die on 2nd full day with probability 0.2, or lives to 3rd day with probability 0.8.

The scenario on 3rd day is the same as described in Excercise 19, i.e., no death on 3rd day but only division into new individuals.

Again find matrix A that expresses x^{k+1} as Ax^k here, and give an expression for x^k in terms of A and x^1.

21: A 4-year community college has 1st, 2nd, 3rd, 4th year students. From past data they found that every year, among the students in each year's class, 80% move on to the next year's class (or graduate for 4th year students); 10% drop out of school; 5% are forced to repeat that year; and the remaining 5% are asked to take the next year off from school to work and get strong motivation to continue studies, and return to college into the same year's class after that work period. Let $x(n) = (x_1(n), x_2(n), x_3(n), x_4(n))^T$ denote the vector of the number of 1st year, 2nd year, 3rd year, 4th year students in the college in the nth year, treat each of these as continuous variables. Assume that the college admits 100 first year students, and 20 transfer students from other colleges into the 2nd year class each year. Find matrices A, b that expresses $x(n+1)$ as $Ax(n)+b$.

22: Olive harvest: Harvested olives are classified into sizes 1 to 5. Harvest season usually lasts 15 days, with harvesting carried out only on days 1, 8, and 15 of the season. Of olives of sizes 1 to 4 on trees not harvested on days 1, 8 of the season, 10% grow to the next size, and 5% get rotten and drop off, in a week.

At the beginning of the harvest season an olive farm estimated that they have 400, 400, 300, 300, 40 kg of sizes 1 to 5 respectively on their trees at that time. On days 1, 8 of the harvest season they plan to harvest all the size 5 olives on the trees, and 100, 100, 50, 40 kg of sizes 1 to 4 respectively; and on the final 15th day of the season, all the remaining olives will be harvested.

Let $x^r = (x_1^r, x_2^r, x_3^r, x_4^r, x_5^r)^T$ denote the vector of kgs of olives of sizes 1 to 5 on the trees, before harvest commences, on the rth harvest day, $r = 1, 2, 3$.

Find matrices A^1, A^2, b^1, b^2 such that $x^2 = A^1(x^1 - b^1), x^3 = A^2(x^2 - b^2)$. Also determine the total quantity of olives of various sizes that the farm will harvest that year.

23. A population model: (From A. R. Majid, *Applied Matrix Models*, Wiley, 1985) **(i):** An animal herd consists of three cohorts: newborn (age 0 to 1 year), juveniles (age 1 to 2 years), and adults (age 2 or more years.

Only adults reproduce to produce 0.4 newborn per adult per year (a herd with 50 male and 50 female adults will produce 40 newborn calves annually, half of each sex). 65% of newborn survive to become juveniles, 78% of juveniles survive to become adults, and 92% of adults survive to live another year.

Let $x^k = (x_1^k, x_2^k, x_3^k)^T$ denote the vector of the number of newborn, juvenile, and adults in the kth year. Find a matrix k that expresses x^{k+1} as Ax^k for all k.

A system like this is called a *discrete time linear system* with A as the transition matrix from k to $k+1$. Eigenvalues and eigenvectors of square matrices discussed in Chapter 6 are very useful to determine the solutions of such systems, and to study their limiting behavior.

(ii): Suppose human predators enter and colonize this animal habitat. Let x_4^k denote the new variable representing the number of humans in the colony in year k. Human population grows at the rate of 2.8% per year, and each human eats 2.7 adult animals in the herd per year. Let $X^k = (x_1^k, x_2^k, x_3^k, x_4^k)^T$. Find the matrix B such that $X^{k+1} = BX^k$.

24. Fibonacci numbers: Consider the following system of equations: $x_1 = 1$, $x_2 = 1$, and $x_{n+1} = x_n + x_{n-1}$ for $n \geq 2$.

Write the 3rd equation for $n = 2, 3, 4$; and express this system of 5 equations in $(x_1, \ldots, x_5)^T$ in matrix form.

For general n, find a matrix B such that $(x_{n+1}, x_{n+2})^T = B(x_n, x_{n+1})^T$. Then show that $(x_{n+1}, x_{n+2})^T = B^n(x_1, x_2)^T$.

Also for general n find a matrix A such that $(x_{n+1}, x_{n+2}, x_{n+3})^T = A(x_n, x_{n+1}, x_{n+2})^T$ Hence show that $(x_{n+1}, x_{n+2}, x_{n+3})^T = A^n(x_1, x_2, x_3)^T$.

x_n is known as the nth Fibonacci number, all these are positive integers. Find x_n for $n \leq 12$, and verify that x_n grows very rapidly. The growth of x_n is similar to the growth of populations that seem to have no limiting factors.

25. For any u, v between 1 to n define E_{uv} to be the square matrix of order n with the (u, v)th entry $= 1$, and every other entry $= 0$.

Let $(j_1, \ldots, j_n)$ be a permutation of $(1, \ldots, n)$, and let $P = (p_{r,s})$ be the permutation matrix of order n corresponding to it (i.e., $p_{rj_r} = 1$ for all r, and all other entries in P are 0).

Then show that $PE_{uv}P^{-1} = E_{j_u j_v}$. (Remember that for a permutation matrix P, $P^{-1} = P^T$.)

26. Find conditions on b_1, b_2, b_3 so that the following system has at least one solution. Does $b_1 = 5$, $b_2 = -4$, $b_3 = -2$ satisfy this condition? If it does find the general solution of the system in this case in parametric form.

x_1	x_2	x_3	x_4	
1	−1	−1	1	b_1
2	−3	0	1	b_2
8	−11	−2	5	b_3

27. Fill in the missing entries in the following matrix product.

$$\begin{pmatrix} 1 & 0 & 1 \\ & 2 & -1 \\ 1 & & 3 \end{pmatrix}\begin{pmatrix} & & 1 \\ 0 & & \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & 0 \\ 2 & -2 & 9 \end{pmatrix}$$

28. If A, B are square matrices of order n, is it true that $(A - B)(A + B) = A^2 - B^2$? If not under what conditions will it be true? Expand $(A + B)^3$.

29. Determine the range of values of a, b, c for which the following matrix is triangular.

$$\begin{pmatrix} 5 & 6 & b & 0 \\ 0 & a & 2 & -1 \\ c & 0 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

30. In each case below, you are given a matrix A and its product Ax with an unknown vector x. In each case determine whether you can identify the vector x unambiguously.

$$(i.)\ \ A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}, Ax = \begin{pmatrix} -1 \\ 11 \end{pmatrix}$$

$$(ii.)\ \ A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & 5 & 1 \end{pmatrix}, Ax = \begin{pmatrix} 4 \\ 2 \\ 14 \end{pmatrix}$$

$$(iii.)\ \ A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix}, Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(iv.)\quad A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 3 & 4 \end{pmatrix}, Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

31. Inverses of partitioned matrices: Given two square matrices of order n, to show that B is the inverse of A, all you need to do is show that $AB = BA =$ the unit matris of order n.

Consider the partitioned square matrix M of order n, where A, D are square submatrices of orders r, $n-r$ respectively. In the following questions, you are asked to show that M^{-1} is of the following form, with the entries as specified in the question.

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

(i.) If M, A are nonsingular, then $E = A^{-1} - FCA^{-1}$, $F = -A^{-1}BH$, $G = -HCA^{-1}$, and $H = (D - CA^{-1}B)^{-1}$.

(ii.) If M, D are nonsingular, then $E = (A - BD^{-1}C)^{-1}$, $F = -EBD^{-1}$, $G = -D^{-1}CE$, $H = D^{-1} - GBD^{-1}$.

(iii.) If $B = 0$, $r = n - 1$, and $D = \alpha$ where $\alpha = \pm 1$, and A is invertible, then $E = A^{-1}$, $F = 0$, $G = -\alpha CA^{-1}$, $H = \alpha$.

32. Find the determinant of the following matrices.

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ p_1 & p_2 & p_3 & \dots & p_n \end{pmatrix}, \begin{pmatrix} a+b & c & 1 \\ b+c & a & 1 \\ c+a & b & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1+\alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_1 & 1+\alpha_2 & \dots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \dots & 1+\alpha_n \end{pmatrix}$$

33. Consider following matrices:

$A_n = (a_{ij})$ of order n with all diagonal entries $a_{ii} = n$, and all off-diagonal entries $a_{ij} = 1$ for all $i \neq j$.

$B_n = (b_{ij}$ of order n with all diagonal entries $b_{ii} = 2$; $b_{i+1,i} = b_{i,i+1} = -1$ for all i= 1 to $n-1$; and all other entries 0.

$C_n = (c_{ij})$ of order n with $c_{i,i+1} = 1$ for all $i = 1$ to $n-1$; and all other entries 0.

Show that $C_n + C_n^T$ is nonsingular iff n is even.

Develop a formula for determinant(A_n) in terms of n.

Show that determinant(B_n) = 2 determinant(B_{n-1}) - determinant(B_{n-2}).

34. If A, B, C are all square matrices of the same order, C is nonsingular, and $B = C^{-1}AC$, show that $B^k = C^{-1}A^kC$ for all positive integers k.

If A is a square matrix and AA^T is singular, then show that A must be singular too.

If $u = (u_1, \dots, u_n)^T$ is a column vector satisfying $u^Tu = 1$, show that the matrix $(I - 2uu^T)$ is symmetric and that its square is I.

If A, B are square matrices of the same order, and A is nonsingular, show that determinant(ABA^{-1}) = determinant(B).

35. Consider the following system of equations. Without computing the whole solution for this system, find the value of the variable x_2 only in that solution.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 13 \\ x_1 + x_2 + 2x_2 &= -9 \\ 2x_1 + x_2 + x_3 &= -12. \end{aligned}$$

2.10 References

[2.1]. M. Golubitsky and M. Dellnitz, *Linear Algebra and Differential Equations Using MATLAB*, Brooks/Cole, Pacific Grove, CA, 1999.

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