Chapter 6

COMPUTATIONAL COMPLEXITY OF COMPLEMENTARY PIVOT METHODS

In this Chapter, we discuss the worst case behavior of the computational growth requirements of the complementary and principal pivot methods for solving LCPs, as a function of n, the order, and the size of the LCP. These results are from K. G. Murty [6.5]. We construct a class of LCPs with integer data, one of order n for each $n \geqq 2$, and prove that the pivotal methods discussed in Chapters 2, 4, and 5 require $2^{\mathbf{n}} - 1$ or $2^{\mathbf{n}}$ pivot steps to solve the problem of order n in the class. The size of the n^{th} problem in this class, defined to be the total number of bits of storage needed to store all the data in the problem in binary form is $\leqq 4n^{\mathbf{2}} + 3n$. These results establish that in the worst case, the computational growth requirements of complementary pivot methods are not bounded above by any polynomial in the order or size of the LCP.

To study the worst case computational complexity of complementary pivot methods, we look at the following question: What is the maximum number of complementary cones through which a straight line in $\mathbf{R}^n$ can cut across? For a problem of order n, the answer turns out to be $2^{\mathbf{n}}$, that is, there may exist straight lines which cut across the interiors of every one of the $2^{\mathbf{n}}$ complementary cones.

Let $\widetilde{M}(n) = (\widetilde{m}_{ij})$ be the lower triangular matrix of order n, defined by $\widetilde{m}_{ij} = 1$ for $i = 1$ to n, $\widetilde{m}_{ij} = 0$ for all $j > i$, and $\widetilde{m}_{ij} = 2$ for all $j < i$. See (1.15), page 19. Since $\widetilde{M}(n)$ is lower triangular, all principal subdeterminants of $\widetilde{M}(n)$ are equal to 1, and hence $\widetilde{M}(n)$ is a P-matrix. Since $\widetilde{M}(n) + (\widetilde{M}(n))^T$ is a matrix all of whose entries are 2, it is singular, and clearly it is a PSD matrix. Hence $\widetilde{M}(n)$ is a P-matrix, PSD matrix (and hence a copositive plus matrix), but not a PD matrix. Let e_n be the column vector in $\mathbf{R}^n$ all of whose entries are equal to 1. Let:

$$\begin{aligned}\tilde{q}(n) =&(2^{\mathbf{n}}, 2^{\mathbf{n-1}}, \dots, 2)^T\\ \hat{q}(n) =&(-2^{\mathbf{n}}, -2^{\mathbf{n}} - 2^{\mathbf{n-1}}, -2^{\mathbf{n}} - 2^{\mathbf{n-1}} - 2^{\mathbf{n-2}}, \dots,\\ &- 2^{\mathbf{n}} - 2^{\mathbf{n-1}} - \dots - 2^{\mathbf{2}} - 2)^T \qquad (6.1)\\ a(s) =&2^{\mathbf{s}} - 1, \text{ for any } s \geqq 2\\ \mathbf{L}(n) =&\{x : x = \tilde{q}(n) + \gamma(-e_n) : \gamma \text{ a real parameter}\}\end{aligned}$$

Theorem 6.1 . *The straight line $\mathbf{L}(n)$ cuts across the interior of every one of the $2^{\mathbf{n}}$ complementary cones in the class $\mathcal{C}(\widetilde{M}(n))$ for any $n \geqq 2$.*

Proof. Consider the class of parametric LCPs $(\tilde{q}(n)+\gamma(-e_n), \widetilde{M}(n))$ for $n \geqq 2$, where γ is a real valued parameter. Consider the case $n = 2$ first. The following can be verified in this case :

Tableau 6.1

Complementary Cone corresponding to the Complementary Basic Vector	Portions of $\mathbf{L}(2)$ corresponding to values of the Parametr γ which lie in this Complementary Cone
(w_1, w_2)	$2 = \gamma$
(w_1, z_2)	$2 \leqq \gamma \leqq 4$
(z_1, z_2)	$4 \leqq \gamma \leqq 6$
(z_1, w_2)	$6 \leqq \gamma$

Also whenever γ is an interior point of one of these intervals, all the basic variables are strictly positive in the complementary BFS of $(\tilde{q}(2)+\gamma(-e_n), \widetilde{M}(2))$; and this implies that the point $\tilde{q}(2) + \gamma(-e_n)$ corresponding to that value of γ is in the interior of the corresponding complementary cone. Hence, the statement of this Theorem is true when $n = 2$. We now make an induction hypothesis.

Induction Hypothesis: The theorem is true for the LCP of order $n-1$ in the class. Specifically, the complementary basic vectors for the parametric LCP $(\tilde{q}(n-1)+\gamma(-e_n), \widetilde{M}(n-1))$ can be ordered as a sequence $v_0, v_1, \dots, v_{a(n-1)}$, such that the complementary cone corresponding to the complementary basic vector v_r contains the portion of the straight line $\mathbf{L}(n-1)$ corresponding to $\gamma \leqq 2$ if $r = 0$; $2r \leqq \gamma \leqq 2(r+1)$, if $1 \leqq r \leqq 2^{\mathbf{n-1}} - 2$; and $2^{\mathbf{n}} - 2 \leqq \gamma$ if $r = 2^{\mathbf{n-1}} - 1$. Also the straight line $\mathbf{L}(n-1)$ cuts across the interior of each of these complementary cones.

Now consider the parametric LCP of order n in the class, namely $(\tilde{q}(n)+\gamma(-e_n), \widetilde{M}(n))$, the original tableau for which is Tableau 6.2

Tableau 6.2

w_1	w_2	$\dots$	w_n	z_1	z_2	$\dots$	z_n	
1	0	$\dots$	0	-1	0	$\dots$	0	$2^{\mathbf{n}} - \gamma$
0	1	$\dots$	0	-2	-1	$\dots$	0	$2^{\mathbf{n-1}} - \gamma$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
0	1	$\dots$	1	-2	-2	$\dots$	-1	$2 - \gamma$

The principal subproblem of this in the variables $(w_2, \dots, w_n)$, $(z_2, \dots, z_n)$, is the same as the parametric LCP of order $n-1$ in the class we are discussing, with the exception that the variables in it are called as $w_2, \dots, w_n$; $z_2, \dots, z_n$. By induction hypothesis, the complementary basic vectors of this principal subproblem can be ordered in a sequence as $v_0, v_1, \dots, v_{a(n-1)}$, where $v_0 = (w_2, \dots, w_n)$, $v_1 = (w_2, \dots, w_{n-1}, z_n)$, etc. such that the complementary cone for this principal subproblem, corresponding to the complementary basic vector v_r, contains the portion of the straight line $\mathbf{L}(n-1)$ corresponding to $\gamma \leqq 2$ if $r = 0$; $2r \leqq \gamma \leqq 2(r+1)$ if $1 \leqq r \leqq 2^{\mathbf{n-1}} - 2$, and $\gamma \geqq 2^{\mathbf{n}} - 2$ if $r = 2^{\mathbf{n-1}} - 1$; and as long as γ is in the interior of one of these intervals, the corresponding point on $L(n-1)$ is in the interior of the corresponding complementary cone. Notice that in the original problem in Tableau 6.2, $q_1(\gamma) = 2^{\mathbf{n}} - \gamma$ remains nonnegative for all $\gamma \leqq 2^{\mathbf{n}}$ and strictly positive for all $\gamma < 2^{\mathbf{n}}$. This, together with the result for the principal subproblem, implies that the complementary cone corresponding to the complementary basic vector $V_r = (w_1, v_r)$ of the original problem (Tableau 6.2) contains the portion of the line $\mathbf{L}(n)$ corresponding to values of γ satisfying $\gamma \leqq 2$, if $r = 0$; $2r \leqq \gamma \leqq 2r + 2$, if $1 \leqq r \leqq -1 + 2^{\mathbf{n-1}} = a(n-1)$. It also implies that in each case, the straight line $\mathbf{L}(n)$ cuts across the interior of these complementary cones.

Now perform a single principal pivot step in Position 1 in the original problem in Tableau 6.2. This leads to Tableau 6.3

Tableau 6.3

w_1	w_2	$\dots$	w_n	z_1	z_2	$\dots$	z_n	q
-1	0	$\dots$	0	1	0	$\dots$	0	$\gamma - 2^{\mathbf{n}}$
-2	1	$\dots$	0	0	-1	$\dots$	0	$(-2^{\mathbf{n+1}} + \gamma) + 2^{\mathbf{n-1}}$
-2	0	$\dots$	0	0	-2	$\dots$	0	$(-2^{\mathbf{n+1}} + \gamma) + 2^{\mathbf{n-2}}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
-2	0	$\dots$	1	0	-2	$\dots$	-1	$(-2^{\mathbf{n+1}} + \gamma) + 2$

Let $-\lambda = -2^{\mathbf{n+1}} + \gamma$ and treat λ as the new parameter. As γ increases from $2^{\mathbf{n}}$ to $2^{\mathbf{n+1}} - 2$, λ decreases from $2^{\mathbf{n}}$ to 2. As a function of λ, the vector of the right hand side constants in Tableau 6.3 is $(2^{\mathbf{n}} - \lambda, 2^{\mathbf{n-1}} - \lambda, \dots, 2 - \lambda)^T$.

Now look at the principal subproblem of the parametric LCP in Tableau 6.3 in the variables $w_2, \dots, w_n$; $z_2, \dots, z_n$. This principal subproblem considered with λ as the parameter can be verified to be the same as the parametric LCP of order $n-1$ in the class we are discussing, with the exception that the variables in it are called as $w_2, \dots, w_n$; $z_2, \dots, z_n$, and the parameter is λ.

Using arguments similar to those as above on these problems, and translating everything to the original parameter γ again, we conclude that the complementary cone corresponding to the complementary basic vector $V_r = (z_1, v_{b(r)})$ of the original problem, where $b(r) = 2^{\mathbf{n}} - r - 1$, contains the portion of the straight line $\mathbf{L}(n)$ corresponding to values of γ satisfying $2r \leqq \gamma \leqq 2r+2$, if $2^{\mathbf{n-1}} \leqq r \leqq 2^{\mathbf{n}} - 2$; and $\gamma \geqq 2^{\mathbf{n+1}} - 2$, if $r = 2^{\mathbf{n}} - 1$.

Thus if $v_0, \dots, v_{a(n-1)}$ is the ordered sequence of complementary basic vectors for the principal subproblem of the parametric LCP in Tableau 6.2 in the variables $w_2, \dots, w_n$; $z_2, \dots, z_n$; let the ordered sequence of complementary basic vectors for the parametric LCP in Tableau 6.2 be

$$\begin{aligned} V_0 = &(w_1, v_0), (w_1, v_1), \dots, (w_1, v_{a(n-1)}); \\ &(z_1, v_{a(n-1)}), (z_1, v_{a(n-1)-1}), \dots, (z_1, v_0) = V_{a(n)}. \end{aligned} \tag{6.2}$$

Then the induction hypothesis implies the result that the complementary cone corresponding to the complementary basic vector V_r contains the portion of the straight line $\mathbf{L}(n)$ corresponding to $\gamma \leqq 2$, if $r = 0$; $2r \leqq \gamma \leqq 2r+2$, if $1 \leqq r \leqq 2^{\mathbf{n}} - 2$; $\gamma \geqq 2^{\mathbf{n+1}} - 2$, if $r = 2^{\mathbf{n}} - 1$. Also in each case, the straight line cuts across the interior of the complementary cone. Hence the induction hypothesis implies that the statement of Theorem 6.1 also holds for the parametric LCP of order n in the class we are discussing. The statement of Theorem 6.1 has already been verified to be true from $n = 2$. Hence it is true for all $n \geqq 2$.

$\square$

6.1 Computational Complexity of the Parametric LCP Algorithm

Theorem 6.2 *Consider the class of parametric LCPs $(\tilde{q}(n) + \gamma(-e_n), \widetilde{M}(N))$, for $n \geqq 2$. The parametric LCP algorithm discussed in Chapter 5 requires $2^{\mathbf{n}}$ pivot steps to solve the nth problem in the class for all real values of the parameter γ.*

Proof. Let $V_0, V_1, \dots, V_{a(n)}$ be the sequence of complementary basic vectors for the parametric LCP of order n in this class obtained in the proof of Theorem 6.1. From the proof of Theorem 6.1, we conlcude that the complementary basic vector V_r is feasible to the parametric LCP $(\tilde{q}(n) + \gamma(-e_n), \widetilde{M}(n))$ in the interval $\gamma \leqq 2$ if $r = 0$; $2r \leqq \gamma \leqq 2r+2$, if $1 \leqq r \leqq 2^{\mathbf{n}} - 2$; $\gamma \geqq 2^{\mathbf{n+1}} - 2$, if $r = 2^{\mathbf{n}} - 1$. Hence, when the parametric LCP algorithm is applied to solve $(\tilde{q}(n) + \gamma(-e_n), \widetilde{M}(n))$ for all values of the parameter γ, it terminates only after going through all the complementary basic vectors, $V_0, V_1, \ldots, V_{a(n)}$; and thus requires $a(n) + 1 = 2^{\mathbf{n}}$ pivot steps.

□

Example 6.1

See Example 5.1 in Chapter 5. There the parametric LCP $(\tilde{q}(3) + \gamma(-e_3), \widetilde{M}(3))$ is solved for all values of the parameter γ (there the parameter is denoted by λ intead γ) using the parametric LCP algorithm and verify that it took $2^{\mathbf{3}} = 8$ pivot steps in all.

6.2 Geometric Interpretation of a Pivot Step in the Complementary Pivot Method

Let M be a given square matrix of order n, and q a column vector in $\mathbf{R}^n$. Consider the LCP (q, M). The original tableau for solving it by the complementary pivot method is (2.3) of Section 2.2.1.

Let $(y_1, \ldots, y_{r-1}, y_{r+1}, \ldots, y_n, z_0)$ be a basic vector obtained in the process of solving this LCP by the complementary pivot method where $y_j \in \{w_j, z_j\}$ for all j. Let $A_{.j}$ denote the column vector associated with y_j in (2.3) for each j. If $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_n)^T$ is the update right hand constants vector in the canonical tableau of (2.3) with respect to the basic vector $(y_1, \ldots, y_{r-1}, y_{r+1}, \ldots, y_n, z_0)$, then $\bar{q} \geqq 0$ (since this basic vector must be a feasible basic vector) and we have

$$q = A_{.1}\bar{q}_1 + \ldots + A_{.r-1}\bar{q}_{r-1} + A_{.r+1}\bar{q}_r + \ldots + A_{.n}\bar{q}_{n-1} + (-e_n)\bar{q}_n \qquad (6.3)$$

$\bar{q}_n$ is the value of z_0 in the present BFS. If $\bar{q}_n = 0$, the present BFS is a complementary feasible solution and the method would terminate. So assume $\bar{q}_n > 0$ and denote it by the symbol $\tilde{z}_0$. Then (6.3) implies that $q + \tilde{z}_0 e_n \in \operatorname{Pos}\{A_{.1}, \ldots, A_{.r-1}, A_{.r+1}, \ldots, A_{.n}\}$. The present left out complementary pair is (w_r, z_r), and one of the variables from this pair will be choosen as the entering variable at this stage, let us denote it by $y_r \in \{w_r, z_r\}$ and let $A_{.r}$ denote the column vector associated with y_r in (2.3). If y_r replaces z_0 from the basic vector in theis step, we get a complementary feasible basic vector at the end of this pivot step, and the method terminates. Suppose the dropping variable is not z_0, but some y_i for $i \in \{1, \ldots, r-1, r+1, \ldots, n\}$. Let $\hat{z}_0 > 0$ be the value of z_0 in the new BFS obtained after this pivot step. Then using the same arguments as before we conclude that $q + \hat{z}_0 e_n \in \operatorname{Pos}\{A_{.1}, \ldots, A_{.i-1}, A_{.i+1}, \ldots, A_{.n}\}$.

Under these conditions, clearly $(y_1, \ldots, y_n)$ is itself a complementary basic vector, and let $\mathbf{K} = \operatorname{Pos}(A_{.1}, \ldots, A_{.n})$ be the complementary cone associated with it. The net effect in this pivot step is therefore that of moving from the point $q + \tilde{z}_0 e_n$ contained on the facet $\text{Pos}\{A_{.1}, \dots, A_{.r-1}, A_{.r+1}, \dots, A_{.n}\}$ of $\mathbf{K}$ to the point $q + \hat{z}_0 e_n$ on the facet $\text{Pos}\{A_{.1}, \dots, A_{.i-1}, A_{.i+1}, \dots, A_{.n}\}$ of $\mathbf{K}$, along the half-line $\{x : x = q + \lambda e_n,$ λ a nonnegative real number$\}$. See Figure 6.1. The complementary pivot method continues in this manner walking along the half-line $\{x : x = q + \lambda e_n, \lambda \geqq 0\}$ cutting across different complementary cones, until at some stage it enters a complementary cone containing the point q on this half-line.

We will now use this geometric interpretation, to establish the computational complexity of the complementary pivot method in the worst case.

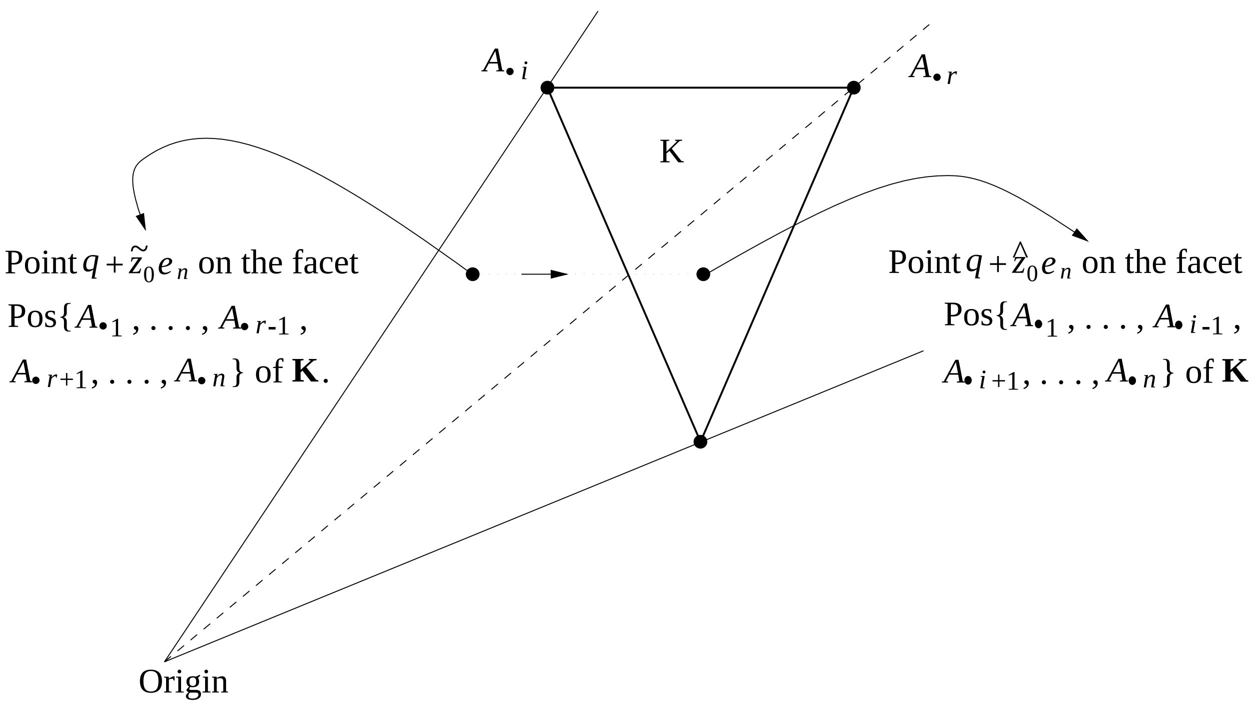

Figure 6.1 Geometric interpretation of a pivot step in the complementary pivot method as a walk from one facet of a complementary cone to another facet of the same cone along the half-line $\{x : x = q + \lambda e_n, \lambda \geqq 0\}$ as λ varies from $\tilde{z}_0$ to $\hat{z}_0$.

6.3 Computational Complexity of the Complementary Pivot Method

Theorem 6.3 *For any $n \geqq 2$, the complementary pivot method requires $2^{\mathbf{n}}$ pivot steps before termination when applied on the LCP $(\hat{q}(n), \widetilde{M}(n))$.*

Proof. Notice that $\tilde{q}(n) + \gamma(-e_n) = \hat{q}(n) + \lambda e_n$ where $\lambda = 2^{\mathbf{n+1}} - \gamma$. Hence the straight line $\{x : x + \hat{q}(n) + \lambda e_n, \lambda \text{ a real number}\}$ is the same as the line $\mathbf{L}(n)$ defined in equation (6.1) but in the reverse direction.

The original tableau for applying the complementary pivot method to solve the LCP $(\hat{q}(n), M(n))$ is shown in Tableau 6.4.

Tableau 6.4

w_1	w_2	$\dots$	w_n	z_1	z_2	$\dots$	z_n	z_0	q
1	0	$\dots$	0	-1	0	$\dots$	0	-1	$-2^{\mathbf{n}}$
0	1	$\dots$	0	-2	-1	$\dots$	0	-1	$-2^{\mathbf{n}} - 2^{\mathbf{n-1}}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
0	0	$\dots$	1	-2	-2	$\dots$	-1	-1	$-2^{\mathbf{n}} - 2^{\mathbf{n-1}} - \dots - 2$

The initial basic vector obtained in the complementary pivot method is $(w_1, \dots, w_{n-1}, z_0)$ and in the solution corresponding to this basic vector, the value of z_0 is $2^{\mathbf{n}} + 2^{\mathbf{n-1}} + \dots + 2 = 2^{\mathbf{n+1}} - 2$. The entering variable into the basic vector at this initial stage is z_n.

Let $V_0, \dots, V_{a(n)}$ be the ordering of the complementary basic vectors for this problem, obtained in the proof of Theorem 6.1. $V_0 = (w_1, \dots, w_n)$, $V_1 = (w_1, \dots, w_{n-1}, z_n)$, etc. Let $\mathbf{K}_r$ be the complementary cone corresponding to the complementary basic vector V_r for the LCP $(\hat{q}(n), \widetilde{M}(n))$. Using the geometric interpretation discussed above, the effect of the initial pivot step of bringing z_n into the basic vector $(w_1, \dots, w_{n-1}, z_0)$ can be interpreted as a walk through the complementary cone $\mathbf{K}_1$, beginning with the point $\hat{q}(n) + \lambda_1 e_n$ (where $\lambda_1 = 2^{\mathbf{n+1}} - 2$) on the facet of $\mathbf{K}_1$ corresponding to the Pos cone of the columns of $w_1, \dots, w_{n-2}, w_{n-1}$ in Tableau 6.4, to the point $\hat{q}(n) + \lambda_2 e_n$ (where $\lambda_2 = 2^{n+1} - 4$) on the facet of $\mathbf{K}_1$ corresponding to the Pos cone of the columns of $w_1, \dots, w_{n-2}, z_n$ in Tableau 6.4 along the half-line $\{x : x = \hat{q}(n) + \lambda e_n, \lambda \geqq 0\}$. Here λ_1, λ_2 are the values of z_0 in the basic solution of Tableau 6.4 corresponding to the basic vectors $(w_1, \dots, w_{n-1}, z_0)$ and $(w_1, \dots, w_{n-2}, z_n, z_0)$ respectively. Thus the initial pivot step of introducing z_n into the basic vector $(w_1, \dots, w_{n-1}, z_0)$ can be interpreted as the walk across the complementary cone $\mathbf{K}_1$, starting at the value $\lambda = \lambda_1$ to the value $\lambda = \lambda_2$ along the half-line $\{x : x = \hat{q}(n) + \lambda e_n, \lambda \geqq 0\}$. Similarly the r^{th} pivot step performed during the complementary pivot method applied on this problem, can be interpreted as the walk through the complementary cone $\mathbf{K}_r$ along the half-line

$\{x : x = \hat{q}(n) + \lambda e_n, \lambda \geqq 0\}$, for $r \geqq 2$. Since the straight line $\{x : x = \hat{q}(n) + \lambda e_n$, λ a real number$\}$ is the same as the line $\mathbf{L}(n)$ defined in equation (6.1), from the results in the proof of Theorem 6.1 and the geometric interpretation of the pivot steps in the complementary pivot method discussed above, we reach the following conclusions: the complementary pivot method starts with a value for z_0 of $2^{\mathbf{n+1}} - 2$ in the initial step. All pivot steps are nondegenerate and the value of z_0 decreases by 2 in every pivot step. Hence the method terminates when the value of z_0 becomes zero after the $(2^{\mathbf{n}} - 1)$th pivot step. This last pivot step in the method corresponds to a walk into the complementary cone $\mathbf{K}_{a(n)}$ associated with the complementary basic vector $V_{a(n)} = (z_1, w_2, \ldots, w_n)$ along the half-line $\{x : x = \hat{q}(n) + \lambda e_n, \lambda \geqq 0\}$. Hence the terminal basic vector obtained in the complementary pivot method applied on this problem will be $(z_1, w_2, \ldots, w_n)$ and it can be verified that the solution of the LCP $(\hat{q}(n), \widetilde{M}(n))$ is $(w = (0, 2^{\mathbf{n-1}}, \ldots, 2),\ z = (2^{\mathbf{n}}, 0, \ldots, 0))$. Therefore counting the first pivot step in which the canonical tableau with respect to the initial basic vector $(w_1, \ldots, w_{n-1}, z_0)$ is obtained, the complementary pivot method requires $2^{\mathbf{n}}$ pivot steps for solving the LCP $(\hat{q}(n), \widetilde{M}(n))$, for any $n \geqq 2$.

□

Example 6.2

See Example 2.10 in Section 2.2.7 where the LCP $(\hat{q}(3), \widetilde{M}(3))$ of order 3 is solved by the complementary pivot method and verify that it required $2^{\mathbf{3}} = 8$ pivot steps before termination.

6.4 Computational Complexity of Principal Pivoting Method I

Theorem 6.4 *Principal pivoting Method I requires $2^{\mathbf{n}} - 1$ pivot steps before termination, when applied on the LCP $(-e_n, \widetilde{M}(n))$, for any $n \geqq 2$.*

Proof. Proof is by induction on n. The original tableau for this problem is shown in Tableau 6.5

Tableau 6.5

w_1	w_2	$\ldots$	w_n	z_1	z_2	$\ldots$	z_n	q
1	0	$\ldots$	0	-1	0	$\ldots$	0	-1
0	1	$\ldots$	0	-2	-1	$\ldots$	0	-1
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
0	0	$\ldots$	1	-2	-2	$\ldots$	-1	-1

It can be verified that $(z_1, w_2, \dots, w_n)$ is a complementary feasible basis for Tableau 6.5 and the solution of this LCP is $(w_1, \dots, w_n) = (0, 1, \dots, 1)$, $(z_1, \dots, z_n) = (1, 0, \dots, 0)$.

In Example 4.1 of Section 4.1 the LCP $(-e_3, \widetilde{M}(3))$ was solved by Principal Pivoting Method I, and it required $2^{\mathbf{3}} - 1 = 7$ pivot steps, thus verifying the theorem for $n = 3$. The theorem can also be verified to be true when $n = 2$. We now set up an induction hypothesis.

Induction Hypothesis: When applied on the LCP $(-e_{n-1}, \widetilde{M}(n-1))$, the Principal Pivoting Method I requires $2^{\mathbf{n-1}} - 1$ pivot steps before termination.

We will now prove that the induction hypothesis implies that the Principal Pivoting Method I requires $2^{\mathbf{n}} - 1$ pivot steps before termination when apllied on the LCP $(-e_n, \widetilde{M}(n))$ of order n.

When it is applied on the LCP in Tableau 6.5 the initial basic vector in Principal Pivoting Method I is $(w_1, \dots, w_n)$. The entering variable into this initial complementary basic vector is z_n. Since $\widetilde{M}(n)$ is a P-matrix, by the results in Section 4.1, the method terminates when all the updated right hand side constants become nonnegative.

By the pivot row choice rule used in Principal Pivoting Method I, the question of using Row 1 in Tableau 6.5 as the pivot row does not arise until a complementary basic vector satisfying the property that the entries in Rows 2 to n of the updated right hand side constant vectors corresponding to it are all nonnegative, is reached. So until such a complementary basic vector is reached, the pivot steps choosen are exactly those that will be choosen in solving the principal subproblem of Tableau 6.5 in the variables $(w_2, \dots, w_n)$; $(z_2, \dots, z_n)$. This principal subproblem is actually the LCP $(-e_{n-1}, \widetilde{M}(n-1))$ of order $n-1$, with the exception that the variables in it are called $(w_2, \dots, w_n)$; $(z_2, \dots, z_n)$. By the induction hypothesis, to solve this principal subproblem, Principal Pivoting Method I takes $2^{\mathbf{n-1}} - 1$ pivot steps. By the results discussed above $(z_2, w_3, \dots, w_n)$ is the unique complementary feasible basic vector for this principal subproblem.

Hence when Principal Pivoting Method I is applied on Tableau 6.5, after $2^{\mathbf{n-1}} - 1$ pivot steps it reaches the complementary basic vector $(w_1, z_2, , w_3, \dots, w_n)$. The canonical tableau of Tableau 6.5 corresponding to the complementary basic vector $(w_1, z_2, w_3, \dots, w_n)$ is given in Tableau 6.6.

Tableau 6.6 Canonical Tableau after $2^{\mathbf{n-1}} - 1$ Pivot Steps are carried out, beginning with Tableau 6.5 in Principal Pivoting Method I.

Basic Variable	w_1	w_2	w_3	$\dots$	w_n	z_1	z_2	z_3	$\dots$	z_n	q
w_1	1	0	0	$\dots$	0	$\boxed{-1}$	0	0	$\dots$	0	-1
z_2	0	-1	0	$\dots$	0	2	1	0	$\dots$	0	1
w_3	0	-2	1	$\dots$	0	2	0	-1	$\dots$	0	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
w_n	0	-2	0	$\dots$	1	2	0	-2	$\dots$	-1	1

Since the update right hand side constant in Row 1 is $-1 < 0$, the method now continues by making a single principal pivot step in position 1 in Tableau 6.6 (this replaces w_1 in the basic vector by z_1). The pivot element is inside a box. This leads to the canonical tableau in Tableau 6.7.

Tableau 6.7 Canonical Tableau after $2^{\mathbf{n-1}}$ Pivot Steps are Carried Out, beginning with Tableau 6.5 in Principal Pivoting Method I.

Basic Variable	w_1	w_2	w_3	$\dots$	w_n	z_1	z_2	z_3	$\dots$	z_n	q
z_1	-1	0	0	$\dots$	0	1	0	0	$\dots$	0	1
z_2	2	-1	0	$\dots$	0	0	1	0	$\dots$	0	-1
w_3	2	-2	1	$\dots$	0	0	0	-1	$\dots$	0	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
w_n	2	-2	0	$\dots$	1	0	0	-2	$\dots$	-1	-1

Since some of the updated right hand side constants in Tableau 6.7 are still negative, the method continues. By the arguments mentioned above, when Principal Pivoting Method I is continued from Tableau 6.7, z_1 remains the basic variable in the first row until another complementary basic vector satisfying the property that the entries in Rows 2 to n in the updated right hand side constants vector corresponding to it are all nonnegative, is again reached. It can be verified that the principal subproblem obtained by eliminating Row 1 and the columns corresponding to the complementary pair of variables w_1, z_1 in Tableau 6.7 and interchanging the columns of the variables w_2 and z_2; is exactly the LCP $(-e_{n-1}, \widetilde{M}(n-1))$ with the exception that the variables in it are called $(z_2, w_3, \dots, w_n)$; $(w_2, , z_3, \dots, z_n)$. When Principal Pivoting Method I is continued from Tableau 6.7 the pivot steps obtained are exactly those that occur when this principal subproblem is solved by Principal Pivoting Method I, until this principal subproblem is solved. Since this principal subproblem is the LCP $(-e_{n-1}, \widetilde{M}(n-1))$, by the induction hypothesis, this leads to an additional $2^{\mathbf{n-1}} - 1$ pivot steps from Tableau 6.7. Since the variables in this principal subproblem are $(z_2, w_3, \dots, w_n)$, $(w_2, z_3, \dots, z_n)$, in that order, by the results mentioned earlier, $(w_2, w_3, \dots, w_n)$ is the unique complementary feasible basic vector for this principal subproblem. So after continuing for an additional $2^{\mathbf{n-1}} - 1$ pivot steps from Tableau 6.7, Principal Pivoting Method I reaches the complementary basic vector $(z_1, w_2, w_3, \dots, w_n)$, which was verified to be a complementary feasible basic vector for the LCP in Tableau 6.5 and then the method terminates. So it took $2^{\mathbf{n-1}}$ pivot steps to reach Tableau 6.7 and an additional $2^{\mathbf{n-1}} - 1$ pivot steps afterwards, before termination. Thus it requires a total of $2^{\mathbf{n-1}} + 2^{\mathbf{n-1}} - 1 = 2^{\mathbf{n}} - 1$ pivot steps before termination, when applied on the LCP of order n in Tableau 6.5. Thus under the induction hypothesis, the statement of the theorem also holds for n. The statement of the theorem has already been verified for $n = 2, 3$. Hence, by induction, Theorem 6.4 is true for all $n \geqq 2$.

$\square$

Exercise

6.1 Prove that the sequence of complementary basic vectors obtained when Principal Pivoting Method I is applied on the LCP in Tableau 6.5 is exactly the sequence $V_0, V_1, \dots, V_{a(n)}$, obtained in the proof of Theorem 6.1. (Hint: Use an inductive argument as in the proof of Theorem 6.4).

So far, we have discussed the worst case computational complexity of complementary and principal pivot methods, which can handle a large class of LCPs. These results may not apply to other special algorithms for solving LCPs (q, M), in which the matrix M has special structure. An example of these is the algorithm of R. Chandrasekaran which can solve the LCP (q, M) when M is a Z-matrix (a square matrix $M = (m_{ij})$ is said to be a Z-matrix if $m_{ij} \leqq 0$ for all $i \neq j$) discussed in Section 8.1. This special algorithm for this special class of LCPs has been proved to terminate in at most n pivot steps.

The matrix $\widetilde{M}(n)$ used in the examples contructed above is lower triangular, it is a P-matrix, a nonnegative matrix, it is copositive plus and also PSD. So it has all the nice properties of matrices studied in LCP literature. In spite of it, complementary pivot methods take $2^{\mathbf{n}} - 1$ or $2^{\mathbf{n}}$ pivot steps to solve the LCP of order n in the examples constructed above, all of which are associated with the matrix $\widetilde{M}(n)$.

We have shown that the computational requirements of the well known complementary and principal pivot methods exhibit an exponential growth rate in terms of the order of the LCP. Our analysis applies only to the worst case behavior of the methods on specially constructed simple problems. The performance of the algorithms on average practical problems using practical data may be quite different. The analysis here is similar to the analysis of the worst case computational requirements of the simplex method for solving linear programs in Chapter 14 of [2.26].

The class of LCPs (q, M) where M is a PD and symmetric matrix is of particular interest because of the special structure of these problems, and also because they appear in many practical applications. It turns out that even when restricted to this special class of LCPs, the worst case computational requirements of complementary pivot methods exhibit an exponential growth rate in terms of the order of the LCP. See reference [6.3] of Y. Fathi and Exercises 6.2 to 6.5.

As mentioned in Section 2.8 the exponential growth of the worst case computational complexity as a function of the size of the problem does not imply that these algorithms are not useful for solving large scale practical problems. The exponential growth has been mathematically established on specially constructed problems with a certain pathological structure. This pathological structure does not seem to appear often in practical applications. As discussed in Section 2.8 and in Reference [2.36], the probabilistic average (or expected) computational complexity of some versions of the complementary pivot algorithm grows at most quadratically with n. Empirical computational tests seem to indicate that the number of pivot steps needed by these algorithms before termination grows linearly with n on an average.

6.5 Exercises

6.2 For $n \geqq 2$, let $M(n) = \big(\widetilde{M}(n)\big)\big(\widetilde{M}(n)\big)^T$.

$$M(n) = \begin{pmatrix} 1 & 2 & 2 & \dots & 2 & 2 \\ 2 & 5 & 6 & \dots & 6 & 6 \\ 2 & 6 & 9 & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 2 & 6 & \vdots & \dots & 1+4(n-2) & 2+4(n-2) \\ 2 & 6 & \vdots & \dots & 2+4(n-2) & 1+4(n-1) \end{pmatrix}$$

Prove that $M(n)$ is PD and symmetric. Solve the LCP $(-e_3, M(3))$ by Principal Pivoting Method I and verify that it takes $2^{\mathbf{3}} - 1 = 7$ pivot steps before termination. Solve The LCP $((-4, -7)^T, M(2))$ by the complementary pivot method and verify that it takes $2^{\mathbf{2}} = 4$ pivot steps before termination. Solve the parametric LCP $((4 - \gamma, 1 - \gamma)^T, M(2))$ by the parametric LCP algorithm and verify that it produces all the $2^{\mathbf{2}} = 4$ complementary basic vectors of this problem before solving the problem for all the values of the parameter γ.
(Y. Fathi [6.3])

6.3 Prove that Principal Pivoting Method I requires $2^{\mathbf{n}} - 1$ steps before termination when applied on the LCP $(-e_n, M(n))$, for any $n \geqq 2$.
(Y. Fathi [6.3])

6.4 Prove that there exists a column vector $q(n) \in \mathbf{R}^n$ (actually an uncountable number of such $q(n)$s exist) such that the straight line $\{x : x = q(n) - \gamma e_n, \gamma \text{ a real number}\}$ cuts across the interior of every one of the $\mathbf{2^n}$ complementary cones in the class $\mathcal{C}(M(n))$ for any $n \geqq 2$.
(Y. Fathi [6.3])

6.5 Prove that the parametric algorithm obtains all the $\mathbf{2^n}$ complementary basic vectors before termination, when applied to solve the LCP $(q(n) - \gamma e_n, M(n))$ for all γ for any $n \geqq 2$, where $q(n)$ is the column vector in $\mathbf{R}^n$ constructed in Exercise 6.4.
(Y. Fathi [6.3])

6.6 Prove that the complementary pivot method requires $\mathbf{2^n}$ pivot steps before termination when applied on the LCP $(q(n), M(n))$, for $n \geqq 2$, where $q(n)$ is the column vector in $\mathbf{R}^n$ constructed in Exercise 6.4.
(Y. Fathi [6.3])

6.7 Construct a class of LCPs with integer data, containing one problem of order n for each $n \geqq 2$, each associated with a PD matrix, such that the number of pivot steps required by Graves' principal pivoting method (Section 4.2) to solve the n^{th} problem in this class is an exponentially growing function of n.

6.8 Let $q(n) = (\mathbf{2^n} + 2, \mathbf{2^n} + 4, \ldots, \mathbf{2^n} + \mathbf{2^j}, \ldots, \mathbf{2^n} + \mathbf{2^{n-1}}, -\mathbf{2^n})^T$ and

$$\overline{M}(n) = \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 & -2 \\ 0 & 1 & 2 & \ldots & 2 & -2 \\ \vdots & 0 & 1 & \ldots & 2 & -2 \\ \vdots & \vdots & 0 & \ldots & \vdots & \vdots \\ \vdots & \vdots & 0 & \ldots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -2 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

Prove that the Dantzig-Cottle principal pivoting method of Section 4.3 requires $\mathbf{2^{n-1}}$ steps to solve the LCP $(q(n), \overline{M}(n))$.
(A. Gana [6.4])

6.9 Show that the variable dimension algorithm of Secton 2.6 requires $\mathbf{2^n} - 1$ steps to solve the LCP $\big(\tilde{q}(n), (\widetilde{M}(n))^T\big)$.
(A. Gana [6.4])

6.10 Define the matrix $M = (m_{ij})$ of order $n \times n$ by the following

$$\begin{aligned} m_{ii} &= 1 \quad \text{for all } i = 1 \text{ to } n \\ m_{ij} &= 2 \quad \text{if } j > i \text{ and } i + j \text{ is odd} \\ &= -1 \quad \text{if } j > i \text{ and } i + j \text{ is even} \\ &= -1 \quad \text{if } j < i \text{ and } i + j \text{ is odd} \\ &= 2 \quad \text{if } j < i \text{ and } i + j \text{ is even.} \end{aligned}$$

For example, the matrix M defined above, is the following for $n = 4$

$$M = \begin{pmatrix} 1 & 2 & -1 & 2 \\ -1 & 1 & 2 & -1 \\ 2 & -1 & 1 & 2 \\ -1 & 2 & -1 & 1 \end{pmatrix}.$$

Show that M is a P-matrix and a PSD matrix.

Let e be the column vector of all 1s in $\mathbf{R}^n$. Consider the LCP $(-e, M)$, where M is the matrix defined above. Show that the complementary feasible basic vector for this problem is

$$\begin{aligned} &(w_1, z_2, \ldots, z_n) \quad \text{if } n \text{ is even} \\ &(z_1, z_2, \ldots, z_n) \quad \text{if } n \text{ is odd.} \end{aligned}$$

Study the computational complexity of the various algorithms for solving LCPs discussed so far, on the LCP $(-e, M)$, where M is the matrix defined above.
(R. Chandrasekaran, J. S. Pang and R. Stone)

6.6 References

6.1 J. R. Birge and A. Gana, "Computational Complexity of Van der Heyden's Variable Dimension Algorithm and Dantzig-Cottle's Principal Pivoting Method for Solving LCPs", *Mathematical Programming,* 26 (1983) 316–325.

6.2 R. W. Cottle, "Observations on a Class of Nasty Linear Complementary Problems", *Discrete Applied Mathematics,* 2 (1980) 89–111.

6.3 Y. Fathi, "Computational Complexity of LCPs Associated with Positive Definite Symmetric Matrices", *Mathematical Programming*, 17 (1979) 335–344.

6.4 A. Gana, "Studies in the Complementary Problem", Ph.D. Dissertation, Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, Michigan (1982).

6.5 K. G. Murty, "Computational Complexity of Complementary Pivot Methods", *Mathematical Programming Study 7*, (1978) 61–73.