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Chapter 4

The Simplex Method for Solving LPs

This is Chapter 4 of “Junior Level Web-Book for Optimization Models for decision Making” by Katta G. Murty.

In this chapter we will discuss the **simplex method** for solving LPs, (also called **primal simplex method**) which is a computational method developed by George B. Dantzig in 1947. There are many different versions of this method, we will discuss the original version which is a direct extension of the GJ method for solving linear equations, because of its simplicity for understanding the main principles on which the method is based. This version which operates with the canonical tableau that is updated after every pivot step, just as in the GJ method, is also a great tool for learning the optimality criterion in the simplex method; and using it to analyze the problem and draw useful conclusions for planning purposes. It is eminently suitable for solving small problems by hand computation.

This version is not computationally efficient for solving LPs which are not small. That’s why all commercial computer programs for LP are based on versions of better computational efficiency discussed in graduate level books. High quality and highly reliable software for the simplex method is widely available from many sources. Software for

LP is reported to be the most heavily used software among all applied mathematics programs.

The Importance of Learning the Details of the Simplex Method

Students are always asking me why it is important to learn the details of the simplex method when there are so many easy to use computer programs readily available for it. Here are some important reasons.

- 1) A good knowledge of the details of the simplex method and how it works, makes it easier for you to appreciate the optimality criterion for LP, and why it makes sense.
- 2) The relative cost coefficients obtained in the simplex method have very nice practical meaning which plays a role in many planning applications. Learning the simplex method is tantamount to learning the practical meaning of relative cost coefficients.
- 3) Many applications involve models with very special structure, in which one may be able to guess the optimum solution. A good knowledge of the simplex method and its termination conditions may enable you to check directly whether the guess is correct, without having to use a computer program.
- 4) In many real world applications, the initial model for the problem may be infeasible, or may produce a solution that does not seem practical. In this case it may be necessary to make changes in the model after analyzing it. Good knowledge of the simplex method is often very useful in this kind of analysis.
- 5) The simplex method is one of the most fundamental algorithms in optimization. Understanding it in detail, and learning how it works is very helpful in understanding other optimization algorithms.
- 6) Because there are so many easy to use computer programs to solve LPs, it is true that it may be unnecessary to understand the

details of the algorithms on which they are based, to solve an LP. But you need to have a clear idea of the process and how the algorithm works, if you want to make more than a simple clerical contribution to the application.

4.1 Transformations to be Carried Out On an LP Model Before Applying the Simplex Method On It

The main computational tool used in the version of the simplex method discussed in this chapter is the GJ pivot step. This step transforms a system of linear equations into another equivalent system of linear equations with the same set of solutions.

The GJ pivot step involves a sequence of row operations. If row operations are carried out directly on a system of linear inequality constraints, then the system changes. As an example, consider the following two systems:

$$\begin{array}{r} \text{(I)} \\ \hline x_1 \geq 0 \\ x_2 \geq 0 \\ \hline \end{array} \qquad \begin{array}{r} \text{(II)} \\ \hline x_1 \geq 0 \\ x_1 + x_2 \geq 0 \\ \hline \end{array}$$

System (II) is obtained from (I) by adding the first constraint to the second. By plotting the sets of feasible solutions it can be verified that the set of feasible solutions of (II) is strictly larger than that of (I). That's why it is not valid to perform row operations or GJ pivot steps directly on a system of linear constraints including some inequalities.

Hence, before applying the simplex method on an LP, all the constraints on which pivot operations are to be carried out must be transformed into equality constraints. In fact the version of the simplex method discussed in this chapter is applied only after transforming the LP into a form known as **standard form**, which in matrix notation is:

$$\begin{aligned} \text{Minimize } z(x) &= cx + \alpha \\ \text{subject to } Ax &= b \\ x &\geq 0 \end{aligned}$$

where α is a constant. This is expressed in detached coefficient **tableau form** as below. In it, the objective function $z = z(x)$ is expressed through the equation $cx - z = -\alpha$ (this is the equation corresponding to the last row in the tableau).

Tableau representation		
x	$-z$	
A	0	b
c	1	$-\alpha$
$x \geq 0$, minimize z		

Differences in Notation in LP Textbooks

Here we included the equation defining the objective function as the bottom row of the tableau. Some LP textbooks include it as the top row of the tableau instead.

For another difference, we expressed the objective function through the equation $cx - z = -\alpha$. Some LP textbooks record in as $z - cx = \alpha$ instead.

Also, we consider the problem as a minimization problem for minimizing z subject to the constraints given. Mathematically, this problem is equivalent to maximizing $-z$ subject to the same constraints (both of these problems have the same set of optimum solutions; and the optimum objective value in one will be the negative of the optimum objective value in the other). Some books discuss the maximization problem.

These minor differences account for the difference in the way the optimality criterion is stated in different text books. So, while reading another book, if you see that their optimality criterion differs from that in this book, check their problem statement and the way they enter the objective equation in the tableau.

Transformations

In standard form all the constraints other than the nonnegativity restrictions on the variables are linear equations; and all the variables are restricted to be nonnegative. Any LP can be transformed into standard form by carrying out the following steps (it is convenient to carry them out in the order listed). After stating each step we include an example for it.

Step 1. Transforming a variable (x_1 say), subject to only one bound (lower or upper, but not both): If the bound is $x_1 \geq 0$, leave it unchanged.

Suppose the bound is $x_1 \geq \ell_1$ where $\ell_1 \neq 0$. This can be written as $x_1 - y_1 = \ell_1, y_1 \geq 0$, where y_1 is the slack variable. So, $x_1 = y_1 + \ell_1$. In all the constraints and the objective functions, replace x_1 by $y_1 + \ell_1$. This eliminates x_1 from the problem replacing it by the nonnegative variable y_1 .

If the bound is $x_1 \leq u_1$ where u_1 is any real number, it is $x_1 + s_1 = u_1, s_1 \geq 0$. In this case we replace x_1 wherever it appears in the model by $u_1 - s_1$.

Example: To illustrate this step consider the LP

$$\begin{aligned} \text{Maximize } z &= 2x_1 - 3x_2 - 2x_3 + 7x_4 \\ \text{s. to } -x_1 + 2x_2 + x_3 - x_4 &= -6 \\ x_1 - x_2 + 3x_3 + x_4 &\geq -5 \\ x_1 \geq -2, x_2 \leq 7, x_3 \leq 0, & \quad x_4 \geq 0 \end{aligned}$$

Here x_1, x_2, x_3 are all variables subject to a single bound that need to be transformed. As described above, we substitute $x_1 = s_1 - 2$, $x_2 = 7 - s_2$, $x_3 = -s_3$, where s_1, s_2, s_3 are the appropriate slack variables. The transformed problem is

$$\text{Maximize } z = 2(s_1 - 2) - 3(7 - s_2) - 2(-s_3) + 7x_4$$

$$\begin{aligned} \text{s. to} \quad & -(s_1 - 2) + 2(7 - s_2) + (-s_3) - x_4 = -6 \\ & (s_1 - 2) - (7 - s_2) + 3(-s_3) + x_4 \geq -5 \\ & s_1, s_2, s_3, x_4 \geq 0 \end{aligned}$$

or

$$\begin{aligned} \text{Maximize} \quad & z = 2s_1 + 3s_2 + 2s_3 + 7x_4 - 25 \\ \text{s. to} \quad & -s_1 - 2s_2 - s_3 - x_4 = -22 \\ & s_1 + s_2 - 3s_3 + x_4 \geq 4 \\ & s_1, s_2, s_3, x_4 \geq 0 \end{aligned}$$

Once the solution of the transformed problem is obtained in terms of s_1, s_2, s_3, x_4 , the corresponding solution of the original problem can be obtained in terms of x_1, x_2, x_3, x_4 by using the formulas for them given above.

Step 2. Transforming a variable (x_2 say), subject to both an upper and a lower bound : Suppose the bounds are $l_2 \leq x_2 \leq u_2$, where $u_2 > l_2$.

Suppose $l_2 = 0$. In this case $x_2 \leq u_2$ is equivalent to $x_2 + s_2 = u_2, s_2 \geq 0$; here s_2 is the slack variable corresponding to this restriction. Include $x_2 + s_2 = u_2$ as an equality constraint with s_2 as a new nonnegative variable in the model, replacing the upper bound on x_2 . Leave $x_2 \geq 0$ as it is.

If $l_2 \neq 0$, $l_2 \leq x_2$ is equivalent to $x_2 - y_2 = l_2, y_2 \geq 0$. Substitute $x_2 = y_2 + l_2$ in all the constraints and the objective function, thus eliminating x_2 from the problem. In this process $x_2 \leq u_2$ becomes $y_2 + l_2 \leq u_2$, or $y_2 \leq u_2 - l_2$ which is equivalent to $y_2 + s_2 = u_2 - l_2, s_2 \geq 0$ (here s_2 is the slack variable corresponding to the upper bound on y_2). Include $y_2 + s_2 = u_2 - l_2$ as an equality constraint with s_2 as a new nonnegative variable in the model.

Example: To illustrate this step consider the LP

$$\text{Minimize} \quad z = -x_1 + 3x_2 + 4x_3$$

$$\begin{aligned} \text{s. to } 4x_1 - 3x_2 + 5x_3 &= 18 \\ -x_1 + x_2 - 3x_3 &\leq 100 \\ 0 \leq x_1 \leq 10, -2 \leq x_2 \leq 5, & \quad x_3 \geq 0 \end{aligned}$$

We enter $x_1 \leq 10$ as the new constraint $x_1 + s_1 = 10, s_1 \geq 0$.

We write $-2 \leq x_2$ as $x_2 - y_2 = -2, y_2 \geq 0$. So we substitute $x_2 = y_2 - 2$ and eliminate x_2 from the problem. $x_2 \leq 5$ becomes $y_2 - 2 \leq 5$ or $y_2 \leq 7$; which we enter as the new equality constraint $y_2 + s_2 = 7, s_2 \geq 0$.

So, the transformed problem is

$$\begin{aligned} \text{Minimize } z &= -x_1 + 3(y_2 - 2) + 4x_3 \\ \text{s. to } 4x_1 - 3(y_2 - 2) + 5x_3 &= 18 \\ -x_1 + (y_2 - 2) - 3x_3 &\leq 100 \\ x_1 + s_1 &= 10 \\ y_2 + s_2 &= 7 \\ x_1, y_2, s_1, s_2, x_3 &\geq 0 \end{aligned}$$

or

$$\begin{aligned} \text{Minimize } z &= -x_1 + 3y_2 + 4x_3 - 6 \\ \text{s. to } 4x_1 - 3y_2 + 5x_3 &= 12 \\ -x_1 + y_2 - 3x_3 &\leq 102 \\ x_1 + s_1 &= 10 \\ y_2 + s_2 &= 7 \\ x_1, y_2, s_1, s_2, x_3 &\geq 0 \end{aligned}$$

From a solution of the transformed problem, the corresponding solution of the original problem can be obtained by using $x_2 = y_2 - 2$.

Step 3. Transforming All Remaining Inequalities Into Equations: This is done by introducing a nonnegative slack variable corresponding to each of those constraints.

Step 4. Put Objective Function in Minimization Form: If the objective function is required to be minimized, leave it as it is. If it is required to be maximized, replace it by the equivalent problem of minimizing its negative, subject to the same constraints.

Step 5. Eliminate Unrestricted Variables: At this stage all the constraints are either equality constraints, or nonnegativity restrictions on individual variables. If all the variables in the problem at this stage are nonnegative variables, the problem is now in standard form; put it in detached coefficient tableau form and stop.

Otherwise there must be some unrestricted variables (i.e., those on whose value there is no explicit lower or upper bound stated at this stage) in the model.

In the previous steps we have eliminated variables subject to one or two bounds, but each of them was always replaced in the model by another nonnegative variable. It turns out that unrestricted variables can be truly eliminated from the problem altogether without another variable replacing them.

If x_3 is an unrestricted variable, use one of the equations in which it appears with a nonzero coefficient, to get an expression for it in terms of other variables. Eliminate this equation from the model, but store this expression for x_3 some where to get the value of x_3 in the solution after the values of the other variables are obtained from the remaining model. Substitute the expression for x_3 where ever it appears in the remaining model, this eliminates x_3 from the remaining model.

A convenient computational tool for eliminating an unrestricted variable at this stage in the GJ pivot step.

Suppose there are M equality constraints in N variables, x_1, \dots, x_N , at this stage, of which some are unrestricted and the others are nonnegative. If the objective at this stage is to minimize $z = \sum_{j=1}^N c_j x_j + \alpha$, where α is a constant, augment the system of equality constraints by the additional equation $\sum_{j=1}^N c_j x_j - z = \alpha$ defining the objective function. Express all the equality constraints in a detached coefficient tableau form as given below.

Tableau: System of equality constraints and objective row

x_1	\dots	x_N	$-z$	
a_{11}	\dots	a_{1N}	0	b_1
\vdots		\vdots	\vdots	\vdots
a_{M1}	\dots	a_{MN}	0	b_M
c_1	\dots	c_N	1	α

Suppose x_N is an unrestricted variable. To eliminate it, select its column as the pivot column for a GJ pivot step on this tableau. Select a row with a nonzero coefficient in the pivot column as the pivot row, and perform the GJ pivot step. In the new tableau, use the equation corresponding to the pivot row to express x_N as a function of the other variables, store this expression somewhere. Now delete the pivot row, and the column of the variable x_N from this tableau. This deletion eliminates x_N from the optimization portion of the problem. The remaining tableau represents the remaining problem to be solved.

Thus elimination of an unrestricted variable, reduces the problem into one with one less variable, and one less equation.

If there are some more unrestricted variables in the remaining tableau, repeat the same procedure on it.

When all the unrestricted variables are eliminated from the optimization portion of the problem this way, the remaining tableau represents an LP in standard form that is equivalent to the original one.

Example: To illustrate this step consider the LP in detached coefficient form. each row of the tableau represents an equality constraint on the variables.

x_1	x_2	x_3	PC x_4	$-z$	
1	-1	2	0	0	5
-2	4	0	-2	0	6 PR
3	4	1	1	0	17
10	15	-7	2	1	100

$x_1, x_2, x_3 \geq 0, x_4$ unrestricted, min z

To eliminate the unrestricted variable x_4 , we choose the column of x_4 as the pivot column (PC), and select row 2 with a nonzero entry of -2 in the PC as the pivot row (PR). The pivot element in the PC and the PR is boxed. Performing the GJ pivot step leads to the next tableau.

BV	x_1	x_2	x_3	x_4	$-z$	
	1	-1	2	0	0	5
x_4	1	-2	0	1	0	-3
	2	6	1	0	0	20
	8	19	-7	0	1	106

The equation corresponding to 2nd row (row which was the PR for the pivot operation) in this tableau is $x_1 - 2x_2 + x_4 = -3$. From this we get the expression for x_4 to be $x_4 = 2x_2 - x_1 - 3$.

Now eliminate the 2nd row and the column of x_4 from this tableau. The remaining problem in detached coefficient form is given below. It is in standard form.

x_1	x_2	x_3	$-z$	
1	-1	2	0	5
2	6	1	0	20
8	19	-7	1	106

$x_1, x_2, x_3 \geq 0, \min z$

Example: In this example we will illustrate all the above steps. Consider the LP

$$\begin{aligned}
 \text{Maximize } z' &= 2x_1 + 7x_2 - x_3 + 2x_4 - x_5 \\
 \text{s. to } x_1 + x_2 - x_3 - x_4 + x_5 &\geq 2 \\
 x_2 + 2x_3 + x_4 - 2x_5 &\leq 11 \\
 x_1 - x_2 + x_3 + x_4 &= 14 \\
 x_1 \geq 2; x_2 \leq 0; 0 \leq x_3 \leq 10; & \quad x_4, x_5 \text{ unrestricted}
 \end{aligned}$$

The bound restriction $x_1 \geq 2$ becomes $x_1 = 2 + y_1, y_1 \geq 0$ where y_1 is the associated slack variable. $x_2 \leq 0$ becomes $x_2 = -y_2, y_2 \geq 0$. We

use these expressions to replace x_1, x_2 in the problem by y_1, y_2 . Also, the bound $x_3 \leq 10$ is recorded as the constraint $x_3 + y_3 = 10, y_3 \geq 0$. These lead to

$$\begin{aligned}
 \text{Maximize } z' &= 2y_1 - 7y_2 - x_3 + 2x_4 - x_5 + 4 \\
 \text{s. to } y_1 - y_2 - x_3 - x_4 + x_5 &\geq 0 \\
 -y_2 + 2x_3 + x_4 - 2x_5 &\leq 11 \\
 y_1 + y_2 + x_3 + x_4 &= 12 \\
 x_3 + y_3 &= 10 \\
 x_4, x_5 \text{ unrestricted, } y_1, y_2, y_3, x_3 &\geq 0.
 \end{aligned}$$

Now we transform the first two constraints into equations by introducing the slack variables y_4, y_5 , and also change the objective function from maximizing z' to minimizing $z = -z'$. The resulting problem in detached coefficient form is the one at the top. In this tableau we perform GJ pivot steps in the columns of the unrestricted variables x_4, x_5 . The PC (pivot column), PR (pivot row), BV (basic variable selected) are indicated and the pivot element for each step is boxed.

BV	y_1	y_2	y_3	y_4	y_5	x_3	x_4	x_5	$-z$	
	1	-1	0	-1	0	-1	-1	1	0	0 PR
	0	-1	0	0	1	2	1	-2	0	11
	1	1	0	0	0	1	1	0	0	12
	0	0	1	0	0	1	0	0	0	10
-2	7	0	0	0	1	-2	1	1	4	
							PC			
x_4	-1	1	0	1	0	1	1	-1	0	0
	1	-2	0	-1	1	1	0	-1	0	11 PR
	2	0	0	-1	0	0	0	1	0	12
0	0	1	0	0	1	0	0	0	10	
	-4	9	0	2	0	3	0	-1	1	4
							PC			

BV	y_1	y_2	y_3	y_4	y_5	x_3	x_4	x_5	$-z$
x_4	0	3	0	2	-1	0	1	0	-11
x_5	-1	2	0	1	-1	-1	0	1	-11
	3	-2	0	-2	1	1	0	0	23
	0	0	1	0	0	1	0	0	10
	-5	11	0	3	-1	2	0	0	-7

$y_1, y_2, y_3, y_4, y_5, x_3 \geq 0; x_4, x_5$ unrestricted

From rows 1, 2 of the final tableau we have the expressions $x_4 = -11 - 3y_2 - 2y_4 + y_5$, and $x_5 = -11 + y_1 - 2y_2 - y_4 + y_5 + x_3$. Now we eliminate these two rows, and the columns of x_4, x_5 leading to the remaining problem in standard form.

y_1	y_2	y_3	y_4	y_5	x_3	$-z$
3	-2	0	-2	1	1	0
0	0	1	0	0	1	0

y_1 to $y_5, x_3 \geq 0, \min z$

Exercises

4.1.1: Transform the following LPs into standard form.

(i) Maximize $3x_1 + 5x_2 - 2x_3 + 2x_4 + 4x_5 - x_6$
s. to $x_1 + 2x_2 - x_3 + x_4 + x_5 = 13$
 $x_1 - x_2 + x_3 - x_4 - x_5 + x_6 \geq 2$
 $x_2 - 2x_3 + x_4 + x_5 - 2x_6 \leq 11$
 $-x_1 + x_3 + 2x_4 + 2x_6 = 5$
 $x_1 \geq 2, x_2 \leq 5, x_3, x_4 \geq 0, x_5, x_6$ unrestricted

(ii) Maximize $2x_1 - 3x_2 + 3x_3 - x_4 - x_5 - 5x_6$
s. to $4x_1 + x_3 - x_4 + x_5 - 3x_6 \geq 100$
 $2x_1 + 2x_2 + x_3 + 2x_4 + x_6 = 80$
 $-2x_1 + 2x_2 + 2x_3 - 2x_4 + x_5 - 2x_6 \leq 200$

$$x_1 \geq 20, x_2 \leq 50, x_3, x_4 \geq 0, x_4 \leq 15,$$

$$x_5, x_6 \text{ unrestricted}$$

(iii) Maximize $2x_2 + x_3 + x_4$
 s. to $2x_1 - x_2 - x_3 + x_4 \leq -8$
 $2x_2 + x_3 - x_4 \geq 4$
 $x_1 - x_2 + x_4 = 13$
 $-3 \leq x_1 \leq 4, \quad x_2 \geq 2, \quad x_3 \text{ unrestricted}, \quad x_4 \leq 0$

(iv) Minimize $3x_1 - x_2 + x_3 - 2x_4$
 s. to $x_1 + x_2 + 2x_3 + x_4 = 12$
 $x_2 - x_3 + x_4 \geq 6$
 $2x_1 + x_3 - x_4 \leq 10$
 $1 \leq x_1 \leq 5, \quad x_2 \leq 10, \quad x_3 \geq 0, \quad x_4 \text{ unrestricted.}$

4.2 Definitions of Various Types of Basic Vectors for the Problem

Note: The definition of basic vector, basis given below are linear algebra concepts referring to systems of linear equations (i.e., no inequalities). In the discussion below they refer to the system of equality constraints in an LP in standard form. Hence these definitions cannot be used directly for LPs in general form.

The concept of a basic feasible solution given below is specialized to the LP in standard form. It has been generalized to LPs in general form, but we will not discuss this general definition here.

Consider the detached coefficient representation of the LP in standard form:

Tableau 1: Original Tableau

x_1		x_j		x_n	$-z$	
a_{11}	\dots	a_{1j}	\dots	a_{1n}	0	b_1
\vdots		\vdots		\vdots	\vdots	\vdots
a_{i1}	\dots	a_{ij}	\dots	a_{in}	0	b_i
\vdots		\vdots		\vdots	\vdots	\vdots
a_{m1}	\dots	a_{mj}	\dots	a_{mn}	0	b_m
c_1		c_j		c_n	1	α

$$x_j \geq 0 \text{ for all } j, \min z.$$

The top m rows in this tableau are the **constraint rows**. For $i = 1$ to m , the i th row represents the equality constraint $a_{i1}x_1 + \dots + a_{in}x_n = b_i$. The a_{ij} are called the **input-output (I/O) coefficients** or **technology coefficients**; and the b_i are called the **right hand side (RHS) constants**. $b = (b_i)^T$ is called the **RHS constants vector**.

The last row is the **objective row** because it represents the equation $c_1x_1 + \dots + c_nx_n - z = -\alpha$ that defines the objective function to be minimized in the problem subject to the constraints. The entries c_j in the objective row are called **original cost coefficients**, c_j being the original cost coefficient of x_j .

m is the number of constraints in this LP, and n is the number of decision variables (all required to be nonnegative). $A = (a_{ij})$ is the $m \times n$ **I/O coefficient matrix** for the problem. A includes only the constraint rows and not the objective row. Letting $c = (c_j)$, $x = (x_1, \dots, x_n)^T$, in matrix notation this LP is

$$\begin{aligned} \text{Minimize } & z = cx \\ \text{s. to } & Ax = b \\ & \text{and } x \geq 0 \end{aligned}$$

In some textbooks, a vector \bar{x} satisfying $A\bar{x} = b$, but may or may not satisfy $x \geq 0$ is referred to as a **solution** for this problem. If it also satisfies $x \geq 0$, then it satisfies all the constraints and the bound restrictions on the problem and is called a **feasible solution** of the problem.

An **optimum solution** is a feasible solution that makes the objective function $z(x)$ assume its smallest value among all feasible solutions.

The reason for using the adjective original when referring to the coefficients in the original tableau is because they will change when GJ pivot steps are carried out while solving the problem by the simplex method. After some pivot steps are carried out, the current tableau is referred to as the **updated tableau** and the cost coefficient vector in it is called the **updated cost vector**.

Redundant equality constraints can be eliminated without changing the problem. In Section 4.5 we will discuss how redundant constraints can be detected and eliminated while solving the problem by the simplex method. After all the redundant constraints are eliminated, the rank of the coefficient matrix will equal the number of rows in it. So, without any loss of generality, we will assume that $A = (a_{ij})$ in the original tableau is of full row rank.

The symbol A_j denotes the j th column $(a_{1j}, \dots, a_{mj})^T$ of A , it is the column vector of x_j in the original tableau.

A **basic vector** for the problem is a vector of m of the variables x_j whose set of column vectors is a linearly independent set. For example the vector of m variables $x_B = (x_1, x_2, \dots, x_m)$ is a basic vector iff the associated set of column vectors $\{A_1, \dots, A_m\}$ is linearly independent.

When referring to a particular basic vector x_B , variables in it are called **basic variables**, those not in x_B are called **nonbasic variables**.

Given a basic vector $x_{B_1} = (x_{j_1}, \dots, x_{j_m})$, the submatrix B_1 of the original I/O coefficient matrix A consisting of the column vectors of basic variables (in the order in which these variables are listed in x_{B_1}) is called the **basis** corresponding to x_{B_1} . Thus every basic vector is associated with a unique basis and vice versa.

If the original coefficient matrix A of order $m \times n$ contains the unit matrix of order m as a submatrix, by selecting the vector of variables associated with columns of this unit matrix, we get a basic vector x_B for which the basis B is the unit matrix I , known as the **unit basis**. A basic vector associated with the unit basis is known as a **unit basic vector**.

Canonical (or Updated) Tableau WRT a Basic Vector

Given a basic vector x_B associated with the basis B , the updated tableau obtained by converting the matrix B into the unit matrix of order m by GJ pivot steps, and also reducing the coefficients of all the basic variables in the objective row to 0 by row operations on the objective row is called the **canonical (or updated tableau)** WRT x_B . In this process, after each GJ pivot step, the variable associated with the pivot column is recorded as the **basic variable in the pivot row** in the tableau. These basic variables associated with the rows are normally recorded in a column in the tableau. Let x_B denote the vector of basic variables in the order in which they appear in this column in the canonical tableau. Then for $i = 1$ to m , the i th variable in x_B is called the **i -th basic variable**, or the **basic variable in the i -th row** in the canonical tableau, its column in the canonical tableau will be the i -th unit vector.

The cost coefficients of the variables in the canonical tableau WRT x_B are called the **updated or reduced or relative cost coefficients** WRT the basic vector x_B .

During the entire computational process of the simplex method, the column vector of $-z$ remains the $(m + 1)$ th column vector of the unit matrix of order $(m + 1)$. So, in the canonical tableau WRT x_B , the column vectors of the variables in $(x_B, -z)$ constitute the unit matrix of order $(m + 1)$. That's why we always consider $-z$ as the basic variable in the $(m + 1)$ -th row (i.e., the objective row). This way every row in the canonical tableau including the objective row has a basic variable associated with it, and the basic variable in the objective row will always be $-z$. For every basic vector x_B , the vector $(x_B, -z)$ is called the **augmented basic vector** corresponding to it, for the canonical tableau.

Example 4.2.1: Consider the following LP in standard form.

x_1	x_2	x_3	x_4	x_5	$-z$	
1	-1	0	1	1	0	10
2	-1	1	1	-2	0	-5
1	2	3	1	-1	0	8
3	4	-5	-2	4	1	100

$x_j \geq 0$ for all j , $\min z$

To check whether (x_1, x_2, x_3) is a basic vector for this problem, we try to perform GJ pivot steps to convert the matrix consisting of the column vectors of these variables into the unit matrix of order 3. In each tableau given below, PC, PR indicate the pivot column and the pivot row for the GJ pivot step performed on it, and the pivot element is boxed. BV indicates the basic variables selected in the various rows so far.

BV	x_1	x_2	x_3	x_4	x_5	$-z$	
	1	-1	0	1	1	0	10 PR
	2	-1	1	1	-2	0	-5
	1	2	3	1	-1	0	8
$-z$	3	4	-5	-2	4	1	100
		PC					
x_1	1	-1	0	1	1	0	10
	0	1	1	-1	-4	0	-25
	0	3	3	-2	-2	0	-2
$-z$	0	7	-5	-5	1	1	70
		PC					
x_1	1	0	1	0	-3	0	-15
x_2	0	1	1	-1	-4	0	-25
	0	0	0	1	10	0	23
$-z$	0	0	-12	2	29	1	105

The entry in the column of x_3 in the remaining row 3 in the last tableau is 0, this implies that the column vectors of (x_1, x_2, x_3) form a linearly dependent set; so (x_1, x_2, x_3) is not a basic vector for this problem.

For another example, we will now check whether (x_1, x_2, x_4) is a

basic vector for this problem. Since we already performed GJ pivot steps in the columns of x_1, x_2 , we continue from that stage. We copy that tableau below and continue from it.

BV	x_1	x_2	x_3	x_4	x_5	$-z$	
x_1	1	0	1	0	-3	0	-15
x_2	0	1	1	-1	-4	0	-25
	0	0	0	1	10	0	23 PR
$-z$	0	0	-12	2	29	1	105
PC							
Canonical tableau							
x_1	1	0	1	0	-3	0	-15
x_2	0	1	1	0	6	0	-2
x_4	0	0	0	1	10	0	23
$-z$	0	0	-12	0	21	1	55

So, (x_1, x_2, x_4) is indeed a basic vector for the problem, and the canonical tableau for it is the last tableau given above.

Basic Solution, Basic Feasible Solution (BFS), Feasible and Infeasible Basic Vectors

Tableau 2: Canonical Tableau WRT (x_1, \dots, x_m)

BV	x_1	x_i	x_m	x_{m+1}	x_n	$-z$				
x_1	1	...	0	...	0	$\bar{a}_{1,m+1}$...	\bar{a}_{1n}	0	\bar{b}_1
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
x_i	0	...	1	...	0	$\bar{a}_{i,m+1}$...	\bar{a}_{in}	0	\bar{b}_i
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
x_m	0	...	0	...	1	$\bar{a}_{m,m+1}$...	\bar{a}_{mn}	0	\bar{b}_m
$-z$	0	...	0	...	0	\bar{c}_{m+1}	...	\bar{c}_n	1	$-\bar{z}$

Suppose $x_B = (x_1, \dots, x_m)$ is a basic vector for the general LP in standard form given in Tableau 1, associated with the basis B . Entries in the canonical tableau are usually denoted by symbols with a bar on them to distinguish them from those in the original tableau. So, the

canonical tableau WRT x_B is of the form given in Tableau 2.

The **basic solution** of the problem corresponding to this basic vector x_B is obtained by setting all the nonbasic variables to 0, and equating the i th basic variable to the i th updated RHS constant in the canonical tableau. So, it is $\bar{x} = (\bar{b}_1, \dots, \bar{b}_m, 0, \dots, 0)^T$.

This basic solution is called a **basic feasible solution (BFS)** if it also satisfies the nonnegativity restriction, i.e., $\bar{x} \geq 0$. In this case x_B , B are called **(primal) feasible basic vector and basis** respectively for the problem. The canonical tableau WRT a feasible basic vector is referred to as a **feasible canonical tableau**.

If $\bar{x} \not\geq 0$, it is not a feasible solution of the problem, in this case x_B , B are called **(primal) infeasible basic vector, basis** respectively for the problem.

There is also another important classification. Whether they are feasible or not, the basic vector x_B and basis B are said to be

nondegenerate if all basic variables are nonzero in the basic solution

degenerate if at least one basic variable is 0 in the basic solution.

So, the basic vector (x_1, \dots, x_m) in Tableau 2 is nondegenerate if all of \bar{b}_1 , to \bar{b}_m are different from 0, degenerate if at least one of them is 0.

Example 4.2.2: For the LP in Example 1 above, the basic solution corresponding to the basic vector (x_1, x_2, x_4) is $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5)^T = (-15, -2, 0, 23, 0)^T$.

It is primal infeasible because the nonnegative variables x_1, x_2 have negative values in it. However, it is a nondegenerate basic vector for the problem as all the basic variables are nonzero in the basic solution.

Example 4.2.3: Consider the following LP in standard form at the top of the following series of tableaus (so, all x_j are nonnegative variables, and z is to be minimized). Check whether (x_1, x_2, x_3) is a basic vector for this problem, and if so whether it is feasible or infeasible, degenerate or nondegenerate. We try to compute the canonical

tableau WRT (x_1, x_2, x_3) . PC, PR indicate pivot column, pivot row respectively, and the pivot elements are boxed. BV indicates the basic variable selected in the row.

BV	x_1	x_2	x_3	x_4	x_5	x_6	$-z$		
	1	0	-1	1	0	0	0	6	PR
	-1	1	2	0	1	1	0	2	
	0	-1	-2	0	1	-2	0	-8	
$-z$	3	4	-2	-6	8	-7	1	-100	
	PC								
x_1	1	0	-1	1	0	0	0	6	
	0	1	1	1	1	1	0	8	PR
	0	-1	-2	0	1	-2	0	-8	
$-z$	0	4	1	-9	8	-7	1	-118	
	PC								
x_1	1	0	-1	1	0	0	0	6	
x_2	0	1	1	1	1	1	0	8	
	0	0	-1	1	2	-1	0	0	PR
$-z$	0	0	-3	-13	4	-11	1	-174	
	PC								
Canonical Tableau									
x_1	1	0	0	0	-2	1	0	6	
x_2	0	1	0	2	3	0	0	8	
x_3	0	0	1	-1	-2	1	0	0	
$-z$	0	0	0	-16	-2	-8	1	-174	

Since we performed all the required pivot steps and obtained the canonical tableau, we conclude that (x_1, x_2, x_3) is a basic vector for the problem. Its basic solution is $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6)^T = (6, 8, 0, 0, 0, 0)^T$; and from the last row in the canonical tableau we see that the value of $-z$ at this solution is -174 , or $z = 174$. Since all the variables are ≥ 0 in \bar{x} , it is a BFS, and (x_1, x_2, x_3) is a feasible basic vector. It is a degenerate basic vector as the basic variable x_3 is zero in the basic solution.

In a similar way one can verify that (x_4, x_5, x_6) is another feasible basic vector to this problem that is nondegenerate.

Classification of Unit Basic Vectors

A **unit basic vector** is one associated with the unit matrix as its basis. If the original tableau has a unit basic vector, the basic solution corresponding to it can be found directly without doing any pivot steps. Suppose (x_1, \dots, x_m) is a unit basic vector. Then the original tableau has the following form (BV = basic variable selected in row).

Original Tableau										
BV	x_1	\dots	x_i	\dots	x_m	x_{m+1}	\dots	x_n	$-z$	
x_1	1	\dots	0	\dots	0	$a_{1,m+1}$	\dots	a_{1n}	0	b_1
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
x_i	0	\dots	1	\dots	0	$a_{i,m+1}$	\dots	a_{in}	0	b_i
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
x_m	0	\dots	0	\dots	1	$a_{m,m+1}$	\dots	a_{mn}	0	b_m
$-z$	c_1	\dots	c_i	\dots	c_m	c_{m+1}	\dots	c_n	1	$-\bar{z}$

So, the basic solution corresponding to this basic vector is $x = (b_1, \dots, b_m, 0, \dots, 0)^T$ which is feasible iff all the RHS constants in the original tableau are ≥ 0 . Therefore, for an LP in standard form, a unit basic vector in the original tableau is a feasible unit basic vector iff all the RHS constants in constraint rows are ≥ 0 in the original tableau.

Also, in this case since the basis is already a unit matrix, the canonical tableau WRT a unit basic vector can be found by **pricing out the basic columns** in the objective row, i.e., multiplying the i th row by c_i and subtracting it from the objective row to make the entry in the column of the i th basic variable in the objective row into zero for $i = 1$ to m .

Example 4.2.4: Consider the LP in standard form for which the original tableau is given below. The basic vector (x_4, x_1, x_2) is a unit basic vector, and these basic variables are entered on the original tableau under the BV column in their proper order. The corresponding basic solution obtained by setting all the nonbasic variables at 0 is $\bar{x} = (1, 2, 0, -15, 0, 0, 0)^T$, it is infeasible because the nonnegative

Original Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
x_4	0	0	1	1	1	2	1	0	-15
x_1	1	0	1	0	2	1	1	0	1
x_2	0	1	-1	0	3	1	2	0	2
	2	-3	-7	1	-5	6	7	1	-10

$$x_j \geq 0 \text{ for all } j, \min z$$

variable x_4 has a negative value in it. So, this unit basic vector is infeasible to the problem because the first RHS constant is < 0 in the original tableau.

The original cost coefficients of the basic variables in their order are 1, 2, -3. Multiplying rows 1, 2, 3 by these cost coefficients respectively and subtracting from the objective row leads to the canonical tableau WRT this basic vector which is:

Canonical Tableau WRT (x_4, x_1, x_2)

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
x_4	0	0	1	1	1	2	1	0	-15
x_1	1	0	1	0	2	1	1	0	1
x_2	0	1	-1	0	3	1	2	0	2
$-z$	0	0	-13	0	-1	5	10	1	9

$$x_j \geq 0 \text{ for all } j, \min z$$

Optimum Basic Vectors

Consider the following LP in canonical form

Canonical Tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
x_1	1	0	0	1	0	1	0	3
x_2	0	1	0	1	1	0	0	5
x_3	0	0	1	0	1	1	0	7
$-z$	0	0	0	4	5	0	1	-100

$$x_j \geq 0 \text{ for all } j, \min z$$

This is a feasible canonical tableau associated with the BFS $\bar{x} = (3, 5, 7, 0, 0, 0)^T$, with objective value $\bar{z} = 100$.

From the last row of this canonical tableau we have the equation $4x_4 + 5x_5 - z = -100$, or $z = 100 + (4x_4 + 5x_5)$. Since x_4, x_5 are nonnegative variables, $4x_4 + 5x_5 \geq 0$ on the set of feasible solutions of this problem, therefore $z = 100 + (4x_4 + 5x_5) \geq 100$.

Thus 100 is the smallest value that z can have, therefore \bar{x} with an objective value of 100 minimizes z over the set of feasible solutions of this problem, i.e., BFS \bar{x} is an optimum solution of this LP. In this case the basic vector (x_1, x_2, x_3) is known as an **optimum basic vector** and its basis is called an **optimum basis** for this problem.

The generalization of this statement provides the **optimality criterion** used in the simplex algorithm.

Optimality Criterion: Given below is the canonical tableau for an LP, WRT a feasible basic vector x_B . In general the basic variables may be scattered anywhere in the tableau, but for the sake of simplicity we will assume that the column vectors in the tableau are rearranged so that all the basic columns are on the left of the tableau in proper order, so that we can assume $x_B = (x_1, \dots, x_m)$.

The condition for the feasibility of this basic vector is that all the updated RHS constants \bar{b}_i , $i = 1$ to m are nonnegative. Then, the associated BFS is $\bar{x} = (\bar{b}_1, \dots, \bar{b}_m, 0, \dots, 0)^T$ with an objective value of \bar{z} .

If all the nonbasic relative cost coefficients $\bar{c}_j \geq 0$ for all $j = m+1$ to n , then \bar{x} is an optimum solution of the problem, and x_B is an optimum basic vector.

Tableau 2: Canonical Tableau WRT (x_1, \dots, x_m)

BV	x_1	x_i	x_m	x_{m+1}	x_n	$-z$				
x_1	1	...	0	...	0	$\bar{a}_{1,m+1}$...	\bar{a}_{1n}	0	\bar{b}_1
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
x_i	0	...	1	...	0	$\bar{a}_{i,m+1}$...	\bar{a}_{in}	0	\bar{b}_i
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
x_m	0	...	0	...	1	$\bar{a}_{m,m+1}$...	\bar{a}_{mn}	0	\bar{b}_m
$-z$	0	...	0	...	0	\bar{c}_{m+1}	...	\bar{c}_n	1	$-\bar{z}$

The argument for the proof of the optimality criterion is very similar to that in the above numerical example. From the objective row in the canonical tableau we get the equation $z = \bar{z} + (\bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_n x_n)$. If $\bar{c}_{m+1}, \dots, \bar{c}_n$ are all ≥ 0 , since x_{m+1}, \dots, x_n are all required to be ≥ 0 in the problem; $(\bar{c}_{m+1}x_{m+1} + \dots + \bar{c}_n x_n) \geq 0$; i.e., \bar{z} is the smallest value that z can have on the set of feasible solutions of the problem. Since \bar{z} is the objective value at the BFS \bar{x} , \bar{x} is an optimum solution of the problem.

We will now state without proof another result that the simplex algorithm uses to solve LPs.

Result 4.2.1: Existence of an optimum solution that is a BFS: *If an LP in standard form has an optimum solution, then there is a BFS which is an optimum solution for it. In this case there exists an optimum basic vector for this LP, the canonical tableau WRT which satisfies the optimality criterion stated above.*

Using this result, the simplex algorithm searches only among BFSs of the LP for its optimum solution.

Exercises

4.2.1 For the following LP in standard form, check whether the following are basic vectors, and if so whether they are feasible or infeasible. For feasible basic vectors find the associated BFS. Check whether the feasible basic vectors are optimal or not.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	b	
1	0	-1	0	1	1	0	6	
1	2	1	1	1	0	0	0	6
0	2	2	1	2	1	0	0	6
4	10	7	4	12	4	3	1	0
$x_j \geq 0$ for all j , $\min z$								

(i) (not basic vectors) (x_1, x_2, x_3) , (x_1, x_2, x_4) , (x_1, x_3, x_5, x_7) , (x_4, x_5)

(ii) (basic vectors) (x_4, x_3, x_6) (infeasible), (x_7, x_4, x_5) (feasible but not optimal), (x_1, x_4, x_6) (feasible and optimal).

4.3 How Does the (Primal) Simplex Method Work?

There are many versions of the simplex method for solving LPs. The one that we will discuss in this book is called the **primal simplex method using canonical tableaus**. In this book when we refer to the simplex method, we mean this version.

The simplex method follows a path moving from one feasible basic vector of the problem to a better one (better in terms of the objective value) in each move. So, before applying the method we need a feasible basic vector (and consequently its BFS) first. For some problems it is possible to find a feasible basic vector by simple inspection, on those problems the method can be initiated with that basic vector.

On other problems, even finding a feasible basic vector is not easy. To solve these problems the simplex method needs two stages called **Phase I** and **Phase II**. Phase I ignores the objective function, and focuses on finding a feasible basic vector for the problem first. If Phase I is successful in finding a feasible basic vector, beginning with that Phase II tries to find an optimum basic vector.

The portion of the (primal) simplex method beginning with a feasible basic vector is called the **(primal) simplex algorithm**.

So, if a feasible basic vector can be located by simple inspection, then the simplex method begins Phase II directly and solves the problem by the simplex algorithm. Suppose the LP being solved is

$$\begin{aligned} \text{Minimize } z &= cx + \alpha \\ \text{s. to } Ax &= b \\ x &\geq 0 \end{aligned} \tag{4.3.1}$$

where A is an $m \times n$ matrix.

The GJ method of linear algebra can find a basic solution for the constraints $Ax = b$ ignoring the $x \geq 0$ because it cannot handle inequality constraints. In his 1947 paper, G. B. Dantzig has shown that the problem of finding a BFS to this LP (i.e., a basic solution of $Ax = b$ that also satisfies $x \geq 0$) can be posed as another LP for which a feasible basic vector is readily available, by introducing additional variables called **artificial variables** into (4.3.1). This new LP is called the Phase I problem. This Phase I problem is what is solved to find a feasible basic vector for (4.3.1) during Phase I of the simplex method. The way the Phase I problem is set up, there will be a unit feasible basic vector readily available for it, so it can be solved by applying the simplex algorithm on it.

Given a general vector of m variables in (4.3.1), to check whether it is a basic vector, needs checking whether the set of their column vectors in A is linearly independent which itself may need m GJ pivot steps. One set of column vectors which is readily recognized to be linearly independent is the set of column vectors of the unit matrix. That's why the simplex method is usually initiated with a unit feasible basic vector.

Simple Search for a Unit Feasible Basic Vector

In the RHS vector $b = (b_i)$ in (4.3.1), if any of the b_i is < 0 , multiply both sides of the i -th equation by -1 to make the RHS constant in it > 0 . After this operation, the RHS constants vector in the problem will be ≥ 0 . Put it in detached coefficient tableau and look for the i -th unit column vector in the constraint coefficient matrix for $i = 1$ to m . This search can end in two ways:

1: If all are found, they provide the column vectors associated with the initial unit feasible basic vector. Obtain the canonical tableau WRT it, and begin Phase II with it.

2: If at least one of the unit vectors cannot be found in the original tableau, we begin setting up the Phase I problem. To keep the explanation simple, we assume that the 1st to the r -th unit vectors are

found in the original constraint coefficient matrix, and these are the column vectors of the variables x_1, \dots, x_r . So, we do not have a full unit basis in the original tableau. Therefore the original tableau is of the following form.

$$\begin{array}{cccccc|c}
 x_1 & & x_r & x_{r+1} & & x_n & -z & \\
 \hline
 1 & \dots & 0 & a_{1,r+1} & \dots & a_{1n} & 0 & b_1 \\
 \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
 0 & \dots & 1 & a_{r,r+1} & \dots & a_{rn} & 0 & b_r \\
 \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\
 0 & \dots & 0 & a_{m,r+1} & \dots & a_{mn} & 0 & b_m \\
 \hline
 c_1 & \dots & c_r & c_{r+1} & \dots & c_n & 1 & -\alpha \\
 \hline
 & & & x_j \geq 0 \text{ for all } j, & & & \min z &
 \end{array}$$

To get the Phase I problem, introduce a new nonnegative **artificial variable**, call it t_i , associated with the i -th unit vector into the original tableau for each $i = r + 1$ to m , to complete a unit basis together with the columns of x_1, \dots, x_r .

x_1, \dots, x_n are now called **original problem variables** to distinguish them from the newly introduced artificial variables. During Phase I, the original objective function, now called the **Phase II objective function** is kept dormant, i.e., ignored. A new objective row called the **Phase I objective row** is opened as the $(m + 2)$ -th row in the original tableau, corresponding to the Phase I objective function which we denote by w . The various Phase I cost coefficients are defined as below.

$$\text{Phase I cost coefficient of every } \begin{cases} \text{original problem variable } x_j \text{ is } 0 \\ \text{artificial variable } t_i \text{ is } 1 \end{cases}$$

$$\text{Phase II cost coefficient of every artificial variable } t_i \text{ is } 0$$

So, the Phase I objective function $w =$ sum of all the artificial variables introduced, and since the artificial variables are all nonnegative, w is always ≥ 0 . The original tableau for the Phase I problem is therefore of the form given below, where t_{r+1}, \dots, t_m are the artificial variables introduced.

x_1	\dots	x_r	t_{r+1}	\dots	t_m	x_{r+1}	\dots	x_n	$-z$	$-w$	
1	\dots	0	0	\dots	0	$a_{1,r+1}$	\dots	a_{1n}	0	0	b_1
\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
0	\dots	1	0	\dots	0	$a_{r,r+1}$	\dots	a_{rn}	0	0	b_r
0	\dots	0	1	\dots	0	$a_{r+1,r+1}$	\dots	$a_{r+1,n}$	0	0	b_{r+1}
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
0	\dots	0	0	\dots	1	$a_{m,r+1}$	\dots	a_{mn}	0	0	b_m
c_1	\dots	c_r	0	\dots	0	c_{r+1}	\dots	c_n	1	0	$-\alpha$
0	\dots	0	1	\dots	1	0	\dots	0	0	1	0

$x_j, t_i \geq 0$ for all j, i , $\min w$

Since $b = (b_1, \dots, b_m) \geq 0$, the unit basic vector $(x_1, \dots, x_r, t_{r+1}, \dots, t_m)$ is feasible to the Phase I problem, and corresponds to the BFS

$$\begin{aligned} & (x_1, \dots, x_r, t_{r+1}, \dots, t_m, x_{r+1}, \dots, x_n)^T \\ &= (b_1, \dots, b_r, b_{r+1}, \dots, b_m, 0, \dots, 0)^T \end{aligned}$$

with a Phase I objective value of $w_0 = b_{r+1} + \dots + b_m$. This solution is not feasible to the original problem, because some of the artificial variables, which are not part of the original problem, may have positive values in it.

Any solution to the Phase I problem in which $w = 0$ must have all artificial variables = 0, and the x -part in it must therefore be feasible to the original problem. So, to find a feasible solution of the original problem, we need to look for a solution of the Phase I problem in which $w = 0$; this can be done by minimizing w in the Phase I problem. If the minimum value of w in the Phase I problem is > 0 , then it is impossible to find a feasible solution for it which makes $w = 0$; this implies that the original problem has no feasible solution. And since an initial unit feasible basic vector $(x_1, \dots, x_r, t_{r+1}, \dots, t_m)$ is available, the Phase I problem can be solved by applying the simplex algorithm on it.

Since we become feasible to the original problem when the Phase I objective function w becomes 0, its value at any stage during Phase I provides a measure of how far away we are at that stage from feasibility to the original problem. That's why w is called an **infeasibility measure** for the current solution.

4.4 How Does the Simplex Algorithm Move From One Feasible Basic Vector to a Better one?

The general iterative step in the simplex algorithm is known as the **pivot step** because it involves one GJ pivot step.

In a pivot step, the basic vector changes by exactly one variable, i.e., exactly one variable in the basic vector is replaced by a nonbasic variable leading to the next basic vector. The nonbasic variable being made basic is called the **entering variable** in this pivot step, and the basic variable being dropped from the basic vector is called the **dropping** or **leaving basic variable**; these are selected by special rules specified by the algorithm.

The **entering variable choice rule** makes sure that the objective function value improves (i.e., decreases, since we are discussing a minimization problem) in this pivot step.

The **dropping variable choice rule** makes sure that the next basic vector obtained will remain feasible.

We will give the statements of these rules and explain the rationale behind them using numerical examples.

Example 4.4.1: Consider the following feasible canonical tableau for an LP. BV is an abbreviation for “Basic Variable”.

Canonical tableau WRT (x_1, x_2, x_3, x_4)										
BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$-z$	\bar{b}
x_1	1	0	0	0	1	-1	1	0	0	6
x_2	0	1	0	0	2	1	2	1	0	12
x_3	0	0	1	0	0	-1	3	-2	0	3
x_4	0	0	0	1	-2	1	7	-3	0	1
$-z$	0	0	0	0	-3	2	0	-2	1	-100

$x_j \geq 0$ for all j , min z

The present BFS is $\bar{x} = (6, 12, 3, 1, 0, 0, 0, 0)^T$ with an objective value of $\bar{z} = 100$. Here x_5, x_6, x_7, x_8 are the nonbasic variables whose

values are 0 in the present BFS \bar{x} . The relative cost coefficients of the nonbasic variables are: $\bar{c}_5 = -3$, $\bar{c}_6 = 2$, $\bar{c}_7 = 0$, $\bar{c}_8 = -2$.

The equation corresponding to the last row in this canonical tableau is $-3x_5 + 2x_6 - 2x_8 - z = -100$, or $z = 100 - 3x_5 + 2x_6 - 2x_8$.

Remembering that 100 is the objective value at the present BFS, and that all the variables can only take nonnegative values in the problem, we see that the objective value can be decreased from that of the present BFS only if one of the nonbasic variables x_5 or x_8 with a strictly negative relative cost coefficient, has its value increased from its present value of 0. That's why nonbasic variables with negative relative cost coefficients are called **eligible variables** in this canonical tableau (meaning that they are eligible to be entering variables into the basic vector to improve the objective value). As a result we have the following rules used in the simplex algorithm.

Optimality Criterion: In a feasible canonical tableau, if all nonbasic relative cost coefficients $\bar{c}_j \geq 0$ (i.e., no eligible variables), the present BFS, basic vector, are optimal; and the present objective value is the minimum that z can have in the problem.

Entering Variable Choice Rule: If the optimality criterion is violated in the present feasible canonical tableau, select the entering variable to be a nonbasic variable x_j with a negative relative cost coefficient (i.e., one with $\bar{c}_j < 0$, or an eligible variable).

The updated column vector of the entering variable in the *canonical tableau* is called the **entering column** or **pivot column** for this pivot step.

Even when there are several eligible variables, the fundamental principle followed by the simplex algorithm is to select only one of them as the entering variable for increasing its value from 0. The question is, which eligible variable to select as the entering variable. In his original paper Dantzig suggested taking the entering variable to be the nonbasic variable with the most negative relative cost coefficient \bar{c}_j ; this rule is therefore called **Dantzig's entering variable choice rule**.

Dantzig's rule works fine in computer programs, but for solving

a small problem by hand computation, it may result in complicated fractions making hand computation difficult. That's why when solving an LP by hand, I normally look for an entering variable that leads to a pivot step with a pivot element of 1 (this will become clear when we discuss the dropping basic variable choice rule) if possible, or the one that keeps the fractions as simple as possible.

Suppose x_5 has been selected as the entering variable. Since x_5 is the only nonbasic variable selected for a value increase, the remaining nonbasic variables x_6, x_7, x_8 will remain at their present values of 0 in the next solution. We will denote the new value given to x_5 by λ . The next solution will depend on λ , so we denote it by $x(\lambda), z(\lambda)$.

So, the values of x_5, x_6, x_7, x_8 will be $\lambda, 0, 0, 0$ respectively in the next solution. From the canonical tableau we find that the values of the basic variables $x_1, x_2, x_3, x_4, -z$ will equal the (updated RHS constants vector) $-\lambda(\text{updated column vector of } x_5 \text{ in the tableau})$. Therefore we have

$$\begin{aligned} x(\lambda) &= (6 - \lambda, 12 - 2\lambda, 3, 1 + 2\lambda, \lambda, 0, 0, 0)^T \\ -z(\lambda) &= -100 + 3\lambda, \quad \text{so, } z(\lambda) = 100 - 3\lambda \end{aligned}$$

The following facts should be noticed.

1. The relative cost coefficient of the entering variable, -3 , is the rate of objective value change per unit increase in the value of the entering variable from 0. Since it is < 0 , for obtaining the best objective value in this step, λ should be given the highest possible value.
2. As λ increases, the values of basic variables in rows with positive entries in the entering column decreases. We need to make sure that their values do not go below 0 for maintaining feasibility.

That's why every basic variable with a positive entry in the entering column is known as a **blocking variable**, blocking the increase of λ to the **ratio**

$$\frac{(\text{updated RHS Constant in its row})}{(\text{entry in the entering column in its row})}$$

3. So, to keep $x(\lambda)$ feasible, the maximum value that λ can have is the minimum of the ratios computed above, called the **minimum ratio** or **min ratio** in this pivot step; normally denoted by θ .

Here is the general statement. Let x_s denote the entering variable selected. Let \bar{b}_i, \bar{a}_{is} denote the updated RHS constant, and entry in the entering column in the constraint row i in canonical tableau. Let \bar{c}_s be the relative cost coefficient of the entering variable. Then in each constraint row i with $\bar{a}_{is} > 0$, compute

$$\text{Ratio in row } i \text{ with } \bar{a}_{is} > 0 \quad \text{is} \quad \bar{b}_i / \bar{a}_{is}$$

$$\text{Minimum ratio} = \theta = \text{Min}\{\bar{b}_i / \bar{a}_{is} : i \text{ such that } \bar{a}_{is} > 0\}.$$

Then the maximum value of λ that will keep the next solution $x(\lambda)$ feasible is θ . So, we will fix $\lambda = \theta$, and $x(\theta)$ will be the next solution with improved objective value $z(\theta)$; and the change in the objective value in this step is $\theta\bar{c}_s$.

In our example we selected x_5 as the entering variable, the entering column, its column in the canonical tableau has positive entries in only rows 1, 2. We have $\bar{c}_5 = -3$, $\bar{b}_1 = 6$, $\bar{b}_2 = 12$, $\bar{a}_{15} = 1$, $\bar{a}_{25} = 2$. So, we have

$$\text{Ratios in rows 1, 2 : } 6/1 = 6, 12/2 = 6 \text{ respectively}$$

$$\text{Minimum ratio} = \theta = \min\{6, 6\} = 6$$

and the next solution will be $x(6) = (0, 0, 3, 7, 6, 0, 0, 0)^T$ with an objective value of $z(6) = 100 - 18 = 82$. So, the change in the objective value in this step will be $-18 = (-3)6 = \theta\bar{c}_5$.

Here both rows 1, 2 tie for the minimum ratio, and the values of the basic variables, x_1, x_2 in both these rows have dropped to 0 in the next solution. Therefore we can make either of these basic variables into a nonbasic variable to be replaced by the entering variable in the basic vector. Here are the general statements.

Dropping Variable (or Pivot Row) Choice Rule: Select any of the rows in which the minimum ratio is attained, as the pivot row

for this pivot step. The present basic variable in the pivot row is the **dropping** or **leaving basic variable** in this pivot step. The element in the pivot row and the pivot column is the **pivot element** for this pivot step. Here it will be > 0 by the way the pivot row is selected.

Pivot Step: Perform the GJ pivot step with the selected pivot row, and pivot column on the present canonical tableau. In the resulting tableau replace the basic variable in the pivot row by the entering variable x_s . This gives the canonical tableau WRT the new basic vector. The new solution obtained earlier will be the BFS corresponding to this new basic vector. The change in the objective value in this step is $\theta \bar{c}_s$.

This pivot step is called a

nondegenerate pivot step if the minimum ration $\theta > 0$

degenerate pivot step if $\theta = 0$.

In our example the current pivot step will be a nondegenerate pivot step since the minimum ratio $\theta = 6 > 0$.

We will indicate below the whole process of this pivot step in our example. The top tableau is the present canonical tableau with the computed ratios, PC (pivot column), PR (pivot row), boxed pivot element shown; and the bottom tableau is the canonical tableau obtained after the GJ pivot step.

Canonical tableau WRT (x_1, x_2, x_3, x_4)											
BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$-z$	b	Ratio
x_1	1	0	0	0	1	-1	1	0	0	6	6/1 = 6, PR
x_2	0	1	0	0	2	1	2	1	0	12	12/2 = 6
x_3	0	0	1	0	0	-1	3	-2	0	3	
x_4	0	0	0	1	-2	1	7	-3	0	1	
$-z$	0	0	0	0	-3	2	0	-2	1	-100	Min, $\theta = 6$
					PC↑						
x_5	1	0	0	0	1	-1	1	0	0	6	
x_2	-2	1	0	0	0	3	0	1	0	0	
x_3	0	0	1	0	0	-1	3	-2	0	3	
x_4	2	0	0	1	0	-1	9	-3	0	13	
$-z$	3	0	0	0	0	-1	3	-2	1	-82	

Example 4.4.2: In this example we will illustrate the only other possibility that can occur in a pivot step of the simplex algorithm. Consider the following feasible canonical tableau for an LP.

Canonical tableau WRT (x_1, x_2, x_3)

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	b	Ratio
x_1	1	0	0	-1	1	-1	1	0	3	
x_2	0	1	0	1	-2	0	1	0	5	
x_3	0	0	1	2	3	-2	0	0	2	
$-z$	0	0	0	3	-5	-1	-2	1	-10	
						PC↑				

The present BFS is $\bar{x} = (3, 5, 2, 0, 0, 0, 0)^T$ with objective value $\bar{z} = 10$.

Since the relative cost coefficients $\bar{c}_5 = -5$, $\bar{c}_6 = -1$, $\bar{c}_7 = -2$, are negative, the optimality criterion is violated, and we select x_6 with relative cost coefficient $\bar{c}_6 = -1$, as the entering variable and its column is the PC (pivot column).

We try to compute the ratios, but there are no positive entries in the PC and hence no ratios to compute. Giving the entering variable x_6 the value λ , keeping all the other nonbasic variables x_4, x_5, x_7 at their present values of 0, leads to the new solution

$$x(\lambda) = (3 + \lambda, 5, 2 + 2\lambda, 0, 0, \lambda, 0)^T, \quad z(\lambda) = 10 - \lambda$$

We see that as λ increases from 0, none of the variables decreases in $x(\lambda)$. So $x(\lambda)$ remains ≥ 0 and hence feasible for all $\lambda \geq 0$. Also, as λ goes from 0 to ∞ , $z(\lambda)$ diverges to $-\infty$.

So, in this example the minimum value of z is $-\infty$, i.e., there is no finite minimum. We say that z is **unbounded below** in this LP and there is no finite optimum solution, and the method terminates with this unboundedness conclusion. And $\{x(\lambda) : \lambda \geq 0\}$ is the parametric representation of a half-line lying completely in the feasible region of this LP along which the objective value diverges to $-\infty$. This half-line is an **extreme half-line** or an **unbounded edge of the set of**

feasible solutions of this LP along which the objective value diverges to $-\infty$.

Here is the general statement:

Checking for Unboundedness: If the entering column, updated column of the entering variable, contains no positive entries, terminate the algorithm with the conclusion that the objective function is unbounded below on the set of feasible solutions of this LP. In this case if $x(\lambda)$ is the new solution obtained by giving the entering variable the value λ , and keeping all other nonbasic variables fixed at 0, then $\{x(\lambda) : \lambda \geq 0\}$ is the parametric representation of a feasible extreme half-line along which the objective value diverges to $-\infty$.

4.5 The (Primal) Simplex Method

Here we provide the complete statement of the simplex method for solving a general LP in standard form, in terms of the tools, rules, and operations defined in Sections 4.3, 4.4.

This version of the **simplex method** always begins with a unit feasible basic vector. The reason for calling it the **primal simplex method** is that in the literature there is another method called the *dual simplex method* that is discussed in graduate level LP books. We do not discuss the dual simplex method in this book. So, in the sequel, whenever we talk about simplex method (or algorithm), it actually refers to this primal version. The meanings of the words **primal**, **dual** are given in Chapter 5.

From Section 4.2 we know that if a unit basic vector can be found in the original tableau, it will be feasible iff all the original RHS constants in constraint rows are ≥ 0 . That's why before looking for a unit basic vector in the original tableau, in the initialization step (Step 1) all the RHS constants in the constraint rows are made ≥ 0 first.

m denotes the number of constraint rows. So, with the objective row (row $(m + 1)$), the original tableau has $(m + 1)$ rows. n denotes the number of decision variables.

The Simplex Method

BEGIN

Step 1: Initialization:

Step 1.1 Making the RHS constants nonnegative in constraint rows: Multiply each constraint row in the original tableau in which the RHS constant b_i is negative, by -1 . When this operation is completed, the RHS constants in all the constraint rows in the original tableau become ≥ 0 .

Step 1.2 Looking for a unit feasible basic vector: For each $i = 1$ to m , look for a variable associated with the i th unit vector among the constraint rows in the original tableau; if such a variable is found, record it (the one with the smallest cost coefficient if there are several such variables) as the basic variable in the i th row.

In this process if basic variables have been selected in all the constraint rows, let x_B be the resulting basic vector. Since the RHS constants in all the constraint rows are ≥ 0 , x_B is a unit feasible basic vector. Obtain the canonical tableau WRT x_B by pricing out the basic columns in the objective row, and selecting $-z$ as the basic variable in the objective row. With this feasible canonical tableau, go to Step 2 (Phase II).

If basic variables have not been selected in some of the constraint rows in the above process, go to Step 3 with the present tableau for setting up the Phase I problem to find a feasible basic vector for the original problem.

Step 2: Phase II beginning with a feasible canonical tableau:

Step 2.1: Canonical tableau: In a canonical tableau basic columns may appear in any order (from left to right), and be scattered anywhere in the tableau. But for ease of understanding, we will explain the computation assuming that the

present basic vector is $x_B = (x_1, \dots, x_m)$. Let the present canonical tableau be:

Canonical Tableau WRT (x_1, \dots, x_m)											
BV	x_1	...	x_m	x_{m+1}	...	x_s	...	x_n	$-z$	\bar{b}_i	*
x_1	1	...	0	$\bar{a}_{1,m+1}$...	\bar{a}_{1s}	...	\bar{a}_{1n}	0	\bar{b}_1	
\vdots	\vdots	\ddots	\vdots	\vdots		\vdots		\vdots	\vdots	\vdots	
x_m	0	...	1	$\bar{a}_{m,m+1}$...	\bar{a}_{ms}	...	\bar{a}_{mn}	0	\bar{b}_m	
$-z$	0	...	0	\bar{c}_{m+1}	...	\bar{c}_s	...	\bar{c}_n	1	$-\bar{z}$	

*Ratios (\bar{b}_i/\bar{a}_{is}) computed in row i only if $\bar{a}_{is} > 0$

The BFS corresponding to x_B is $\bar{x} = (\bar{b}_1, \dots, \bar{b}_m, 0, \dots, 0)^T$, $z = \bar{z}$. $\bar{c}_{m+1}, \dots, \bar{c}_n$ are the nonbasic relative cost coefficients in this canonical tableau.

Step 2.2: Checking optimality: If all the nonbasic relative cost coefficients in the present canonical tableau are ≥ 0 , the present BFS is an optimum solution of the problem, and its objective value \bar{z} is the minimum objective value; terminate.

Step 2.3: Selecting an entering variable: If the optimality criterion is violated, select an eligible variable (a nonbasic variable with negative relative cost coefficient) as the entering variable. Suppose it is x_s , its column vector in the canonical tableau is the PC (pivot column).

Step 2.4: Checking unboundedness: If the PC has no strictly positive entries, the objective function z is unbounded below on the set of feasible solutions of the problem, so there is no finite optimum solution. Let

$$\begin{aligned}
 x(\lambda) = & (x_1(\lambda), \dots, x_m(\lambda), x_{m+1}(\lambda), \dots, x_{s-1}(\lambda), x_s(\lambda), x_{s+1}(\lambda), \\
 & \dots, x_n(\lambda))^T = (\bar{b}_1 - \lambda\bar{a}_{1s}, \dots, \bar{b}_m - \lambda\bar{a}_{ms}, 0, \dots, 0, \lambda, \\
 & 0, \dots, 0)^T, \quad z(\lambda) = \bar{z} + \bar{c}_s\lambda.
 \end{aligned}$$

be the solution obtained by: giving the entering variable x_s a value of λ , keeping all other nonbasic variables fixed at 0, and evaluating the corresponding values of basic variables.

Then $\{x(\lambda) : \lambda \geq 0\}$ is an extreme half-line in the feasible region along which $z \rightarrow -\infty$ as $\lambda \rightarrow +\infty$.

2.4: Min ratio computation, finding pivot row: If the PC, $(\bar{a}_{1s}, \dots, \bar{a}_{ms}, \bar{c}_s)^T$, has at least one positive entry, then for each row i such that $\bar{a}_{is} > 0$, compute the ratio (\bar{b}_i/\bar{a}_{is}) and enter it in this row in the ratios column. Let $\theta =$ the minimum of all the computed ratios. Select any row that ties for the minimum ratio as the PR (pivot row). The present basic variable in the PR is the dropping basic variable in this pivot step.

Step 2.5: Pivot step to get the new canonical tableau:

Perform the GJ pivot step on the canonical tableau with the PR, PC selected. In the resulting tableau, replace the basic variable in the PR by the entering variable, leading to the canonical tableau WRT the new basic vector. Go to Step 2.1 with this new canonical tableau and begin the next iteration.

Step 3: Setting up the Phase I problem: The search in Step 1.2 may have found a few but not all unit vectors to form a full unit basis of order m . Introduce nonnegative artificial variables associated with the missing unit vectors into the tableau, and make a unit basic vector with them. Make the Phase II cost coefficient 0 in each artificial column. Introduce the equation: (sum of artificial variables introduced) - $w = 0$, as the Phase I objective row. For an example, suppose Step 1.2 found only i -th unit vectors for $i = 1$ to r in the original tableau, and suppose these are the columns of variables x_1, \dots, x_r . Denoting the artificial variables introduced by t_{r+1}, \dots, t_m , the Phase I original tableau is given below.

Select $x_B = (x_1, \dots, x_r, t_{r+1}, \dots, t_m)$ as the initial basic vector and obtain the canonical tableau by pricing out the basic variables in both the objective rows. Go to Phase I (Step 4) with it.

x_1		x_r	t_{r+1}		t_m	x_{r+1}		x_n	$-z$	$-w$	
1	...	0	0	...	0	$a_{1,r+1}$...	a_{1n}	0	0	b_1
\vdots	\ddots	\vdots	\vdots		\vdots	\vdots		\vdots	\vdots	\vdots	\vdots
0	...	1	0	...	0	$a_{r,r+1}$...	a_{rn}	0	0	b_r
0	...	0	1	...	0	$a_{r+1,r+1}$...	$a_{r+1,n}$	0	0	b_{r+1}
\vdots		\vdots	\vdots	\ddots	\vdots	\vdots		\vdots	\vdots	\vdots	\vdots
0	...	0	0	...	1	$a_{m,r+1}$...	a_{mn}	0	0	b_m
c_1	...	c_r	0	...	0	c_{r+1}	...	c_n	1	0	$-\alpha$
0	...	0	1	...	1	0	...	0	0	1	0

$x_j, t_i \geq 0$ for all j, i , $\min w$

Step 4: Phase I: The artificial variables are introduced solely for providing a full basic vector to apply the simplex algorithm, to move towards a feasible basic vector for the original problem. So, at some stage during Phase I, if an artificial variable is replaced from the basic vector by an original problem variable, we delete it by erasing its column from both the current and the Phase I original tableau. So, an artificial variable exists in the problem only as long as it is a basic variable. Thus every nonbasic variable will always be an original problem variable, and the entering variables in every iteration, in fact every variable eligible to enter the basic vector in every iteration, will always be an original problem variable.

Suppose the present canonical tableau at some stage of Phase I is:

BV	Basic cols.			Nonbasic cols.			$-z$	$-w$	RHS
	x_j	...			
	1	...	0	...	\bar{a}_{1j}	...	0	0	\bar{b}_1
\vdots	\vdots		\vdots		\vdots		\vdots	\vdots	\vdots
	0	...	1	...	\bar{a}_{mj}	...	0	0	\bar{b}_m
$-z$	0	...	0	...	\bar{c}_j	...	1	0	$-\bar{z}$
$-w$	0	...	0	...	\bar{d}_j	...	0	1	$-\bar{w}$

The Phase II objective ($(m+1)$ th) row plays no role during Phase

I. The \bar{d}_j are the Phase I relative cost coefficients, these are used for determining Phase I termination, and entering variable selection during Phase I. Phase I is just the application of the simplex algorithm to the problem of minimizing w subject to the constraints in the Phase I original tableau. Here are the steps in Phase I iterations.

Step 4.1: Checking Phase I termination: If all the nonbasic Phase I relative cost coefficients \bar{d}_j are ≥ 0 in the present canonical tableau; \bar{w} , the present value of w , is its minimum; terminate Phase I.

If this termination condition is satisfied, and $\bar{w} > 0$, terminate the method with the conclusion that the original problem is infeasible, i.e., has no feasible solution.

If this termination condition is satisfied, and $\bar{w} = 0$, then any artificial variables remaining in the tableau must have values 0 in the present BFS. Hence the x -part of the present BFS is a feasible solution of the original problem. In this case go to Step 4.5 to move over to Phase II.

Step 4.2: Selecting an entering variable: If Phase I termination condition is not satisfied, select an eligible nonbasic variable (one with negative Phase I relative cost coefficient \bar{d}_j) as the entering variable. Suppose it is x_s . Its column vector in the canonical tableau is the PC (pivot column).

Step 4.3: Determining min ratio, PR, and Dropping variable: The PC is $(\bar{a}_{1s}, \dots, \bar{a}_{ms})^T$. For each i such that $\bar{a}_{is} > 0$ compute the ratio (\bar{b}_i/\bar{a}_{is}) and enter it in the ratios column. Let $\theta = \text{Minimum}\{(\bar{b}_i/\bar{a}_{is}) : i \text{ such that } \bar{a}_{is} > 0\}$. Select any one of the rows which ties for the minimum ratio as the PR (pivot row). The present basic variable in the PR will be the dropping basic variable in the coming pivot step.

Step 4.4: Pivot step, new canonical tableau: Perform the GJ pivot step on the canonical tableau with the PR, PC selected. In the resulting tableau, replace the basic variable in the PR by the entering variable, leading to the canonical

tableau WRT the new basic vector. If the dropping variable is an artificial variable, delete its column vector from both the current tableau and the Phase I original tableau. Go back to Step 4.1 with this new canonical tableau and begin the next Phase I iteration.

Step 4.5: Moving over to Phase II: If there are no artificial variables in the present basic vector, then it consists of original problem variables only, and is primal feasible to the original problem. Delete the last (i.e., $(m + 2)$ th) row and the column of $-w$ from the present tableau and the original tableau. The present tableau is now the canonical tableau for the original problem WRT the present feasible basic vector. With it move over to Step 2.1 to begin Phase II.

Suppose there are some artificial variables in the basic vector at this stage. Since, $\bar{w} =$ the value of w is 0, the values of all the artificial variables in the present BFS must be 0, and as long as this property holds, the x -part of the BFS will be feasible to the original problem.

The equation corresponding to the $(m + 2)$ th row in the present canonical tableau is:

$$-w + \sum(\bar{d}_j x_j : \text{over } j \text{ such that } x_j \text{ is nonbasic}) = \bar{w} = 0$$

or: $w = \sum(\bar{d}_j x_j : \text{over } j \text{ such that } x_j \text{ is nonbasic}).$

All the \bar{d}_j are ≥ 0 in the present canonical tableau. Let $F = \{j : \bar{d}_j > 0\}$. From the above equation we see that if any nonbasic variable x_j for $j \in F$ assumes a positive value, then the infeasibility measure w will become positive, i.e., the solution becomes infeasible to the original problem again. This implies that all x_j for $j \in F$ must be 0 in every feasible solution of the original problem.

So, fix all x_j for $j \in F$ at 0, and delete their columns from the original tableau and present canonical tableau. After this deletion, for all remaining variables $\bar{d}_j = 0$, so the equation

corresponding to the $(m + 2)$ th row in the tableau is $w = 0$, i.e., w remains 0 during Phase II iterations even though some artificial variables are still in the basic vector. During Phase II the values of these artificials will remain 0 (if one of them becomes > 0 during Phase II, it is an indication that you made computational errors).

So, now delete the $(m+2)$ th row and the column of $-w$ from both the Phase I original tableau and the present canonical tableau, and move over to Step 2.1 to begin Phase II.

END

We will now discuss a couple of important points before presenting numerical examples.

Discussion

1. How are redundant equality constraints discovered in the simplex method: The original problem is

$$\begin{aligned} \text{Minimize } z &= cx + \alpha \\ \text{s. to } Ax &= b \\ x &\geq 0 \end{aligned}$$

where A is an $m \times n$ matrix. By definition, the i -th constraint in the problem, $A_i x = b_i$, can be deleted as a redundant equality constraint if it is a linear combination of the remaining equality constraints.

If Step 1.2 locates a unit basic vector for the problem, or if the feasible basic vector obtained at the end of Phase I (Step 4.4) has no artificial variables in it, then the rank of A is m , and there are no redundant equality constraints in the problem.

When Phase I terminates with the value of $w = 0$, but with some artificial variables in the final basic vector (Step 4.4); any row in the final tableau with only a single nonzero entry of 1 in the column of

an artificial variable, corresponds to a redundant equality constraint in the original problem that can be deleted. Any such rows, and the column of the artificial variable with the entry of 1 in it can be deleted before moving over to Phase II.

2. How to check whether the optimum solution of the problem is unique: If Phase II terminates by satisfying the optimality criterion, the BFS in the final tableau is an optimum solution of the problem. If the relative cost coefficients of all the nonbasic variables are all strictly positive, then this optimum solution is the unique optimum solution for the problem.

If some of the nonbasic relative cost coefficients are 0 in the final tableau, the problem may have alternate optima. Carrying out a pivot step in an optimum canonical tableau, with the column of a nonbasic variable with 0 relative cost coefficient as the entering column, will lead to an alternate optimum BFS if that pivot step is a nondegenerate pivot step.

3. How to find a feasible solution to a system of linear constraints that includes some inequalities: The GJ elimination method of linear algebra can be used to solve a system of linear equations.

Suppose we have to solve a system of linear constraints that consists of not just linear equations, but maybe some linear inequalities, sign restrictions or lower and/or upper bound constraints on variables. How does one solve such a general system? Notice that we are not required to optimize an objective function here, but just to find a feasible solution to the system if one exists. Using the techniques discussed in Section 4.1, this general system can be transformed into a system of linear equations in nonnegative variables. And the problem of finding a nonnegative solution to a system of linear equations can be solved by applying Phase I of the simplex method discussed above, to a Phase I formulation of the problem. In applying this Phase I method, we will not have a Phase II objective row, or the Phase II objective function $-z$, that is the only difference.

4. Entering and dropping variable choice rules used in computer programs for the simplex method: In solving large linear programs by the simplex method using floating point arithmetic on a digital computer, round-off error accumulation poses a serious problem in getting accurate results. Software engineers have found that finding better sized pivot elements in pivot steps to be crucial to the numerical stability of the simplex method. When there are several eligible variables, software engineers have developed efficient rules for choosing the entering variable among them to make sure that the pivot element for the ensuing pivot step will be the largest in magnitude possible. Also, the pivot row is selected among the tied rows in the ratio test to choose the largest pivot element among the possible.

4.6 Numerical Examples of the Simplex Method

In the various pivot steps in all these numerical examples, we select the entering variable among those eligible, to have a pivot element of 1, or one that will keep the fractions simple as far as possible. The following abbreviations are used: BV = basic variable, PC = pivot column (or entering column), PR = pivot row (or row of the dropping basic variable). In each tableau, θ denotes the minimum ratio computed.

Example 4.6.1: An LP with a unique optimum solution, on which Phase II can be initiated

Consider the following LP in standard form

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
-1	0	-1	-2	0	-1	-1	0	-3
-1	0	0	-3	-1	0	0	0	-3
-1	1	0	1	0	3	0	0	1
-2	4	3	10	6	9	-2	1	10

$$x_j \geq 0 \text{ for all } j, \min z$$

Original Tableau

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
1	0	1	2	0	1	1	0	3
1	0	0	3	1	0	0	0	3
-1	1	0	1	0	3	0	0	1
-2	4	3	10	6	9	-2	1	10

$x_j \geq 0$ for all j , $\min z$

In the problem given, we need to make the top 2 RHS constants nonnegative, so we multiply both rows 1 and 2 in it by -1 , leading to the original tableau for the problem, given above.

1st canonical tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	RHS	Ratio
x_7	1	0	1	2	0	1	1	0	3	3 PR
x_5	1	0	0	3	1	0	0	0	3	3
x_2	-1	1	0	1	0	3	0	0	1	
$-z$	-2	0	5	-8	0	-1	0	1	-6	min $\theta = 3$
	PC \uparrow									
x_1	1	0	1	2	0	1	1	0	3	3/2
x_5	0	0	-1	1	1	-1	-1	0	0	0 PR
x_2	0	1	1	3	0	4	1	0	4	4/3
$-z$	0	0	7	-4	0	1	2	1	16	min $\theta = 0$
	PC \uparrow									
x_1	1	0	3	0	-2	3	3	0	3	1
x_4	0	0	-1	1	1	-1	-1	0	0	
x_2	0	1	4	0	-3	7	4	0	4	1 PR
$-z$	0	0	3	0	4	-3	-2	1	16	min $\theta = 1$
	PC \uparrow									
Final canonical tableau										
x_1	1	-3/4	0	0	1/4	-9/4	0	0	0	
x_4	0	1/4	0	1	1/4	3/4	0	0	1	
x_7	0	1/4	1	0	-3/4	7/4	1	0	1	
$-z$	0	1/2	5	0	5/2	1/2	0	1	18	

Searching the original tableau we find that both x_3 and x_7 are associated with the first unit vector; among them we select x_7 corresponding to the smaller cost coefficient of -2 as the basic variable in the 1st row. Similarly selecting x_5 , x_2 as basic variables in rows 2, 3 respectively,

we get the initial unit feasible basic vector (x_7, x_5, x_2) . So, there is no need to apply Phase I, we can initiate Phase II on this problem with the canonical tableau WRT this basic vector. To price out these basic variables, we subtract from the objective row: $(-2)\text{Row 1}$, then $6(\text{Row 2})$, and finally $4(\text{Row 3})$. This yields the 1st canonical tableau at the top of the list of tableaus given above. The list shows the various canonical tableaus obtained during Phase II. The pivot elements in the various steps are boxed.

As all the relative cost coefficients in the final tableau are ≥ 0 , the optimality criterion is satisfied. So, the BFS in the final tableau, $\bar{x} = (0, 0, 0, 1, 0, 0, 1)^T$ with objective value of $\bar{z} = -18$ is an optimum solution for the problem. Since all the nonbasic relative cost coefficients in the final tableau are > 0 , \bar{x} is the unique optimum solution to the problem.

Example 4.6.2: An unbounded LP, on which Phase II can be initiated

Consider the following LP

$$\begin{aligned} \text{Maximize } z' &= x_1 + 2x_2 - x_3 \\ \text{s. to } x_1 - 2x_2 - x_3 &\leq 2 \\ -x_1 + 3x_2 - 2x_3 &\geq -4 \\ -x_1 + x_2 + 3x_3 &\leq 1 \\ x_j &\geq 0 \quad \text{for all } j \end{aligned}$$

To put in standard form we introduce the slack variables x_4, x_5, x_6 in the three constraints in that order. In the resulting equations, we multiply both sides of each one with a negative RHS constant by -1 . Then we transform the objective function to minimize $z = -z'$. This results in the original tableau, which is the top one in the following list.

The slack vector (x_4, x_5, x_6) is a unit feasible basic vector for which the original tableau is in fact the canonical tableau. x_1, x_2 are both eligible to enter the basic vector, among them we select x_1 as the en-

tering variable. Continuing in a similar way the following tableaus are obtained.

BV	x_1	x_2	x_3	x_4	x_5	x_6	$-z$	RHS	Ratio
x_4	<u>1</u>	-2	-1	1	0	0	0	2	2
x_5	1	-3	2	0	1	0	0	4	4
x_6	-1	1	3	0	0	1	0	1	
$-z$	-1	-2	1	0	0	0	1	0	min $\theta = 2$
PC \uparrow									
Final canonical tableau									
x_1	1	-2	-1	1	0	0	0	2	
x_5	0	-1	3	-1	1	0	0	2	
x_6	0	-1	2	1	0	1	0	3	
$-z$	0	-4	0	1	0	0	1	2	
		\uparrow							

In the final tableau, x_2 is eligible, so we select it as the entering variable. But its updated column vector has no positive entry, so the unboundedness criterion is satisfied. We terminate with the conclusion that z diverges to $-\infty$ on the set of feasible solutions of the problem.

Making $x_2 = \lambda$, and leaving other nonbasic variables at 0 leads to the solution $x(\lambda) = (x_1(\lambda) \text{ to } x_6(\lambda))^T = (2 + 2\lambda, \lambda, 0, 0, 2 + \lambda, 3 + \lambda)^T$, with objective value $z(\lambda) = -2 - 4\lambda$. $x(\lambda)$ remains feasible for all $\lambda \geq 0$, and as $\lambda \rightarrow +\infty$, $z(\lambda) \rightarrow -\infty$. So, $\{x(\lambda) : \lambda \geq 0\}$ is an extreme half-line in the feasible region along which the objective function z diverges to ∞ .

In fact, given any negative number with very large absolute value, we can find a feasible solution on this half-line which corresponds to that objective value. As an example, to make $z = -4,000,002$ we need to select λ to satisfy $-2 - 4\lambda = -4,000,002$, which yields $\lambda = 1,000,000$. So, $x(1,000,000) = (2 + 2,000,000, 1,000,000, 0, 0, 2 + 1,000,000, 3 + 1,000,000)^T$ is a feasible solution on this half-line with the objective value for $z = -4,000,002$.

To state the result in terms of the variables and objective function in the original statement of the problem, all we have to do is to drop the slack variables x_4, x_5, x_6 from the solution, and use $z' = -z$. This shows that on the half-line $\{(x_1(\lambda), x_2(\lambda), x_3(\lambda))^T = (2 + 2\lambda, \lambda, 0)^T : \lambda \geq 0\}$,

$z' = 2 + 4\lambda$, every point is feasible, and as $\lambda \rightarrow +\infty$, $z' \rightarrow +\infty$; so the objective function is unbounded above in the original problem, and hence does not have a finite maximum.

Example 4.6.3: An LP with a unique optimum that requires both Phase I, II to solve

Consider the following LP

$$\begin{aligned} \text{Minimize } z &= x_1 + 2x_2 - 2x_3 \\ \text{s. to } 2x_1 + 3x_2 + x_3 &\geq 9 \\ x_1 + 2x_2 - x_3 &\leq 5 \\ x_1 + x_2 + 2x_3 &= 4 \\ x_j &\geq 0 \quad \text{for all } j \end{aligned}$$

Introducing the slack variables x_4, x_5 in the top two constraints, we get the following original tableau.

x_1	x_2	x_3	x_4	x_5	$-z$	RHS
2	3	1	-1	0	0	9
1	2	-1	0	1	0	5
1	1	2	0	0	0	4
1	2	-2	0	0	1	0

$x_j \geq 0$ for all j , min z

Searching for unit vectors, we find only the 2nd unit vector in the column of x_5 , which is selected as the basic variable in row 2. So, we introduce two nonnegative artificial variables t_1, t_3 associated with the 1st and 3rd unit vectors and go to Phase I. The phase I objective function to be minimized is $w = t_1 + t_3$. Here is the Phase I original tableau.

BV	t_1	t_3	x_1	x_2	x_3	x_4	x_5	$-z$	$-w$	RHS
t_1	1	0	2	3	1	-1	0	0	0	9
x_5	0	0	1	2	-1	0	1	0	0	5
t_3	0	1	1	1	2	0	0	0	0	4
	0	0	1	2	-2	0	0	1	0	0
	1	1	0	0	0	0	0	0	1	0

$x_j, t_i \geq 0$ for all i, j , min w

Pricing out the basic variables in both the Phase II and Phase I objective rows leads to the 1st Phase I canonical tableau given at the top of the following sequence of tableaus. The Phase I BFS corresponding to the 1st (unit) basic vector (t_1, x_2, t_3) is $(t_1, t_3, x_1, \text{to } x_5)^T = (9, 4, 0, 0, 0, 0, 5)^T$, $w = 13$. Since $w = 13$ (because $t_1 = 9$, $t_2 = 4$, so $w = t_1 + t_3 = 13$) this solution is not feasible to the original problem. By minimizing w in Phase I we try to reduce its value to 0 if possible. During Phase I, the Phase II objective row is updated but not used. Once artificial variables leave the basic vector, we delete them and their columns in the sequel.

BV	t_1	t_3	x_1	x_2	x_3	x_4	x_5	$-z$	$-w$	RHS	Ratio
t_1	1	0	2	3	1	-1	0	0	0	9	9/2
x_5	0	0	1	2	-1	0	1	0	0	5	5
t_3	0	1	1	1	2	0	0	0	0	4	4 PR
$-z$	0	0	1	2	-2	0	0	1	0	0	
$-w$	0	0	-3	-4	-3	1	0	0	1	-13	min $\theta = 4$
			PC \uparrow								
t_1	1		0	1	-3	-1	0	0	0	1	1
x_5	0		0	1	-3	0	1	0	0	1	1 PR
x_1	0		1	1	2	0	0	0	0	4	4
$-z$	0		0	1	-4	0	0	1	0	-4	
$-w$	0		0	-1	3	1	0	0	1	-1	min $\theta = 1$
			PC \uparrow								
Final Phase I tableau											
t_1	1		0	0	0	-1	-1	0	0	0	
x_2	0		0	1	-3	0	1	0	0	1	
x_1	0		1	0	5	0	-1	0	0	3	
$-z$	0		0	0	-1	0	-1	1	0	-5	
$-w$	0		0	0	0	1	1	0	1	0	

Since all the Phase I relative cost coefficients \bar{d}_j are now ≥ 0 Phase I terminates. Since the value of w is 0, we have a feasible solution for the original problem, but the artificial variable t_1 is in the final basic vector with a value of 0 in the BFS. The Phase I relative cost coefficients of x_4, x_5 are both strictly > 0 . Therefore, as explained in Step 4.5, in every feasible solution of the original problem, x_4, x_5 must have values 0. So, to move to Phase II we fix both x_4, x_5 at 0 and delete them from the tableau. Then we delete the Phase I objective row and the $-w$ column, and begin Phase II.

BV	t_1	x_1	x_2	x_3	$-z$	RHS	Ratio
t_1	1	0	0	0	0	0	
x_2	0	0	1	-3	0	1	
x_1	0	1	0	5	0	3	3/5 PR
$-z$	0	0	0	-1	1	-5	min $\theta = 3/5$
PC \uparrow							
Final Phase II tableau							
t_1	1	0	0	0	0	0	
x_2	0	3/5	1	0	0	14/5	
x_3	0	1/5	0	1	0	3/5	
$-z$	0	1/5	0	0	1	-22/5	

Now all the Phase II relative cost coefficients are ≥ 0 , so the present BFS is an optimum solution of the problem. Remembering that x_4, x_5 have 0 values in every feasible solution of the problem, and dropping the artificial variable t_1 with 0 value from the solution we see that the current BFS is $(x_1, \text{ to } x_5)^T = (0, 14/5, 3/5, 0, 0)^T$ with the minimum objective value of $z = 22/5$.

Also, since the relative cost coefficient of the only nonbasic variable x_1 is positive in the final tableau, we conclude that this optimum solution is unique.

Example 4.6.4: Another LP with an optimum solution that requires both Phase I, II to solve

Consider the following LP:

Original tableau

x_1	x_2	x_3	x_4	x_5	$-z$	b
-2	0	0	0	0	0	0
1	0	1	0	-1	0	3
0	2	1	1	1	0	4
-40	-14	0	0	-14	1	0

$x_j \geq 0$ for all j , minimize z

Phase I original tableau

x_1	x_2	x_3	x_4	x_5	t_1	t_2	$-z$	$-w$	b
-2	0	0	0	0	1	0	0	0	0
1	0	1	0	-1	0	1	0	0	3
0	2	1	1	1	0	0	0	0	4
-40	-14	0	0	-14	0	0	1	0	0
0	0	0	0	0	1	1	0	1	0

$x_j \geq 0$ for all j ; $t_1, t_2, t_3 \geq 0$ artificials; minimize w

BV	x_1	x_2	x_3	x_4	x_5	t_1	t_2	$-z$	$-w$	b	Ratio
t_1	-2	0	0	0	0	1	0	0	0	0	
t_2	1	0	1	0	-1	0	1	0	0	3	3
x_4	0	2	1	1	1	0	0	0	0	4	4
$-z$	-40	-14	0	0	-14	0	0	1	0	0	min
$-w$	1	0	-1	0	1	0	0	0	1	-3	$\theta = 3$
			PC \uparrow								
t_1	-2	0	0	0	0	1		0	0	0	
x_3	1	0	1	0	-1	0		0	0	3	
x_4	-1	2	0	1	2	0		0	0	1	
$-z$	-40	-14	0	0	-14	0		1	0	0	
$-w$	2	0	0	0	0	0		0	1	0	

The RHS constants vector in the problem is already nonnegative, so it is the original tableau for the problem. Here we only have the 3rd unit vector in the column of x_4 . So, we select x_4 as the basic variable in row 3, and introduce artificial variables t_1, t_2 as basic variables in rows 1, 2 for the initial basic vector. The Phase I original tableau is the 2nd tableau given above.

(t_1, t_2, x_4) is the initial feasible basic vector for the Phase I problem. The various canonical tableaus obtained during Phase I are given next.

Phase I terminates and we have $w = 0$. The artificial variable t_1 is still in the basic vector, but its value in the final solution is 0. This final solution leads to the feasible solution $(x_1 \text{ to } x_5)^T = (0, 0, 3, 1, 0)^T$ for the original problem.

Now we need to go to Phase II. We look for original problem variables with positive Phase I relative cost coefficients at Phase I termination. Only x_1 satisfies this property. So, $x_1 = 0$ in every feasible solution of the original problem. We fix $x_1 = 0$, and delete it from the problem. In Phase II we only consider variables x_2 to x_6 as candidates to enter the basic vector. The initial Phase II canonical tableau is the one given at the top.

BV	x_2	x_3	x_4	x_5	t_1	$-z$	b	Ratio
t_1	0	0	0	0	1	0	0	
x_3	0	1	0	-1	0	0	3	
x_4	2	0	1	2	0	0	1	1/2
$-z$	-14	0	0	-14	0	1	0	
PC↑								
Final Phase II canonical tableau								
t_1	0	0	0	0	1	0	0	
x_3	1	1	1/2	0	0	0	7/2	
x_6	1	0	1/2	1	0	0	1/2	
$-z$	0	0	7	0	0	1	7	

The optimality criterion is satisfied in the final Phase II canonical tableau, so the BFS for the original problem in it (obtained by dropping the 0-valued artificial variable t_1), $(\bar{x}_1 \text{ to } \bar{x}_5)^T = (0, 0, 7/2, 0, 0, 1/2)^T$ with objective value $\bar{z} = -7$ is an optimum solution to the problem.

Example 4.6.5: An infeasible problem

Consider the following LP:

$$\begin{aligned} \text{Minimize } & z = 10x_1 - 2x_2 - 4x_3 \\ \text{subject to } & x_1 + x_2 + x_3 \leq 1 \end{aligned}$$

$$\begin{aligned}
 x_1 + 2x_2 + 2x_3 &\leq 3 \\
 2x_1 - x_2 - x_3 &\leq 4 \\
 -2x_1 + x_2 - x_3 &\geq 3 \\
 x_1, x_2, x_3 &\geq 0
 \end{aligned}$$

Phase I original tableau

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	t_4	$-z$	$-w$	b
x_4	1	1	1	1	0	0	0	0	0	0	1
x_5	1	1	2	0	1	0	0	0	0	0	3
x_6	2	-1	-1	0	0	1	0	0	0	0	4
t_4	2	-1	1	0	0	0	-1	1	0	0	3
$-z$	10	-2	-4	0	0	0	0	0	1	0	0
	0	0	0	0	0	0	0	1	0	1	0

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	t_4	$-z$	$-w$	b	Ratio
x_4	1	1	1	1	0	0	0	0	0	0	1	1
x_5	1	1	2	0	1	0	0	0	0	0	3	3
x_6	2	-1	-1	0	0	1	0	0	0	0	4	2
t_4	2	-1	1	0	0	0	-1	1	0	0	3	
$-z$	10	-2	-4	0	0	0	0	0	1	0	0	min
$-w$	-2	1	-1	0	0	0	1	0	0	1	-3	$\theta = 1$
	PC↑											
x_1	1	1	1	1	0	0	0	0	0	0	1	
x_5	0	0	1	-1	1	0	0	0	0	0	2	
x_6	0	-3	-3	-2	0	1	0	0	0	0	2	
t_4	0	-3	-1	-2	0	0	-1	1	0	0	1	
$-z$	0	-2	-4	0	0	0	-1	1	0	0	3	
$-w$	0	3	1	2	0	0	1	0	0	1	-1	

Introducing slack variables x_4 to x_7 in that order, we get the original tableau for this problem. In the original tableau, the RHS constants vector is already ≥ 0 , and variables x_4, x_5, x_6 correspond to 1st, 2nd, and 3rd unit vectors; but the 4th unit vector needed to make a full unit basic vector is missing (x_7 is associated with the negative 4th unit vector, and hence is not suitable). So, we set up the Phase I problem by introducing the artificial variable t_4 associated with the 4th unit vector. The Phase I original tableau is given above.

Pricing out the basic variable t_4 in the Phase I objective row leads to the 1st Phase I canonical tableau given at the top of the list of tableaus given next. We continue applying Phase I of the simplex method. The pivot elements are boxed, and PR, PC indicate the pivot row, pivot column.

In the final tableau the Phase I termination condition is satisfied, and we see that the minimum value of w in the Phase I problem is $1 > 0$. This implies that the original problem has no feasible solution.

Example 4.6.6: Problem with alternate optimum solutions:

Consider the canonical tableau for the LP in standard form at the top of the following tableaus. All x_j are restricted to be nonnegative, and z has to be minimized.

BV	x_1	x_2	x_3	x_4	x_5	x_6	$-z$	b	Ratio
x_1	1	0	0	1	-1	2	0	3	3
x_2	0	1	0	-1	1	-1	0	4	
x_3	0	0	1	1	0	-1	0	5	5
$-z$	0	0	0	-2	1	-1	1	-100	min $= \theta = 3$
PC \uparrow									
x_4	1	0	0	1	-1	2	0	3	
x_2	1	1	0	0	0	1	0	7	
x_3	-1	0	1	0	1	-3	0	2	2
$-z$	2	0	0	0	-1	3	1	-94	min $= \theta = 2$
PC \uparrow									
Final tableau									
x_4	0	0	1	1	0	-1	0	5	
x_2	1	1	0	0	0	1	0	7	7
x_5	-1	0	1	0	1	-3	0	2	
$-z$	1	0	1	0	0	0	1	-92	min $= \theta = 7$
PC \uparrow									

In the final tableau the optimality criterion holds, so the BFS in it $\bar{x} = (0, 7, 0, 5, 2, 0)^T$, is an optimum solution with the optimum objective value of $\bar{z} = 92$.

Notice that the relative cost coefficient of the nonbasic variable x_6 in the final tableau is $\bar{c}_6 = 0$, this indicates that there may be alternate optimum

solutions to the problem. Choosing x_6 with $\bar{c}_6 = 0$ as the entering variable in that tableau leads to the following alternate optimum canonical tableau to this LP.

BV	x_1	x_2	x_3	x_4	x_5	x_6	$-z$	b
x_4	1	1	1	1	0	0	0	12
x_6	1	1	0	0	0	1	0	7
x_5	2	3	1	0	1	0	0	23
$-z$	1	0	1	0	0	0	1	-92

The BFS in this tableau $\hat{x} = (0, 0, 0, 12, 23, 7)^T$ also has an optimum objective value of $\bar{z} = 92$, and is an alternate optimum BFS to this LP.

Example 4.6.7: To find a feasible solution: Find a feasible solution for the system of linear constraints given below. We introduce the slack variable x_5 corresponding to the last inequality, and get the detached

$$\begin{aligned} x_1 + x_3 - x_4 &= 3 \\ x_1 + x_2 + 2x_3 &= 10 \\ x_1 + x_2 + x_3 - 2x_4 &\geq 14 \\ x_j &\geq 0, \quad \text{for all } j \end{aligned}$$

Detached coefficient tableau form

x_1	x_2	x_3	x_4	x_5	b
1	0	1	-1	0	3
1	1	2	0	0	10
1	1	1	-2	-1	14

$x_j \geq 0$ for all j

Phase I original tableau

BV	x_1	x_2	x_3	x_4	x_5	t_1	t_2	t_3	$-w$	b
t_1	1	0	1	-1	0	1	0	0	0	3
t_2	1	1	2	0	0	0	1	0	0	10
t_3	1	1	1	-2	-1	0	0	1	0	14
	0	0	0	0	0	1	1	1	1	0

coefficient tableau form given next. All the RHS constants are ≥ 0 in it, but none of the three unit vectors needed to make up a unit basis are in the tableau. So, to find a feasible solution we introduce the 3 artificial variables t_1, t_2, t_3 with the unit vectors as coefficient vectors and set up the Phase I problem given next.

We apply Phase I beginning with the Phase I canonical tableau at the top of the following list of tableaus.

BV	x_1	x_2	x_3	x_4	x_5	t_1	t_2	t_3	$-w$	b	Ratio
t_1	1	0	1	-1	0	1	0	0	0	3	3
t_2	1	1	2	0	0	0	1	0	0	10	5
t_3	1	1	1	-2	-1	0	0	1	0	14	14
$-w$	-3	-2	-4	3	1	0	0	0	1	-27	min
			PC \uparrow								$\theta = 3$
x_3	1	0	1	-1	0		0	0	0	3	3
t_2	-1	1	0	2	0		1	0	0	4	4
t_3	0	1	0	-1	-1		0	1	0	11	11
$-w$	1	-2	0	-1	1		0	0	1	-15	min
			PC \uparrow								$\theta = 4$
x_3	1	0	1	-1	0			0	0	3	3
x_2	-1	1	0	2	0			0	0	4	4
t_3	1	0	0	-3	-1			1	0	7	7
$-w$	-1	0	0	3	1			0	1	-7	min
			PC \uparrow								$\theta = 3$
Final Phase I canonical tableau											
x_1	1	0	1	-1	0			0	0	3	
x_2	0	1	1	1	0			0	0	7	
t_3	0	0	-1	-2	-1			1	0	4	
$-w$	0	0	1	2	1			0	1	-4	

The Phase I optimality criterion is now satisfied because all the Phase I relative cost coefficients are ≥ 0 . The minimum value of $w = 4 > 0$. This implies that the original system has no feasible solution.

Notice how the value of w , the Phase I objective function, decreases from 27 to 15, to 7, and finally to its minimum value of 4 in the sequence of Phase I solutions obtained in the method.

Exercises

Solving some of these numerical exercises by hand may help in mastering how the simplex method works, and understanding the work involved in carrying it out. All the exercises can be solved using pivot steps involving pivot elements which are = 1 most of the time, or 2 in the final step in some of them. So, the work does not require dealing with messy fractions.

4.2.1: Solve the following LPs. On all these problems Phase II can be applied directly, and the method terminates with the unique optimum solution for the problem.

$$\text{(a): } \min z = -x_1 - 8x_2, \quad \text{s. to } -x_1 + x_2 \leq 2 \\ x_1 + x_2 \leq 1, \quad 2x_1 + x_2 \leq 5, \quad x_1, x_2 \geq 0.$$

$$\text{(b) } \text{Min } z = -2x_1 + x_2 - 2x_3 + x_4, \quad \text{s. to } x_1 - x_2 + x_4 \leq 2 \\ x_2 + x_3 + 2x_4 \leq 3, \quad x_1 + 2x_2 - x_3 - 2x_4 \leq 12, \quad x_j \geq 0 \text{ for all } j.$$

(c) Solve using Dantzig's rule (most negative \bar{c}_j) for selecting the entering variable in each pivot step.

$$\min z = 3x_1 - 8x_2 + 2x_3 - 7x_4 - 5x_5 + 8x_6, \quad \text{s. to } -x_2 + x_3 + x_4 + x_6 = 3 \\ x_1 + x_2 - x_4 = 6, \quad x_2 + x_4 + x_5 - x_6 = 0, \quad x_j \geq 0 \text{ for all } j.$$

$$\text{(d) } \min z = -8x_1 + 8x_2 + 14x_3 + 4x_4 + 6x_5 - 3x_6 + 3x_7 \\ \text{s. to } x_1 - x_6 + x_7 = 3, \quad -2x_1 - 3x_3 + x_4 + 3x_6 = 2, \quad 4x_3 + x_5 - x_6 = 1 \\ x_2 - x_6 = 4, \quad x_j \geq 0 \text{ for all } j.$$

4.2.2: Solve the following LPs. On all these problems Phase II can be applied directly, and the method terminates with the unboundedness conclusion. Construct an extreme half-line in the feasible region along which the objective function diverges to $-\infty$.

$$\text{(a) } \min z = -3x_1 + 4x_2 + x_3, \quad \text{s. to } x_1 - 2x_2 + 2x_3 \leq 3 \\ x_1 - x_2 - 3x_3 \leq 5, \quad -x_1 + x_2 - x_3 \leq 7, \quad x_j \geq 0 \text{ for all } j.$$

$$\text{(b) : } \min z = -2x_1 - x_2, \quad \text{s. to } -x_1 + x_2 \leq 2 \\ x_1 - 2x_2 \geq -5, \quad x_1 - 3x_2 \leq 2, \quad x_1, x_2 \geq 0.$$

$$\begin{aligned} \text{(c): } \min z &= -2x_1 + 2x_2 + x_3, \quad \text{s. to } x_1 - x_2 - 2x_3 \leq 3 \\ x_1 - x_2 - x_3 &\leq 4, \quad x_1 - 2x_2 \leq 0, \quad x_1, x_2, x_3 \geq 0. \end{aligned}$$

$$\begin{aligned} \text{(d): } \min z &= -3x_1 + 2x_2 - 2x_3, \quad \text{s. to } x_1 - 2x_2 + 2x_3 \leq 0 \\ x_1 - x_2 - 2x_3 &\leq 10, \quad -x_1 + 3x_2 - 4x_3 \leq 2, \quad -x_1 + 2x_2 - 2x_3 \leq 3 \\ x_j &\geq 0 \text{ for all } j. \end{aligned}$$

4.2.3: Solve the following LPs. On all these problems with an optimum solution, Phase I needs to be applied, and you have to move to Phase II after Phase I.

(a) In solving this problem, if there is a tie for the min ratio, always select the bottommost among the rows tied as the pivot row.

$$\begin{aligned} \min z &= -2x_3 - 10x_4, \quad \text{s. to } x_1 + x_3 + 2x_4 \geq 2 \\ x_2 + x_3 + x_4 &\geq 4, \quad x_1 + x_2 + 2x_3 + 3x_4 \leq 6 \\ 2x_1 + x_2 + 3x_3 + 6x_4 &\leq 8, \quad x_j \geq 0 \text{ for all } j. \end{aligned}$$

$$\begin{aligned} \text{(b): (O, I)} \min z &= -x_1 - 2x_2 + 3x_3 + 2x_4 - 3x_5, \quad \text{s. to } x_1 + x_3 + x_4 - x_5 = 4 \\ x_2 + x_3 + x_4 + x_5 &= 5, \quad x_1 + x_2 + x_4 + 2x_5 = 3, \quad x_j \geq 0 \text{ for all } j. \end{aligned}$$

4.2.4: Solve the following LPs. On all these problems both Phase I, II need to be applied, and the method terminates with the feasible and unbounded conclusion. Construct an extreme half-line in the feasible region along which the objective function diverges to $-\infty$.

$$\begin{aligned} \text{(a): } \min z &= -x_1 - 2x_2, \quad \text{s. to } x_1 + x_2 \geq 1, \quad x_1 - x_2 \leq 2 \\ -x_1 + x_2 &\leq 2, \quad x_1, x_2 \geq 0. \end{aligned}$$

(b):

$$\begin{aligned} \text{Minimize } & -2x_1 + 2x_2 + x_3 \\ \text{subject to } & x_2 + x_3 - x_4 + x_5 + 2x_6 \leq 6 \\ & x_1 + x_3 - x_4 + x_5 = 5 \\ & -x_1 + x_2 - x_3 + x_4 + x_6 = 3 \\ & x_j \geq 0 \text{ for all } j \end{aligned}$$

(c):

$$\begin{array}{rllllll}
\text{Minimize} & -2x_1 & +2x_2 & +x_3 & & & & \\
\text{subject to} & & x_2 & +x_3 & -x_4 & +x_5 & +2x_6 & \leq 6 \\
& x_1 & & +x_3 & -x_4 & +x_5 & & = 5 \\
& -x_1 & +x_2 & -x_3 & +x_4 & & +x_6 & = -3 \\
& & & & & & & x_j \geq 0 \text{ for all } j
\end{array}$$

If possible, determine a feasible solution where the objective function has value = -200 .

4.2.5: Solve the following problems. All these problems are infeasible.

(a): Minimize $z = -2x_1 - 3x_2 - 4x_3 - x_4$, s. to $x_1 + x_2 + 2x_3 + x_4 \geq 5$
 $x_2 + 2x_4 \geq 10$, $2x_1 + 2x_2 + 2x_3 + 4x_4 \leq 14$, $x_j \geq 0$ for all j .

(b):

$$\begin{array}{rllll}
\text{Minimize} & -x_1 & +x_2 & & \\
\text{subject to} & x_1 & +x_2 & -x_3 & +x_4 \leq 4 \\
& & -x_2 & +x_3 & +x_4 \leq 6 \\
& x_1 & & & +2x_4 \geq 12 \\
& & & & x_j \geq 0 \text{ for all } j
\end{array}$$

(c): Find a feasible solution to the following system of constraints: $x_1 + x_3 - x_4 = 3$, $x_1 + x_2 + 2x_3 = 10$, $x_1 + x_2 + x_3 - 2x_4 \geq 14$, $x_j \geq 0$ for all j .

4.2.6: Solve the following LP. This problem has alternate optimum solutions. Find several optimum BFSs for this problem.

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	b
1	0	0	1	-1	1	0	3
0	1	0	1	1	-1	0	5
0	0	1	-1	1	1	0	6
3	4	-2	7	1	-5	1	0

$x_j \geq 0$ for all j , minimize z

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