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# Chapter 5

## Duality, Marginal and Sensitivity Analysis in LP

This is Chapter 5 of “Junior Level Web-Book for Optimization Models for decision Making” by Katta G. Murty.

Associated with every linear programming problem, there is another linear program called its **dual**, involving a different set of variables, but sharing the same data. When referring to the dual problem of an LP, the original LP is called **the primal** or **the primal problem**. Together, the two problems are referred to as a **primal, dual pair** of linear programs.

In Chapter 3 we defined the marginal value of an RHS constant in an LP as the rate of change of the optimum objective value, per unit change in that RHS constant from its present value; when that rate is well defined. Associated with each constraint in an LP there will be a dual variable in its dual problem, it can be shown that the marginal values in the primal are well defined only if the dual problem has a unique optimum solution, and in this case the value of a dual variable in that solution will be the marginal value of the associated RHS constant in the primal problem.

The topic of duality in LP lies in the intersection of two subjects, economics and OR, in fact the pioneering work in duality has been

carried out by mathematical economists. The dual of an LP arises from economic considerations that come up in marginal analysis. In LP's, each constraint usually comes from the requirement that the total amount of some item utilized should be  $\leq$  (or  $=$ ) the total amount of this item available, or that the total number of units of some item produced should be  $\geq$  (or  $=$ ) the known requirement for this item. Using the marginal value of that item as the dual variable, the dual problem is constructed through rational economic arguments. These economic arguments become simplified if in the primal problem all the variables are nonnegative variables, and all the remaining constraints are  $\leq$  inequalities ( $\geq$  inequalities) if the primal is a maximization (minimization) problem. The fertilizer problem discussed in Example 3.4.1 is of this type, so we will discuss the derivation of its dual.

## 5.1 Derivation of the Dual of the Fertilizer Problem Through Rational Economic Arguments

In this problem formulated in Section 3.4 of Chapter 3, the fertilizer

Item	Tons required to make one ton of		Maximum amount of item available daily (tons)
	Hi-ph	Lo-ph	
RM 1	2	1	1500
RM 2	1	1	1200
RM 3	1	0	500
Net profit (\$ per ton made	15	10	

manufacturer has a daily supply of 1500 tons of RM 1, 1200 tons of RM 2, and 500 tons of RM 3 from the company's quarries at a cost of \$50, 40, 60/ton respectively for RM1, RM2, RM3. Presently these supplies can be used to manufacture Hi-ph or Lo-ph fertilizers to make profit. Relevant data from Section is tabulated above.

The LP model for this problem is:

$$\begin{array}{rllllll}
 \text{Max.} & z(x) = & 15x_1 & + & 10x_2 & & \text{Item} \\
 \text{S. to} & & 2x_1 & + & x_2 & \leq & 1500 \quad \text{RM 1} \\
 & & x_1 & + & x_2 & \leq & 1200 \quad \text{RM 2} \\
 & & x_1 & & & \leq & 500 \quad \text{RM 3} \\
 & & x_1 & \geq & 0, & x_2 & \geq 0
 \end{array} \quad (5.1.1)$$

where the decision variables are:

$$\begin{array}{ll}
 x_1 & = \text{the tons of Hi-ph made per day} \\
 x_2 & = \text{the tons of Lo-ph made per day}
 \end{array}$$

There is a detergent company in the area that needs supplies of RM 1, 2, 3. The detergent manufacturer wants to persuade the fertilizer manufacturer to give up fertilizer making, and instead sell the supplies of RM 1, 2, 3 to the detergent company. Being very profit conscious, the fertilizer manufacturer will not agree to this deal unless the prices offered by the detergent manufacturer for each of these raw materials fetch at least as much income as each of the options in the fertilizer making business.

In this problem, money is measured in *net profit dollar* units (i.e., after subtracting the cost of raw materials from the real life revenue dollars). Let the offer made by the detergent manufacturer be:

$$\pi_i = \text{price/ton for RM}i, i = 1, 2, 3$$

in these same money units (i.e., in real life dollars, the detergent manufacturer offers to pay  $\$50 + \pi_1$ ,  $\$40 + \pi_2$ ,  $\$60 + \pi_3$  per/ton of RM1, RM2, RM3 respectively). With this understanding, we will continue our discussion in net profit dollar units for money, and dollar will refer

to these units. Clearly, these prices  $\pi_1, \pi_2, \pi_3$  have to be  $\geq 0$  for the deal to be acceptable to the fertilizer manufacturer.

Now consider the Hi-ph fertilizer making process. Manufacturing one ton of this fertilizer yields a net profit of \$15, and uses up 2 tons RM 1, 1 ton RM 2, and 1 ton RM 3. The same basket of raw materials fetches a price of  $2\pi_1 + \pi_2 + \pi_3$  from the detergent manufacturer. So, the fertilizer manufacturer will not find the price vector  $\pi = (\pi_1, \pi_2, \pi_3)$  acceptable unless  $2\pi_1 + \pi_2 + \pi_3 \geq 15$ . Similar economic analysis with the Lo-ph fertilizer process leads to the constraint  $\pi_1 + \pi_2 \geq 10$ . With the price vector  $\pi$ , the cost to the detergent company of acquiring the daily raw material supply is  $1500\pi_1 + 1200\pi_2 + 500\pi_3$ , and the detergent manufacturer would clearly like to see this minimized. Thus the price vector  $\pi = (\pi_1, \pi_2, \pi_3)$  that the detergent manufacturer offers for the supplies of RM 1, 2, 3, should minimize  $v(\pi) = 1500\pi_1 + 1200\pi_2 + 500\pi_3$ , subject to the constraints  $2\pi_1 + \pi_2 + \pi_3 \geq 15$ ,  $\pi_1 + \pi_2 \geq 10$ ,  $\pi_1, \pi_2, \pi_3 \geq 0$ , to make it acceptable to the fertilizer manufacturer. Thus the detergent manufacturer's problem, that of determining the best price vector acceptable to the fertilizer manufacturer, is

$$\begin{array}{llllll} \text{Min. } v(\pi) = & 1500\pi_1 + & 1200\pi_2 + & 500\pi_3 & & \\ \text{S. to} & 2\pi_1 + & \pi_2 + & \pi_3 & \geq & 15 \\ & \pi_1 + & \pi_2 & & \geq & 10 \\ & \pi_1, & \pi_2, & \pi_3 & \geq & 0 \end{array} \quad (5.1.2)$$

(5.1.2) is the dual of (5.1.1) and vice versa. This pair of problems is a primal-dual pair of LPs. When considering the primal (5.1.1), the variables in its dual (5.1.2) are called the **dual variables**, and the slacks in (5.1.2) corresponding to the inequality constraints in it are called the **dual slack variables**.

Since the first constraint in (5.1.2) comes from the economic analysis of the Hi-ph manufacturing process, this dual constraint is said to correspond to the Hi-ph primal variable  $x_1$ . Likewise, the second dual constraint in (5.1.2) corresponds to the primal variable  $x_2$ . In the same way, the dual variable  $\pi_1$ , the detergent manufacturer's price for the item RM 1, is associated with the RM 1 (first) primal constraint

in (5.1.1). Similarly the dual variables  $\pi_2, \pi_3$  are associated with the second (RM 2), and third (RM 3) primal constraints in (5.1.1), respectively. Thus there is a dual variable associated with each primal constraint, and a dual constraint corresponding to each primal variable. Also, verify the following facts.

1. The coefficient matrix in the detergent manufacturer's problem (5.1.2) is just the transpose of the coefficient matrix in the fertilizer manufacturer's problem (5.1.1) and vice versa.
2. The RHS constants in (5.1.2) are the objective coefficients in (5.1.1) and vice versa.
3. Each variable in (5.1.1) leads to a constraint in (5.1.2) and vice versa.
4. (5.1.1) is a maximization problem in which the constraints are  $\leq$  type; and (5.1.2) is a minimization problem in which the constraints are  $\geq$  type.

### Dual Variables Are Marginal Values

The marginal value of RM  $i$  in the fertilizer manufacturer's problem is the rate of change in the maximum profit per unit change in the availability of RM  $i$  from its present value; thus it is the net worth of one additional unit of RM  $i$  over the present supply, for  $i = 1, 2, 3$ , to the fertilizer manufacturer. Hence, if the detergent manufacturer offered to buy RM  $i$  at a price  $\geq$  its marginal value, for  $i = 1, 2, 3$ , the fertilizer manufacturer would find the deal acceptable. Being cost conscious, the detergent manufacturer wants to make the price offered for any raw material to be the smallest value that will be acceptable to the fertilizer manufacturer. Hence, in an optimum solution of (5.1.2), the  $\pi_i$  will be the marginal value of RM  $i$ , for  $i = 1, 2, 3$ , in (5.1.1). Thus the dual variables are the marginal values of the items associated with the constraints in the primal problem. These marginal values depend on the data, and may change if the data does.

## The Dual of the General Problem in this Form

Now consider the general LP in the same form, it is

$$\begin{aligned} \text{Maximize } & z(x) = cx \\ \text{subject to } & Ax \leq b \\ & x \geq 0 \end{aligned} \quad (5.1.3)$$

where  $A$  is an  $m \times n$  matrix. From similar arguments it can be seen that the marginal values of this LP are the dual variables in the dual of this problem given below. Let  $\pi_i$  denote the dual variable associated with the  $i$ th constraint in this LP,  $i = 1$  to  $m$ . If we write the vector of dual variables as a column vector, the statement of the dual problem will involve  $A^T$  as the coefficient matrix. But usually the vector of dual variables is written as the row vector  $\pi = (\pi_1, \dots, \pi_m)$ . Using it, the dual of the above LP is

$$\begin{aligned} \text{Minimize } & v(\pi) = \pi b \\ \text{subject to } & \pi A \geq c \\ & \pi \geq 0 \end{aligned} \quad (5.1.4)$$

We will discuss some of the relationships between the primal and dual problems after we discuss the dual of the LP in standard form in the next section.

## 5.2 Dual of the LP In Standard Form

The economic arguments in the previous section can be applied to derive the dual of the general LP model with general linear constraints (equations, inequalities of the  $\geq$ ,  $\leq$  types) and bounds on the variables, and even unrestricted variables. But that is beyond the scope of this book (it is suitable for a graduate level book). However, since we are discussing the LP in standard form so much, we will state its dual

without elaborating on its derivation from economic principles. The LP in standard form is

$$\begin{aligned} \text{Minimize } & z(x) = cx \\ \text{subject to } & Ax = b \\ & x \geq 0 \end{aligned} \quad (5.2.1)$$

where  $A$  is an  $m \times n$  matrix.

To write its dual, associate a dual variable to each constraint in the primal. Calling the dual variable associated with the  $i$ th primal constraint ( $A_i x = b_i$ ) as  $\pi_i$ , the vector of dual variables is the row vector  $\pi = (\pi_1, \dots, \pi_m)$ . Then the dual of (5.5) is

$$\begin{aligned} \text{Maximize } & v(\pi) = \pi b \\ \text{subject to } & \pi A \leq c \\ & \pi \geq 0 \end{aligned} \quad (5.2.2)$$

$\pi A \leq c$  in matrix notation is a system of  $n$  inequality constraints, the  $j$ th one here being  $\pi A_{.j} \leq c_j$ ; this is the **dual constraint corresponding to the primal variable  $x_j$** , for  $j = 1$  to  $n$ . This constraint can be transformed into an equation by introducing a slack variable for it. This slack variable can be shown to be related to the relative cost coefficient  $\bar{c}_j$  of  $x_j$  when (5.2.1) is solved by the simplex method, so this dual slack variable is usually denoted by the same symbol  $\bar{c}_j$ . Using it, the dual constraint corresponding to the primal variable  $x_j$  is

$$\begin{aligned} \pi A_{.j} + \bar{c}_j &= c_j \\ \bar{c}_j &\geq 0 \end{aligned}$$

For  $j = 1$  to  $n$ , the nonnegative primal variable  $x_j$ , and its associated nonnegative dual slack variable  $\bar{c}_j$  together form the pair  $(x_j, \bar{c}_j = c_j - \pi A_{.j})$  called the  **$j$ th complementary pair** in the primal, dual solutions  $(x, \pi)$  for the primal, dual pair of LPs (5.2.1), (5.2.2).

Every complementary pair in a primal, dual pair of LPs always consists of a variable restricted to be  $\geq 0$  in one problem, and the nonnegative slack variable of the corresponding constraint in the other problem. There are no complementary pairs associated with equality constraints in one problem, and the corresponding unrestricted variables in the other problem. In the primal, dual pair (5.2.1), (5.2.2), the complementary pairs are  $(x_j, \bar{c}_j = c_j - \pi A_j)$  for  $j = 1$  to  $n$ .

### Example 5.2.1

As an example, consider the following LP in standard form

**Tableau 5.2.1: Primal problem**

Associated dual var.	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$b$
$\pi_1$	1	2	3	-2	1	16	17
$\pi_2$	0	1	-4	1	1	1	2
$\pi_3$	0	0	1	-2	1	0	1
Primal obj. row	3	11	-15	10	4	57	= $z$ , minimize

$x_j \geq 0$  for all  $j$ .

**Tableau 5.2.2: Dual problem**

$\pi_1$	$\pi_2$	$\pi_3$		Primal var. corresponding to dual constraint
1	0	0	$\leq$	3 $x_1$
2	1	0	$\leq$	11 $x_2$
3	-4	1	$\leq$	-15 $x_3$
-2	1	-2	$\leq$	10 $x_4$
1	1	1	$\leq$	4 $x_5$
16	1	0	$\leq$	57 $x_6$
17	2	1	=	$v(\pi)$ , maximize

expressed in detached coefficient tableau form in Tableau 5.2.1, involving 3 constraints in 6 nonnegative variables. The last row in the tableau gives the objective function. In a column on the left hand side,

we listed the dual variables associated with the primal constraints. We tabulate the dual problem in Tableau 5.2.2 just after the primal.

Introducing the dual slack variables  $\bar{c}_1$  to  $\bar{c}_6$ , the dual can be written with its constraints as equality constraints as in Tableau 5.2.3.

**Tableau 5.2.3: Dual problem**

$\pi_1$	$\pi_2$	$\pi_3$	$\bar{c}_1$	$\bar{c}_2$	$\bar{c}_3$	$\bar{c}_4$	$\bar{c}_5$	$\bar{c}_6$	
1	0	0	1	0	0	0	0	0	3
2	1	0	0	1	0	0	0	0	11
3	-4	1	0	0	1	0	0	0	-15
-2	1	-2	0	0	0	1	0	0	10
1	1	1	0	0	0	0	1	0	4
16	1	0	0	0	0	0	0	1	57
17	2	1	0	0	0	0	0	0	$= v(\pi)$ , maximize

$\bar{c}_j \geq 0, j = 1$  to  $6$

$\bar{c}_j$  here is the relative cost coefficient of  $x_j$ , for  $j = 1$  to  $6$ . The complementary pairs in these primal, dual problems are  $(x_1, \bar{c}_1 = 3 - \pi_1)$ ,  $(x_2, \bar{c}_2 = 11 - (2\pi_1 + \pi_2))$ ,  $(x_3, \bar{c}_3 = -15 - (3\pi_1 - 4\pi_2 + \pi_3))$ ,  $(x_4, \bar{c}_4 = 10 - (-2\pi_1 + \pi_2 - 2\pi_3))$ ,  $(x_5, \bar{c}_5 = 4 - (\pi_1 + \pi_2 + \pi_3))$ ,  $(x_6, \bar{c}_6 = 57 - (16\pi_1 + \pi_2))$ .

## Optimality Conditions for an LP

We now state without proof, a fundamental result in LP theory that serves as the basis for designing algorithms to solve LPs, and for checking when an algorithm has reached an optimum solution.

**Theorem 5.2.1** *In a primal, dual pair of LPs, let  $x$  be the vector of primal variables, and  $\pi$  the vector of dual variables. A primal vector  $\bar{x}$  is an optimum solution for the primal problem iff it satisfies the following condition (i), and there exists a dual vector  $\bar{\pi}$  satisfying (ii), which together with  $\bar{x}$  also satisfies (iii).*

- (i) **Primal feasibility:** *The vector  $\bar{x}$  must satisfy all the constraints and bound restrictions in the primal problem.*

- (ii) **Dual feasibility:** *The vector  $\bar{\pi}$  must satisfy all the constraints in the dual problem.*
- (iii) **Complementary slackness optimality conditions:** *In every complementary pair for these primal, dual problems, at least one of the two quantities in the pair is zero at the solutions  $(\bar{x}, \bar{\pi})$ . Or, equivalently, the product of the two quantities in every complementary pair is zero.*

*If all three conditions are satisfied,  $\bar{x}$  is an optimum solution for the primal problem, and  $\bar{\pi}$  is an optimum solution of the dual problem, and the optimum objective values in the two problems are equal.*

For a proof of this theorem, see any of the graduate level books on LP. We will use this theorem in the algorithm for the transportation problem discussed in the next chapter.

We now explain what the complementary slackness conditions are for the primal, dual problems (5.2.1), (5.2.2). If  $x, \pi$  are primal and dual solutions for (5.2.1), (5.2.2), and  $(\bar{c}_j) = (c_j - \pi A_{.j})$ , since the complementary pairs in these problems are  $(x_j, \bar{c}_j)$  for  $j = 1$  to  $n$ ; the complementary slackness conditions for these problems can be stated in one of two ways: At least one quantity in each pair  $(x_j, \bar{c}_j)$  is zero; or equivalently,  $x_j \bar{c}_j = 0$  for all  $j$ .

As an example, consider the LP in standard form Tableau 5.2.1. Consider the primal vector  $\bar{x} = (2, 6, 1, 0, 0, 0)^T$ . It satisfies all the constraints and sign restrictions in the primal problem, so it is primal feasible. Consider the dual vector  $\bar{\pi} = (3, 5, -4)$ , which can be verified to be dual feasible. The dual slack vector corresponding to  $\bar{\pi}$  is  $\bar{c} = (\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4, \bar{c}_5, \bar{c}_6) = (0, 0, 0, 3, 0, 4)$ . So the values of the various complementary pairs at  $\bar{x}, \bar{\pi}$ ,  $(\bar{x}_j, \bar{c}_j); j = 1$  to  $6$  are:  $(2, 0)$ ,  $(6, 0)$ ,  $(1, 0)$ ,  $(0, 3)$ ,  $(0, 0)$ ,  $(0, 4)$ . At least one quantity in each pair is zero. So  $\bar{x}, \bar{\pi}$  satisfy all the complementary slackness optimality conditions. Hence, by Theorem 5.2.1,  $\bar{x}$  is an optimum solution of the LP in Tableau 5.2.1,  $\bar{\pi}$  is an optimum solution of its dual in Tableau 5.2.2. Both optimum objective values can be verified to be equal to 57.

In this example we used Theorem 5.2.1 to check whether a given solution to an LP is optimal. Actually Theorem 5.2.1 also provides a guiding light for designing algorithms to try to construct solutions which satisfy the conditions there, and thereby solve both the primal and dual problems together. We will illustrate this for the special case of the balanced transportation problem in the next chapter. In the next section we will discuss the dual of the balanced transportation problem, and the optimality conditions for it, these will be used in the next chapter to develop a specialized version of the simplex method to solve it very efficiently.

### 5.3 The Dual of the Balanced Transportation Problem

Consider the balanced transportation problem for shipping iron ore from mines 1, 2 to plants 1, 2, 3 at minimum cost, formulated in Example 3.7.1. In this problem, the primal variable  $x_{ij}$  = ore (in tons) shipped from mine  $i$  to plant  $j$ ;  $i = 1, 2$ ;  $j = 1, 2, 3$ . Here is the problem in detached coefficient form. In a column on the left hand side we list the dual variables that we associate with the primal constraints for writing the dual problem.

Iron Ore shipping problem								
Associated dual var.	$x_{11}$	$x_{12}$	$x_{13}$	$x_{21}$	$x_{22}$	$x_{23}$		Item
$u_1$	1	1	1	0	0	0	800	Ore/mine 1
$u_2$	0	0	0	1	1	1	300	Ore/mine 2
$v_1$	1	0	0	1	0	0	400	Ore/plant 1
$v_2$	0	1	0	0	1	0	500	Ore/plant 2
$v_3$	0	0	1	0	0	1	200	Ore/plant 3
	11	8	2	7	5	4		= $z$ , minimize

$$x_{ij} \geq 0 \text{ for all } i, j.$$

So, the dual of this problem is the following.

Maximize  $800u_1 + 300u_2 + 400v_1 + 500v_2 + 200v_3$

Associated  
primal var.

subject to

$u_1 + v_1 \leq$	11	$x_{11}$
$u_1 + v_2 \leq$	8	$x_{12}$
$u_1 + v_3 \leq$	2	$x_{13}$
$u_2 + v_1 \leq$	7	$x_{21}$
$u_2 + v_2 \leq$	5	$x_{22}$
$u_2 + v_3 \leq$	4	$x_{23}$

Here  $u_i$  is the dual variable associated with source  $i$  (mines 1, 2 in this problem), and  $v_j$  is the dual variable associated with demand center  $j$  (plants 1, 2, 3 in this problem). If  $c_{ij}$  is the original cost coefficient of the primal variable  $x_{ij}$  in this problem, the corresponding dual constraint is  $u_i + v_j \leq c_{ij}$ ; its dual slack or reduced cost coefficient is  $\bar{c}_{ij} = c_{ij} - u_i - v_j$ . The pairs  $(x_{ij}, \bar{c}_{ij} = c_{ij} - u_i - v_j)$  for various values of  $i, j$  are the complementary pairs in these primal, dual problems.

**Array Representation of the Iron Ore Shipping Problem**

	Steel Plant			$a_i$	Dual var.
	1	2	3		
Mine 1	$\bar{c}_{11}$ $x_{11}$ $c_{11} = 11$	$\bar{c}_{12}$ $x_{12}$ 8	$\bar{c}_{13}$ $x_{13}$ 2	800	$u_1$
Mine 2	$\bar{c}_{21}$ $x_{21}$ 7	$\bar{c}_{22}$ $x_{22}$ 5	$\bar{c}_{23}$ $x_{23}$ 4	300	$u_2$
$b_j$	400	500	200		
Dual var.	$v_1$	$v_2$	$v_3$		

$x_{ij} \geq 0$  for all  $i, j$ . Minimize cost.  $\bar{c}_{ij} = c_{ij} - u_i - v_j$   
 $a_i, b_j$  are availability at mine  $i$ , requirement at plant  $j$  in tons.

In Chapter 3 we mentioned that all the constraints and the decision variables or their values in a particular solution in a balanced transportation problem can be displayed very conveniently in the form of a two dimensional transportation array. In this array representation we can also include the dual variables  $u_i$  associated with the rows of the array (representing sources in the problem) in a right hand column, and the dual variables  $v_j$  associated with the columns of the array (representing demand centers in the problem) in a bottom row. With these things, the array representation of this iron ore shipping problem is given above.

In this array representation it is very convenient to check whether the given dual vector ( $u = (u_i), v = (v_j)$ ) is dual feasible. It is dual feasible if,  $\bar{c}_{ij} = c_{ij} - u_i - v_j \geq 0$  for all  $i, j$ . For this it is convenient to compute  $\bar{c}_{ij}$  and enter it in the top left corner of the cell  $(i, j)$  for all  $i, j$ . When both  $x_{ij}, \bar{c}_{ij}$  are entered this way in each cell of the array, it is easy to check whether the complementary slackness optimality conditions hold (at least one of  $x_{ij}, \bar{c}_{ij}$  have to be zero for each  $(i, j)$ , or equivalently  $x_{ij}\bar{c}_{ij} = 0$  for every  $(i, j)$ ).

The array form of the balanced transportation problem is very convenient for displaying the current primal and dual solutions and the relative cost coefficients.

In a general balanced transportation problem, there may be  $m$  sources, and  $n$  demand centers with the following data

$$\begin{aligned} a_i &= \text{material (in units) available at source } i, i = 1 \text{ to } m \\ b_j &= \text{material required at demand center } j, j = 1 \text{ to } n \\ c_{ij} &= \text{cost (\$/unit) to ship from source } i, \text{ to demand center } j, i = 1 \text{ to } m, j = 1 \text{ to } n \end{aligned}$$

The problem is a balanced transportation problem if the data satisfies

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j. \quad (5.3.1)$$

i.e., the total amount of material required at all the demand centers is equal to the total amount of material available at all the sources. We assume that this condition holds.

The primal variables are:  $x_{ij}$  = units shipped from source  $i$  to demand center  $j$ ,  $i = 1$  to  $m$ ,  $j = 1$  to  $n$ . Associate the dual variable  $u_i$  with the primal constraint of source  $i$ , and the dual variable  $v_j$  with the primal constraint of demand center  $j$ . Then the balanced transportation problem with this data is (5.3.2), and its dual is (5.3.3).

$$\begin{aligned} \text{Minimize} \quad z(x) &= \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \\ \text{subject to} \quad \sum_{j=1}^n x_{ij} &= a_i, \quad i = 1 \text{ to } m \\ \sum_{i=1}^m x_{ij} &= b_j, \quad j = 1 \text{ to } n \\ x_{ij} &\geq 0, \quad \text{for all } i, j \end{aligned} \quad (5.3.2)$$

$$\begin{aligned} \text{Maximize} \quad w(u, v) &= \sum_{i=1}^m a_i u_i + \sum_{j=1}^n b_j v_j \\ \text{subject to} \quad u_i + v_j &\leq c_{ij}, \quad \text{for all } i, j \end{aligned} \quad (5.3.3)$$

$\bar{c}_{ij} = c_{ij} - u_i - v_j$  is the relative cost coefficient of  $x_{ij}$ , i.e., the dual slack associated with it. The various  $(x_{ij}, \bar{c}_{ij})$  are the complementary pairs in these primal, dual problems.

## 5.4 Relationship of Dual Slack Variables to the Relative Cost Coefficients in the Simplex Method

Consider the LP in standard form

$$\begin{array}{c|c|c}
 x & -z & \\
 \hline
 A & 0 & b \\
 \hline
 c & 1 & 0 \\
 \hline
 x \geq 0, \min z & & 
 \end{array} \quad (5.4.1)$$

where  $A$  is a matrix of order  $m \times n$  and rank  $m$ .

A basis  $B$  for this problem is a nonsingular square submatrix of  $A$  of order  $m$ , and let  $x_B$  be the corresponding basic vector. Let  $x_D$  denote the vector of nonbasic variables in some order, and  $D$  the submatrix of  $A$  consisting of the columns of  $A$  associated with these nonbasic variables. Let  $c_B$  be the row vector of original basic cost coefficients, and  $c_D$  the row vector of original nonbasic cost coefficients. Rearranging the variables in (5.4.1) into basic and nonbasic parts, (5.4.1) can be written as

$$\begin{array}{c|c|c|c}
 x_B & x_D & -z & \\
 \hline
 B & D & 0 & b \\
 \hline
 c_B & c_D & 1 & \text{minimize } z \\
 \hline
 x_B, x_D \geq 0. & & & 
 \end{array}$$

To get the canonical tableau WRT  $x_B$ , we need to convert  $B$  into the unit matrix  $I$  (this can be done by multiplying the system of constraint rows by  $B^{-1}$  on the left), and then pricing out the basic columns. Therefore, it is

$$\begin{array}{c}
 \text{Canonical Tableau WRT } x_B \\
 \hline
 \begin{array}{c|c|c|c}
 x_B & x_D & -z & \\
 \hline
 I & B^{-1}D & 0 & B^{-1}b \\
 \hline
 0 & c_D - c_B B^{-1}D & 1 & -c_B B^{-1}b \\
 \hline
 \end{array}
 \end{array}$$

So, the vector of relative cost coefficients of nonbasic variables  $x_D$  WRT the basic vector  $x_B$  is  $\bar{c}_D = c_D - c_B B^{-1}D$ .

The basic vector  $x_B$  defines a **basic solution** for the system of equality constraints “ $Ax = b$ ”. This basic solution is obtained by setting all the nonbasic variables equal to zero ( $x_D = 0$ ) and then solving the remaining system for the values of the basic variables in the solution. This remaining system is  $Bx_B = b$ , and its solution is  $\bar{x}_B = B^{-1}b$ . So, the primal basic solution of (5.4.1) associated with the basic vector  $x_B$ , or the corresponding basis  $B$  is  $\bar{x} = (\bar{x}_B, \bar{x}_D)$ , where  $\bar{x}_D = 0$  and  $\bar{x}_B = B^{-1}b$ . This solution may not be feasible in the sense it may not satisfy the sign restrictions “ $x \geq 0$ ”.

If  $B^{-1}b \geq 0$ , the basic solution  $\bar{x}$  is feasible to (5.4.1) and is called a **basic feasible solution (BFS)** of (5.4.1); and the basic vector  $x_B$  and the basis  $B$  are said to be the **primal feasible basic vector** and **primal feasible basis**, respectively.

If  $B^{-1}b \not\geq 0$ , this solution  $\bar{x}$  satisfies the constraints “ $Ax = b$ ” but not the sign restrictions “ $x \geq 0$ ”, it is infeasible to (5.4.1), and the basic vector  $x_B$  and basis  $B$  are called **primal infeasible basic vector**, **primal infeasible basis** respectively.

Rearranging the constraints in the dual problem in order of the primal variables corresponding to them as arranged in the above tableau, they are

$$\begin{aligned}\pi B &\leq c_B & (5.4.2) \\ \pi D &\leq c_D.\end{aligned}$$

Remember that here  $\pi = (\pi_1, \dots, \pi_m)$  is a row vector. The first line in the constraints above contains the dual constraints corresponding to the  $m$  basic variables in  $x_B$ , and the second line contains those corresponding to the nonbasic variables in  $x_D$ . Denote the row vectors of dual slack variables in these sets by  $s_B, \bar{s}_D$ . Introducing these slack variables, the dual becomes

$$\begin{aligned}\pi B + s_B &= c_B & (5.4.3) \\ \pi D + s_D &= c_D \\ s_B, s_D &\geq 0.\end{aligned}$$

$x_B$  is a basic vector for the primal problem (5.4.1), because it consists of primal variables only, and  $B$  is the basis for the primal problem associated with it.  $\bar{x} = (\bar{x}_B = B^{-1}b, \bar{x}_D = 0)$  is the primal basic solution associated with it. In LP theory, a **dual basic solution** associated with  $x_B, B$  is also defined, even though the basic vector  $x_B$  contains no dual variables. The definition of the dual basic solution associated with  $x_B, B$  is tailored to make sure that it satisfies the complementary slackness conditions together with the primal basic solution  $\bar{x}$  associated with  $x_B, B$ . In  $\bar{x}$ , only basic variables in  $x_B$  can have nonzero values, and the complements of these variables are the dual slacks in the vector  $s_B$ . So, for the dual basic solution to satisfy the complementary slackness conditions with the primal basic solution  $\bar{x}$ , it is enough if we make sure that  $s_B = 0$ , from (5.4.3); this defines the dual basic solution associated with  $x_B, B$  to be the unique solution of

$$\pi B = c_B \quad (5.4.4)$$

or  $\tilde{\pi} = c_B B^{-1}$ .

At the dual solution  $\tilde{\pi}$ , the slack vector  $s_B = \tilde{s}_B = c_B - \tilde{\pi}B = 0$  (this follows from the definition of  $\tilde{\pi}$ ), and  $s_D = \tilde{s}_D = c_D - \tilde{\pi}D = c_D - c_B B^{-1}D = \bar{c}_D$ , the vector of relative cost coefficients of nonbasic variables  $x_D$  in the canonical tableau of the primal (5.4.1) WRT the basic vector  $x_B$ . Also,  $\tilde{s}_B = 0 =$  the vector of relative cost coefficients of the basic variables  $x_B$  in this canonical tableau. Thus, at the dual basic solution  $\tilde{\pi}$ , for each variable  $x_j$ , the dual slack in the dual constraint corresponding to  $x_j$  is equal to  $\bar{c}_j$ , the relative cost coefficient of  $x_j$  WRT the basic vector  $x_B$ . That's why the dual slacks  $s_j$  are denoted by  $\bar{c}_j$ .

We summarize these facts in the following result.

**Result 5.4.1:** *Let  $x_B$  be a basic vector and  $B$  the associated basis for the LP in standard form (5.4.1). The dual basic solution corresponding to  $x_B, B$  is the unique solution of the system of dual constraints corresponding to the basic variables in  $x_B$ , each treated as an equation.*

*Also, the relative cost coefficients  $\bar{c}_j$  in the canonical tableau of (5.4.1) WRT  $x_B$  are exactly the values of the dual slack variables at*

this dual basic solution. That's why, the dual slacks are denoted by  $\bar{c}_j$ .

In the next chapter, we will discuss a special implementation of the simplex algorithm for the balanced transportation problem, that in every step computes the dual basic solution efficiently using the special structure of the basis for this problem, and then computes using it, the relative cost coefficients as dual slacks. Using this strategy, this implementation applies the simplex algorithm on this problem without ever having to compute a canonical tableau.

Efficient variants of the simplex algorithm for general LPs (like the revised simplex algorithm, not discussed in this book) also use the dual basic solution in every step to compute the relative cost coefficients through the dual slacks.

### Relationship Between Dual Feasibility and the Optimality Criterion in the Simplex Algorithm

This dual basic solution  $\tilde{\pi}$  associated with the basic vector  $x_B$ , and basis  $B$  for the LP in standard form (5.4.1), is feasible to the dual problem if it satisfies all the dual constraints (those in (5.4.2)). It satisfies the dual constraints corresponding to the basic variables in  $x_B$  as equations. So to be dual feasible it has to satisfy the dual constraints associated with nonbasic variables; i.e., the relative cost coefficients of all the nonbasic variables,  $\bar{c}_j = c_j - \tilde{\pi}A_{.j}$  have to be  $\geq 0$ . If this happens, the basis  $B$ , and the basic vector  $x_B$  are said to be a **dual feasible basis** and **dual feasible basic vector** for (5.4.1), respectively. This condition is exactly the optimality criterion for a basic vector used in the simplex algorithm as a termination condition for the algorithm.

Therefore the optimality criterion in the (primal) simplex algorithm is actually the dual feasibility condition.

If at least one of the nonbasic relative cost coefficients  $\bar{c}_j = c_j - \tilde{\pi}A_{.j}$  is  $< 0$ , then  $\tilde{\pi}$  is dual infeasible; in this case the basis  $B$ , and the basic vector  $x_B$  are said to be **dual infeasible** for (5.4.1).

To summarize, let  $x_B$  be a basic vector for (5.4.1) associated with the basis  $B$ , nonbasic vector  $x_D$ , basic cost (row) vector  $c_B$ , nonbasic cost (row) vector  $c_D$ . The primal basic solution corresponding to  $x_B$

is obtained by the system on the left of (5.4.5); and the dual basic solution corresponding to  $x_B$  is obtained by the system on the right in (5.4.5).  $x_B$  is primal feasible if  $B^{-1}b \geq 0$ ; it is dual feasible if  $c_j - (c_B B^{-1})A_{.j} \geq 0$  for all nonbasic  $x_j$ .

$$\begin{array}{l} Bx_B = b \\ x_D = 0. \end{array} \qquad \begin{array}{l} \pi B = c_B \end{array} \qquad (5.4.5)$$

### Example 5.4.1

Consider the vector  $x_B = (x_1, x_2, x_3)$  for the LP in standard form in Tableau 5.2.1. The corresponding coefficient submatrix  $B$  is the  $3 \times 3$  coefficient matrix for the system on the left given below; it is nonsingular, and hence a basis and so  $x_B$  is a basic vector. The primal basic solution corresponding to it is obtained from the system of equations on the left given below. It is  $\tilde{x} = (2, 6, 1, 0, 0, 0)^T$ . So, this basic vector  $x_B$  is primal feasible, and it is primal nondegenerate since all the basic variables  $x_1, x_2, x_3$  are nonzero in the basic solution. The dual basic solution corresponding to  $x_B$  is the solution of the system of equations on the right given below. It is  $\tilde{\pi} = (3, 5, -4)$ . By substituting this solution in the dual constraints given in Tableau 5.2.3, we find that the vector of dual slacks at  $\tilde{\pi}$  are  $\bar{c} = (0, 0, 0, 3, 0, 4)$ , since  $\bar{c} \geq 0$ , it is dual feasible. So for this problem,  $x_B$  is both a primal and dual feasible basic vector. Also, verify that  $\tilde{x}, \tilde{\pi}$  satisfy the complementary slackness conditions “ $x_j \bar{c}_j = 0$ ” for all  $j$  (this automatically follows from the manner in which the dual basic solution corresponding to a basic vector is defined).

$x_1$	$x_2$	$x_3$	
1	2	3	17
0	1	-4	2
0	0	1	1
$x_4 = x_5 = x_6 = 0$ .			

$\pi_1$	$\pi_2$	$\pi_3$	
1	0	0	3
2	1	0	11
3	-4	1	-15

As another example, consider the vector  $x_{B_2} = (x_4, x_5, x_6)$  for the LP in standard form in Tableau 5.2.1. The corresponding coefficient submatrix  $B_2$  is the  $3 \times 3$  coefficient matrix for the system on the left given below, it is also nonsingular and hence a basis. The primal basic solution, obtained from the system on the left given below, is  $\hat{x} = (0, 0, 0, 0, 1, 1)^T$ ; it is primal feasible, but since the basic variable  $x_4$  is zero in it, it is primal degenerate. So  $x_{B_2}$  is a degenerate primal feasible basic vector for this problem. The dual basic solution corresponding to  $x_{B_2}$ , obtained from the system on the right given below is  $\hat{\pi} = (51/16, 6, -83/16)$ . By substituting  $\hat{\pi}$  in the dual constraints given in Tableau 4.3, we find that the vector of dual slacks at  $\hat{\pi}$  is  $\bar{c} = (-3/16, -11/8, 37/8, 0, 0, 0)$ . Since the first two components in this vector are  $< 0$ ,  $\hat{\pi}$  is dual infeasible; so  $x_{B_2}$  is a dual infeasible basic vector for the LP in Tableau 5.2.1.

$x_4$	$x_5$	$x_6$	
-2	1	16	17
1	1	1	2
-2	1	0	1
$x_1 = x_2 = x_3 = 0$			

$\pi_1$	$\pi_2$	$\pi_3$	
-2	1	-2	10
1	1	1	4
16	1	0	57

Suppose the basic vector  $x_B$  associated with the basis  $B$  for (5.4.1) is both primal and dual feasible. Let  $\tilde{x}, \tilde{\pi}$  be the corresponding primal and dual basic solutions. Then by their definition  $\tilde{x}, \tilde{\pi}$  satisfy all three conditions for optimality (primal and dual feasibility, and complementary slackness conditions) stated in Theorem 5.2.1. So,  $\tilde{x}$  is optimal to (5.4.1), and  $\tilde{\pi}$  is optimal to its dual. Hence the BFS associated with a basic vector for (5.4.1) which is both primal and dual feasible is always optimal. For this reason a basic vector for (5.4.1) which is both primal and dual feasible, is called an **optimal basic vector**.

## 5.5 Some Primal, Dual Properties

Here we will discuss without proofs, some results on the relationship between the primal and dual problems.

**Result 5.5.1: Duality Theorem:** *When an LP has an optimum solution, the dual also has an optimum solution, and the optimum objective values in the two problems are equal.*

**Result 5.5.2: Condition for the Uniqueness of the Dual Optimum Solution:** *If the primal has a nondegenerate optimum BFS (i.e., if the primal is an LP in standard form, all basic variables are  $> 0$  in that BFS), then the dual has a unique optimum solution.*

### Optimum Dual Solution and the Vector of Marginal Values

Consider the LP in standard form (5.5.1)

$$\begin{aligned} \text{Minimize } & z(x) = cx \\ \text{subject to } & Ax = b \\ & x \geq 0 \end{aligned} \quad (5.5.1)$$

where  $A$  is an  $m \times n$  matrix. The marginal value of  $b_i$  in this problem has been defined to be the rate of change in the optimum objective value per unit change in  $b_i$  from its current value, when this rate exists. Select a  $b_i$ , say  $b_1$ . Suppose we keep all the other data in the problem fixed at their current value, except  $b_1$ . Then as  $b_1$  varies, the optimum objective value in the problem is a function of  $b_1$  which we denote by  $f(b_1)$ . Then the marginal value of  $b_1$  in this problem is exactly

$$\frac{df(b_1)}{db_1} = \lim_{\epsilon \rightarrow 0} \frac{f(b_1 + \epsilon) - f(b_1)}{\epsilon}$$

when this derivative, or limit exists. If the derivative does not exist (i.e., if the function  $f(b_1)$  is not differentiable at the current value of  $b_1$ ) then the marginal value of  $b_1$  in this problem does not exist. Here is the result on the existence of marginal values.

**Result 5.5.3: Existence of Marginal Values:** *If (5.5.1) has a nondegenerate optimum BFS, then the dual problem has a unique*

*optimum solution. In this case, (5.5.1) has marginal values WRT all the RHS constants  $b_i$ , and the optimum dual solution is the vector of marginal values.*

This result says that if the primal optimum BFS obtained for (5.5.1) is nondegenerate, then it is perfectly valid to interpret the optimum dual solution,  $\pi = (\pi_i)$  as the marginal value vector.

Suppose  $\bar{x}, \bar{\pi}$  are the optimal primal and dual basic solutions of (5.5.1) obtained by an algorithm. If  $\bar{x}$  is a degenerate BFS (i.e., some basic variables have zero values in it), then  $\bar{\pi}$  may not be the unique dual optimum solution. In this case the marginal values may not be well defined for (5.5.1).

But in practice, when an optimum solution of an LP model like (5.5.1) is computed using floating point arithmetic on a digital computer, the zero value of a basic variable in the optimum solution  $\bar{x}$  obtained may in reality be a small positive value that has become 0 due to roundoff errors. Because of this, the theoretical possibility of a degenerate BFS is very hard to check in practice. That's why even when some basic variables are 0 in the optimum BFS  $\bar{x}$  obtained, practitioners usually ignore the degeneracy warning and continue to interpret  $\bar{\pi}$  as an approximation to a marginal value vector.

## 5.6 Marginal Analysis

When an optimum BFS obtained for an LP is nondegenerate, the dual problem has a unique optimum solution, and that optimum dual solution is the vector of marginal values (i.e., rates of change in the optimum objective value per unit change in the value of an RHS constant from present level, while all the other RHS constants remain at present levels). Marginal analysis is economic cost/benefit analysis of the various options available to the system based on these marginal values. In Section 3.13 we presented examples of these analyses for planning applications in the fertilizer manufacturer's problem. We will now illustrate marginal analysis using another example.

### Example 5.6.1: Marginal Analysis in a Company Using 4 Processes

A company needs products  $P_1, P_2, P_3$  for its internal use. There are four different processes that the company can use to make these products. When a process is run, it may produce one or more of these products as indicated in the following table.

Product	Output (units)/ unit time of process				Minimum daily requirement for product (in units)
	1	2	3	4	
$P_1$	1	2	0	1	17
$P_2$	2	5	1	2	36
$P_3$	1	1	0	3	8
Cost (\$) of running process/unit time	28	67	12	35	

For  $j = 1$  to 4, let  $x_j$  denote the units of time that process  $j$  is run daily. Let  $x_5, x_6, x_7$  denote the slack variables corresponding to  $P_1, P_2, P_3$  (these are the amounts of the product produced in excess of the minimum daily requirement). Then the model for meeting the requirements of the products at minimum cost is the following LP in standard form.

**Tableau 5.6.1: Original tableau**

Item	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$-z$	$b$
$P_1$	1	2	0	1	-1	0	0	0	17
$P_2$	2	5	1	2	0	-1	0	0	36
$P_3$	1	1	0	3	0	0	-1	0	8
	28	67	12	35	0	0	0	1	0

$x_j \geq 0$  for all  $j$ ;  $x_5, x_6, x_7$  are  $P_1, P_2, P_3$  slacks; minimize  $z$

This problem has been solved by the simplex method, yielding the following optimum canonical tableau. BV stands for “basic variable in the row”.

Tableau 5.6.2: Canonical tableau

BV	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$-z$	$b$
$x_1$	1	0	-2	1	-5	2	0	0	13
$x_2$	0	1	1	0	2	-1	0	0	2
$x_7$	0	0	-1	-2	-3	1	1	0	7
$-z$	0	0	1	7	6	11	0	1	-498

The basis  $B$  associated with the basic vector  $x_B = (x_1, x_2, x_7)$  is

$$B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

So, denoting the dual variables associated with the three equality constraints in that order by  $\pi_1, \pi_2, \pi_3$ , the system to compute the optimum dual basic solution associated with this basic vector is

$\pi_1$	$\pi_2$	$\pi_3$	
1	2	1	28
2	5	1	67
0	0	-1	0

whose solution is  $\bar{\pi} = (6, 11, 0)$ .

So, the optimum primal BFS is  $\bar{x} = (13, 2, 0, 0, 0, 0, 7)^T$ . Hence the optimal policy is to run processes 1, 2 for 13, 2 units of time daily, and not use processes 3, 4. This solution attains the minimum cost of \$498, and produces 17, 36, 15 units of  $P_1, P_2, P_3$  respectively; meeting the minimum daily requirements of  $P_1, P_2$  exactly, but leaving an excess of 7 units of  $P_3$  after meeting its requirement. Since the optimal primal BFS is nondegenerate, the vector of marginal values of  $P_1, P_2, P_3$  is the optimum dual solution =  $(6, 11, 0)$ .

So, the marginal value of  $P_3$  is 0. This means that small changes in its daily requirement in the neighborhood of its present value of 7 units, does not change the cost. At the moment the requirement of  $P_3$  is automatically covered while meeting the requirements of  $P_1, P_2$ , this actually produces an excess of 7 units of  $P_3$  beyond its requirement.

$P_2$  has the highest marginal value of \$11 among the three products. This means that small changes in its requirement from its present

level of 36 units result in a change in the optimum cost at the rate of \$11/unit. And if a reliable outside supplier were to offer to supply  $P_2$ , it is worth considering that supplier for outsourcing this product if the rate is  $\leq$  \$11/unit. Since it has the highest marginal value,  $P_2$  is a critical input for the company.

A similar interpretation can be made for  $P_1$  and its marginal value of \$6/unit.

Suppose the company's research lab has come up with a new process, process 8, which produces  $P_1, P_2$  at the rate of 4, 9 units per unit time it is run, and does not produce any  $P_3$ . Let  $c_8$  be the cost of running process 8 per unit time. For what values of  $c_8$  is it desirable to run process 8? To answer this question, we evaluate the monetary benefit, in terms of the marginal values, of the output by running this process per unit time. Since it is 4, 9 units of  $P_1, P_2$  respectively, and the marginal values of  $P_1, P_2$  are 6, 11; this monetary benefit is  $4 \times 6 + 9 \times 11 = \$123$ /unit time. Comparing this with the cost  $c_8$  of running this process we conclude that process 8 is not worth running if,  $c_8 > 123$ , it breaks even with the present optimum solution if  $c_8 = 123$ , and can save cost if  $c_8 < 123$ .

Marginal analysis is this kind of cost-benefit analysis using the marginal values. It provides very valuable planning information.

Practitioners often use this kind of analysis using an optimum dual solution provided by the simplex method, even when the optimal primal solution is degenerate. As pointed earlier, this may lead to wrong conclusions in this case, so one should watch out.

## 5.7 Sensitivity Analysis

Data such as I/O coefficients, cost coefficients, and RHS constants in LP models for real world applications are normally estimated from practical considerations, and may have unspecified errors in them. Given an optimum basic vector, the **optimality range** of a data element, is the interval within which that element can vary, *when all the other data remain fixed at their current values*, while keeping the present solution or basic vector feasible and optimal. Ranging techniques in sensitivity analysis determine the optimality range of some

of the data elements very efficiently. The robustness of the present optimum solution or optimum basic vector to errors in a data element can be checked using the width of its optimality range and the position of its present value in this range.

Sensitivity analysis also has efficient techniques for finding a new optimum solution beginning with the current one, if the values of a few data elements (typically one) change. Ranging and these other techniques in sensitivity analysis are all based on simple arguments using the optimality criteria. Here we discuss the simplest among sensitivity analysis techniques to provide a flavor of these techniques.

### Ranging a Nonbasic Cost Coefficient, and Finding a New Optimum Solution When Its Value Moves Outside This Range

Consider the LP model (5.7.1), where  $A$  is a matrix of order  $m \times n$  and rank  $m$ .

$$\begin{array}{cc|c} x & -z & \\ \hline A & 0 & b \\ \hline c & 1 & 0 \\ \hline x \geq 0, \min z \end{array} \quad (5.7.1)$$

Suppose an optimum basic vector  $x_B$  for this LP has been found, and let  $B$ ,  $c_B$  be the basis, and row vector of original basic cost coefficients. Rearranging the variables in (5.7.1) into basic, nonbasic parts WRT  $x_B$ , (5.7.1) can be written as below (in the nonbasic part, we show the column vector of a general nonbasic variable denoted by  $x_s$ ).

Basic	Nonbasic		
$x_B$	$\dots x_s \dots$	$-z$	
$B$	$\dots A_{.s} \dots$	0	$b$
$c_B$	$\dots c_s \dots$	1	0, minimize

Suppose the optimum canonical tableau is the one given below.

BV	$x_B$	$\dots x_s \dots$	$-z$	
$x_B$	$I$	$\dots \bar{A}_{.s} \dots$	$0$	$\bar{b}$
$-z$	$0$	$\dots \bar{c}_s \dots$	$1$	$-\bar{z}$

Let  $\bar{x}, \bar{\pi}$  be the optimum primal and dual basic solutions WRT the basic vector  $x_B$  (from Section 5.4 we know that  $\bar{\pi} = c_B B^{-1}$ ). Also, from Section 5.4 we know that for each nonbasic variable  $x_s$ , its relative cost coefficient is  $\bar{c}_s = c_s - \bar{\pi} A_{.s}$ .

**Ranging question:** Suppose  $x_s$  is a nonbasic variable whose cost coefficient  $c_s$  is likely to change, while all the other data remain fixed at present levels. For what range of values of  $c_s$  does  $\bar{x}$  remain an optimum solution to the problem?

**Computation of the range:** To answer this question, notice that a change in  $c_s$  does not affect the primal or dual basic solutions associated with  $x_B$ , nor does it affect the primal feasibility of  $\bar{x}$ . However, for  $\bar{\pi}$  to remain dual feasible, we need  $\bar{c}_s = c_s - \bar{\pi} A_{.s} \geq 0$ , i.e.,  $c_s \geq \bar{\pi} A_{.s}$ . So,  $\bar{x}$  remains an optimum solution to the problem as long as  $c_s \geq \bar{\pi} A_{.s}$ , that is the optimality range for  $c_s$  is  $[\bar{\pi} A_{.s}, \infty]$ .

**Restoring optimality when data changes:** If the new value of  $c_s$  is  $< \bar{\pi} A_{.s}$ , then the new  $\bar{c}_s < 0$ , and the basic vector  $x_B$  is no longer dual feasible. In this case,  $x_s$  is eligible to enter  $x_B$ . To get the new optimum solution, correct the value of  $c_s$  in the original tableau, bring  $x_s$  into the basic vector  $x_B$ , and continue the application of the simplex algorithm until it terminates again.

### Example 5.7.1

As an example, consider the LP model in Tableau 5.6.1 of the company trying to produce the required quantities of  $P_1, P_2, P_3$  using four

available processes at minimum cost, discussed in Example 5.6.1. We reproduce the original tableau for the problem.

Original tableau									
Item	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$-z$	$b$
$P_1$	1	2	0	1	-1	0	0	0	17
$P_2$	2	5	1	2	0	-1	0	0	36
$P_3$	1	1	0	3	0	0	-1	0	8
	28	67	12	35	0	0	0	1	0

$x_j \geq 0$  for all  $j$ ;  $x_5, x_6, x_7$  are  $P_1, P_2, P_3$  slacks; minimize  $z$

The optimum basic vector for this problem is  $x_B = (x_1, x_2, x_7)$ . Here is the optimum canonical tableau.

Optimum canonical tableau									
BV	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$-z$	$\bar{b}$
$x_1$	1	0	-2	1	-5	2	0	0	13
$x_2$	0	1	1	0	2	-1	0	0	2
$x_7$	0	0	-1	-2	-3	1	1	0	7
$-z$	0	0	1	7	6	11	0	1	-498

Suppose the cost coefficient of  $x_4$ , the cost of running process 4 per unit time, is likely to change from its present value of \$35, while all the other data remains fixed. Denote the new value of this cost coefficient by  $c_4$ . For what range of values of  $c_4$  does the primal BFS in the above canonical tableau remain optimal to the problem? The answer: as long as the relative cost coefficient of  $x_4$ ,  $\bar{c}_4 = c_4 - (-6, -11, 0)(1, 2, 3)^T = c_4 - 28$  is  $\geq 0$ , i.e., as long as  $c_4 \geq 28$ . This is the optimality range for  $c_4$ .

If the new value of  $c_4$  is  $< 28$ , say  $c_4 = 27$ , the basic vector  $(x_1, x_2, x_7)$  is no longer dual feasible, because the new relative cost coefficient of  $x_4$  is  $-1$ . So,  $x_4$  is eligible to enter this basic vector. To get the new optimum solution, correct the original cost coefficient of  $x_4$  to its new value of 27, here is the original tableau for the new problem.

**Original tableau**

Item	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$-z$	$b$
$P_1$	1	2	0	1	-1	0	0	0	17
$P_2$	2	5	1	2	0	-1	0	0	36
$P_3$	1	1	0	3	0	0	-1	0	8
	28	67	12	27	0	0	0	1	0

$x_j \geq 0$  for all  $j$ ;  $x_5, x_6, x_7$  are  $P_1, P_2, P_3$  slacks; minimize  $z$

The first tableau in the list below is the canonical tableau for the new problem WRT the basic vector  $(x_1, x_2, x_7)$  (obtained from the canonical tableau of original problem by changing the relative cost coefficient of  $x_4$  to  $-1$ ). In it, we bring  $x_4$  into the basic vector  $(x_1, x_2, x_7)$  and continue the application of the simplex algorithm until it terminates again. PR, PC indicates pivot row, column respectively; and the pivot elements are boxed.

**Canonical tableaus**

BV	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$-z$	$b$	Ratio
$x_1$	1	0	-2	<span style="border: 1px solid black;">1</span>	-5	2	0	0	13	13
$x_2$	0	1	1	0	2	-1	0	0	2	
$x_7$	0	0	-1	-2	-3	1	1	0	7	
$-z$	0	0	1	-1	6	11	0	1	-498	
				PC $\uparrow$						
$x_4$	1	0	-2	1	-5	2	0	0	13	
$x_2$	0	1	<span style="border: 1px solid black;">1</span>	0	2	-1	0	0	2	2
$x_7$	2	0	-5	0	-13	5	1	0	33	
$-z$	1	0	-1	0	1	13	0	1	-485	
				PC $\uparrow$						

**New optimum canonical tableau**

$x_4$	1	2	0	1	-1	0	0	0	17	
$x_2$	0	1	1	0	2	-1	0	0	2	
$x_7$	2	5	0	0	-3	0	1	0	43	
$-z$	1	1	0	0	3	12	0	1	-483	

So,  $(x_4, x_2, x_7)$  is the new optimum basic vector with the optimum BFS  $\hat{x} = (0, 0, 2, 17, 0, 0, 43)^T$  with an optimum cost of 483. In terms

of the company, the new optimum solution involves running processes 3, 4 for 2 and 17 units of time daily.

### How Much Are Sensitivity Analysis Techniques Used in Practice?

The techniques of sensitivity analysis are simple techniques for restoring optimality when one data element changes in an LP model. They are all based on the optimality criteria used in the simplex algorithm. We discussed only the simplest of the sensitivity analysis techniques to give a taste of them to the readers. There are many others which can be looked up in graduate level LP books. These sensitivity analysis techniques offer great learning tools for students, to test how well they understand duality theory and the optimality conditions in LP. So, they have great educational value.

Practical applicability of these sensitivity analysis techniques is limited because in applications it is very rare for only one data element to change in an LP model. Usually, many changes occur in the model, and practitioners find it much simpler to solve the revised model from scratch again.

## 5.8 Exercises

**5.1.** Here is a diet problem to meet the minimum daily requirements (MDR) of two nutrients (thiamin and niacin) using 5 different foods, at minimum cost in an infants diet. Data given below.

Nutrient	nutrient units/oz. of food					MDR (units) for nutrient
	1	2	3	4	5	
Thiamin	0	1	2	1	1	4
Niacin	1	1	1	1	2	7
cost/oz	4	7	8	9	11	

(i) For  $i = 1$  to 5, let  $x_i$  denote the ozs of food  $i$  in the infant's daily diet. Using these decision variables, write the formulation of this

problem as an LP. Put this problem in standard form, and give the interpretation of any new variables you introduced in this process.

Find an optimum solution of this problem using the simplex method.

Do marginal values exist in this problem? Why? If so, compute the marginal values of thiamin and niacin, and clearly explain their economic interpretation.

It is commonly believed that meeting one unit niacin requirement is more expensive than meeting one unit thiamin requirement using these foods. Is this true? If so, determine how many times more expensive one unit niacin is than thiamin.

(ii) By how much should the price of food 5 have to decrease before it becomes competitive with other foods?

(iii) The nutritionist recommends that the MDR for thiamine be increased by 1 unit every 6 months as the infant grows (its requirement is 4 units/day now, it should be 5 units/day after 6 months, 6 units/day after 12 months, etc.), until the child reaches 2 years of age. Will the basic vector  $(x_3, x_1)$  remain optimal to the problem after 2 years from now? Why?

**5.2:** Flintink makes 2 printing inks with code names G and B using 3 raw materials  $R_1, R_2, R_3$ , according to the following data (inks & raw materials are measured in drums).

Raw material	Drums needed/drum of		Supply available /month
	G	B	
$R_1$	1	1	10
$R_2$	0	2	6
$R_3$	1	2	8
Net profit (\$/drum)	50	200	

Formulate the problem of determining how much of G, B to make per month to maximize total net profit as an LP.

Find an optimum solution of this problem, and the maximum net profit for the company, by the simplex method.

Find the marginal values of  $R_1, R_2, R_3$  in this problem showing clearly how they are obtained.

A new ink, P, has been developed. To make P needs 3 drums of  $R_1$ , 1 drum of  $R_2$ , and 1 drum of  $R_3$  per drum. Determine how much net profit P has to fetch/drum to make it worthwhile for the company to manufacture it, explaining your argument very clearly.

**5.3:** Consider the fertilizer manufacturer's problem again ( $z(x)$  = total daily net profit):

$$\begin{array}{rllllll}
 \max & z(x) = 15x_1 & + & 10x_2 & & & \text{Item} \\
 \\ 
 \text{S. to} & 2x_1 & + & x_2 & \leq & 1500 & \text{RM 1} \\
 & x_1 & + & x_2 & \leq & 1200 & \text{RM 2} \\
 & x_1 & + & & \leq & 500 & \text{RM 3} \\
 \\ 
 \text{and} & x_1 & \geq & 0 & & x_2 & \geq & 0
 \end{array}$$

where

$x_1, x_2$  = tons of Hi-ph, Lo-ph manufactured daily  
 15, 10 = net profit coeffs., \$/ton of Hi-ph, Lo-ph  
 RM 1, 2, 3 : Three raw materials used in manufacturing with  
 daily availabilities of 1500 tons, 1200, 500 respectively.

To obtain these net profit coeffs. of Hi-ph, Lo-ph, the cost of raw materials needed to make them, and the manufacturing costs, have been subtracted from their selling price. The raw materials RM-1, 2, 3 come from the company's own quarries, and their costs are \$12, 7, 15/ton respectively.

The optimum solution of the problem is  $\bar{x} = (300, 900)^T$ .

The optimum dual solution = the vector of marginal values of RM 1, 2, 3 in this problem is ( \$ 5 , 5 , 0 ).

Since the company is unable to increase the supply of RM 1, 2, 3 from their quarry, they have started looking for outside suppliers for them. A supplier has offered to sell the company

RM 1 at the rate of \$18/ton

RM 2 at the rate of \$11/ton

RM 1 at the rate of \$15/ton

Discuss whether the company should consider buying any of RM 1, 2, 3 from this supplier, explaining the reasons for your conclusion very carefully.

**5.4:** A company manufactures products  $A$  to  $G$  using two types of machines  $P_1, P_2$ ; and three raw materials  $R_1, R_2, R_3$ . Relevant data is given below. Machines time is measured in machine hours, and each raw material input is measured in its own units. Profit coefficients for each product are given in \$/unit product made.

Item	Item input (units)/ unit of							Max. available per day
	$A$	$B$	$C$	$D$	$E$	$F$	$G$	
$R_1$	0.1	0.3	0.2	0.1	0.2	0.1	0.2	500
$R_2$	0.2	0.1	0.4	0.2	0.2	0.3	0.4	750
$R_3$	0.2	0.1	0.1	0.2	0.1	0.2	0.3	350
$P_1$ time	0.02	0.03	0.01	0.04	0.01	0.02	0.04	60
$P_2$ time	0.04		0.02	0.02	0.06	0.03	0.05	80
Bound on output		$\leq 800$			$\leq 400$			
Profit	10	12	8	15	18	10	19	

**(a):** Let  $x_1$  to  $x_7$  denote the units of products  $A$  to  $G$  in that order/day. Let  $x_8$  to  $x_{12}$  denote the slack variables associated with the supply constraint on  $R_1, R_2, R_3, P_1$ -time,  $P_2$ -time respectively. Let  $x_{13}, x_{14}$  be the slack variables associated with the upper bound constraints on the production of  $B, E$  respectively. Using these decision variables, formulate the product mix problem to maximize total daily profit as an LP.

**(b):** The optimum basic vector for this problem is  $(x_2, x_3, x_4, x_5, x_9,$

$x_{10}, x_{13}$ ) with the values of the basic variables in the optimum BFS as (466.7, 1000.0, 800.0, 400.0, 63.3, 3.3, 333.3)

Let  $\pi_1$  to  $\pi_7$  denote the dual variables associated with the supply constraints on  $R_1, R_2, R_3$ ,  $P_1$ -time,  $P_2$ -time, and upper bound constraint on the daily production of  $B, E$  respectively in that order. Using these dual variables, write the dual problem.

The optimum dual basic solution associated with the basic vector  $(x_2, x_3, x_4, x_5, x_9, x_{10}, x_{13})$  is  $\pi = (12.5, 0, 0, 275.0, 137.5, 0, 4.5)$ . Also, answer each of the following questions about this original problem.

- (i) Are the marginal values of the various items well defined in this problem? If so, what are they?
- (ii) Is it worth increasing the supply of  $R_1$  beyond the present 500 units/day? The current supplier for  $R_1$  is unable to supply any more than the current amount. The procurement manager has identified a new supplier for  $R_1$ , but that supplier's price is \$15/unit higher than the current suppliers'. Should additional supplies of  $R_1$  be ordered from this new supplier?
- (iii) The production manager has identified an arrangement by which some extra hours/day of either  $P_1$ - or  $P_2$ -time can be made available at a cost of \$150/day. Is it worth accepting this arrangement, and if so for which of these machines?
- (iv) The sales manager would like to know the relative contributions of the various products in the company's total profit. What are they?
- (v) The production manager claims that the manufacturing process for  $G$  can be changed so that its need for  $P_1$ -time goes down by 50% without affecting quality, demand or selling price. What will be the effect of this change on the optimum product mix and total profit?
- (vi) The company's research division has formulated a new product,  $H$ , which they believe can yield a profit of \$8-10/unit made. The input requirements to make one unit of this product will be

Item	$R_1$	$R_2$	$R_3$	$P_1$ -time	$P_2$ -time
Input	0.1	0.2	0.1	0.02	0.02

Is this product worth further consideration?

- (vii) The sales manager feels that the selling price/unit of product  $F$  can be increased by \$2 without affecting the demand for it. Would this lead to any changes in the optimum production plan? What is the effect of this change on the total profit?

(D. C. S. Shearn, "Postoptimal Analysis in Linear Programming-The Right Example", *IIE Transactions* 16, no. 1(March 1984)99-101.)

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